Quantum interferometric power and the role of nonclassical correlations in quantum metrology for a special class of two-qubit states

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Abstract

We analyze the effects of quantum correlations on the parameter precision in an interferometric configuration. As probe states, we consider a class of two-qubit states for which the analytical expression of the quantum interferometric power, quantifying the quantum correlations, is explicitly derived. Also, we give and analyze the local quantum Fisher information, which evaluates the sensitivity of the probe state to the phase shift, for some relevant local Hamiltonians. The discord-like quantum correlations based on the notion of quantum interferometric power is compared with the original quantum discord based on von Neumann entropy. We also examine the significance of quantum correlations in enhancing the precision of the phase estimation. Our study corroborates the recent series of investigations focusing on the role of quantum correlations other than entanglement on the efficiency of quantum metrology protocols.

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1 Introduction

Quantum metrology is an emerging field in quantum information science [1, 2, 3, 4]. This discipline allows to achieve high precision measurements that are unattainable with classical laws of physics. During the last decade, different metrology protocols have been theoretically investigated and experimentally implemented to enhance the precision in estimating various parameters [5-18] (see also the references quoted in [4]). This special interest is mainly motivated by the fact that in quantum metrology the parameter estimation goes beyond the classical limit and in some cases tends to the Heisenberg limit imposed by the laws of quantum mechanics. The quantum metrology originates from the theory of quantum estimation which has been the subject of important investigations during the last 20 years [19, 20, 21]. The key ingredient in quantum metrology is the notion of quantum Fisher information [22] and especially its inverse which depicts the lower bound in statistical estimation of an unknown parameter according to Cramér-Rao theorem [19, 20, 23]. Now, it is well established that quantum metrology offers the possibility to surpass the restrictions imposed by classical laws of physics and there is a common belief that the fabrication of sensing devices, based on the quantum mechanical laws of physics, would be a priori possible.

Recently, it has been found that the use of entangled states can provide a better sensitivity in estimating an unknown phase shift. Indeed, the estimation of a parameter encoded in a unitary transformation (e.g. a phase shift) of probe states involving n non-entangled qubits, the precision scales as $\frac{1}{\sqrt{n\nu}}$ where ν is the number of repeated measurements which ameliorates the classical scaling given by $\frac{1}{\sqrt{\nu}}$. In the presence of n entangled qubits the optimal scaling rewrites $\frac{1}{n\sqrt{\nu}}$ enhancing the standard scaling limit by a factor of $\sqrt{n}$ [2, 3, 4]. In this sense, it is natural to ask if the quantum correlation other than entanglement can ameliorate the precision in metrology protocols. This issue was recently addressed in [24, 25, 26, 27, 28]. In fact, to understand the role of quantum correlation beyond entanglement in a black-box quantum metrology task, a quantum correlation quantifier termed as quantum interferometric power was recently introduced [24] (see also [26]). This discord-like measure of quantum correlations is defined in term of quantum Fischer information and gives also the precision in an interferometric phase estimation. This adds a new tool to the existing list of quantum correlations quantifiers which have been the subject of numerous studies from different perspectives and for various purposes. The most familiar one is the entropic quantum discord introduced in 2001 to describe the quantum correlations which are not limited to entanglement [29, 30]. However, it must be noticed that the derivation of the analytical expressions of the quantum discord based on von Neumann entropy involves an optimization procedure which is very challenging and only partial results were obtained for some special two-qubit states. In this respect, the quantum interferometric power offers a promising tool to characterize the quantum correlations in multipartite systems. This is essentially due to its reliability and its ease of computability.

In a given estimation protocol, the determination of some unknown parameter can be achieved by measuring a probe system whose quantum state depends on that parameter. Hence, one should first prepare the input state (i.e. ρ) which has to be sensitive to the parameter variations. The second step consists in encoding of the information about the unknown parameter (i.e. θ). This encoding can be realized by be a unitary evolution (i.e. $\rho \longrightarrow \rho_\theta$). The final part of this protocol concerns the measurement of an appropriate observable (i.e. H) in the output state (i.e. ρ_θ). Adopting this picture, we we shall examine in this work the role of quantum correlations in quantum phase estimation when the generic probe states are of the form

$$\rho = \begin{pmatrix} c_1 & 0 & 0 & \sqrt{c_1c_2} \\ 0 & \frac{1}{2}(1-c_1-c_2) & \frac{1}{2}(1-c_1-c_2) & 0 \\ 0 & \frac{1}{2}(1-c_1-c_2) & \frac{1}{2}(1-c_1-c_2) & 0 \\ \sqrt{c_1c_2} & 0 & 0 & c_2 \end{pmatrix} \tag{1}$$

in the computational basis $\mathcal{B} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The real parameters c_1 and c_2 satisfy the conditions $0 \leq c_1 \leq 1$, $0 \leq c_2 \leq 1$ and $0 \leq c_1 + c_2 \leq 1$. Examples of quantum systems described by states of this type were investigated in many works especially ones concerning collective spin models. One may quote for instance bipartite density matrices extracted from symmetric multi-qubit states such as the superpositions of Dicke states [31, 32] and even and odd spin coherent states [33]. States of type (1) are also relevant in investigating the pairwise quantum correlation in balanced superpositions of multipartite Glauber coherent states which interpolates continuously between the generalized Greenberger-Horne-Zeilinger and Werner states [34, 35, 36, 37]. It is important to note that we have deliberately chosen two-qubit states of rank 2. This will simplify the derivation of the quantum discord based on the von Neumann entropy by employing the method reported in [38].

This paper is organized as follows. In Section 2, the local quantum Fisher information is derived for the states (1) when the dynamics of the first qubit is governed by a local Hamiltonian. We determine the minimum of the local quantum Fisher information when the first qubit is driven by an arbitrary local Hamiltonian. This analysis is similar to the derivation of the quantum interferometric power discussed in Section 3. Indeed, the quantum interferometric power is defined by minimizing the quantum Fisher information over all local Hamiltonians. We give the explicit form of this discord-like quantifier and we show the role of quantum correlations in enhancing the phase precision in an interferometric setup. More precisely, we discuss how the quantum correlations in the probe states (1) are responsible of better better sensitivity. We also compare the amount of quantum correlations in the states (1) measured by the quantum interferometric power with the entropy based quantum discord. This comparison is essential to test the reliability of quantum interferometric power as indicator of quantumness. In section 4, tighter bounds on the phase precision in the presence of quantum correlations are investigated in the spirit of the recent results obtained in [24, 26]. Concluding remarks close this paper.

2 Local quantum Fisher information

The process of estimating the value of an unknown parameter consists in three steps: (*i*) the preparation of the probe state (input-state), (*ii*) the interaction of the initialized state with the system (target) encoding the physical quantity to be estimated and finally (*iii*) the measure of the state (output-state) resulting from the interaction of the input-state and the system. In the situations where the Hamiltonian governing the dynamics of the probe state is known one can determine the value of the unknown parameter. In quantum metrology, the interferometric configuration constitutes one of the most interesting scenarios which are widely used in phase estimation (see for instance [4] and [39]). As previously mentioned, the two-qubit states (1) are considered as probe states and we assume that the dynamics of the first qubit is governed by the local phase shift transformation $e^{-i\theta H} \equiv e^{-i\theta\ H_1 \otimes \mathbb{I}}$ where H_1 is a local Hamiltonian acting on the qubit 1 and $\mathbb{I}$ denotes the 2×2 identity matrix. In this picture, the output states are given by

$$\rho_\theta = e^{-i\theta H} \rho e^{+i\theta H}.$$

From the measurement of the observable H in the output states, the parameter θ can be estimated through an (unbiased) estimator $\hat{\theta}$. The quantum mechanics imposes the fundamental limit of the variance of the estimator $\hat{\theta}$. This is given by the quantum Cramér-Rao bound:

$$\text{var } \hat{\theta} \geq \frac{1}{\nu F(\rho, H)}$$

where ν is the number of times the estimation protocol is repeated and $F(\rho, H)$ is the quantum Fisher information. For the parameter dependent states ρ_θ, the quantum Fisher information is defined by

$$F(\rho_\theta) \equiv F(\rho, H) = \text{Tr}(\rho_\theta L_\theta^2) \tag{2}$$

where L_θ is the symmetric logarithmic derivative determined by the algebraic equation

$$\partial_\theta \rho = \frac{1}{2}(\rho_\theta L_\theta + L_\theta \rho_\theta). \tag{3}$$

It is clear that the spectral decomposition of the density matrix and its derivative with respect to the parameter θ provides us with the expression of the quantum Fisher information. The eigenvalues of the density matrices ρ (1) write

$$\lambda_1 = c_1 + c_2, \quad \lambda_2 = 1 - (c_1 + c_2), \quad \lambda_3 = 0 \quad , \lambda_4 = 0, \tag{4}$$

and the corresponding eigenstates are respectively given by

$$|\psi_1\rangle = \sqrt{\tfrac{c_1}{c_1+c_2}}(1, 0, 0, \sqrt{\tfrac{c_2}{c_1}}), \quad |\psi_2\rangle = \tfrac{\sqrt{2}}{2}(0, 1, 1, 0), \quad |\psi_3\rangle = \tfrac{\sqrt{2}}{2}(0, 1, -1, 0), \quad |\psi_4\rangle = -\sqrt{\tfrac{c_2}{c_1+c_2}}(1, 0, 0, \sqrt{\tfrac{c_1}{c_2}}). \tag{5}$$

The explicit expression of quantum Fisher information was derived in [40] for density matrices with arbitrary ranks. For the states under consideration, it is simple to check that the quantum Fisher information takes the form

$$F(\rho, H) = \sum_{i=1}^{2} \lambda_i F(|\psi\rangle_i, H) - 8 \sum_{i \neq j}^{2} \frac{\lambda_i \lambda_j}{\lambda_i + \lambda_j} |\langle \psi_i | H | \psi_j \rangle|^2. \tag{6}$$

The quantity $F(|\psi\rangle_i, H)$ is expressed in term of the variance of the operator H on the eigenstate $|\psi\rangle_i$ as

$$F(|\psi\rangle_i, H) = 4(\Delta H)^2_{|\psi\rangle_i}, \tag{7}$$

where the variance of the Hamiltonian H is given by $(\Delta H)^2_{|\psi\rangle_i} = \langle \psi_i | H^2 | \psi_i \rangle - |\langle \psi_i | H | \psi_i \rangle|^2$. It is interesting to note that the quantum Fisher information involves only the non vanishing eigenvalues of ρ and their corresponding eigenstates. The dynamics of the probe state is assumed to be governed by local unitary transformations acting on the first qubit while leaving the second qubit unchanged. The general form of such local Hamiltonians is

$$H_1 = \overrightarrow{r}.\overrightarrow{\sigma} := r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3 \tag{8}$$

where $\overrightarrow{r} = (\sin\alpha\ \cos\beta, \sin\alpha\ \sin\beta, \cos\alpha)$ and the components of $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the usual Pauli matrices ($\sigma_1 = |0\rangle\langle 1| + |1\rangle\langle 0|,\ \sigma_2 = i(|1\rangle\langle 0| - |0\rangle\langle 1|),\ \sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|$). Using the equation (6), one verifies that the local quantum Fisher information in the states (1) writes as

$$F(\rho, H) = 4 - 4\cos^2\alpha \frac{(c_1 - c_2)^2}{c_1 + c_2} - 8\sin^2\alpha \bigg(1 - (c_1 + c_2)\bigg)\bigg((c_1 + c_2) + 2\sqrt{c_1 c_2}\cos 2\beta\bigg). \tag{9}$$

Now, we shall we shall consider three particular forms of the local Hamiltonian (8) governing the dynamics of the first qubit: $H_1 = \sigma_1$, $H_1 = \sigma_2$ and $H_1 = \sigma_3$ and we shall derive the bounds to precision in each scenario to determine the probe state in the family (1) which guarantees a minimum estimation efficiency. Clearly, by setting the form of H_1 we assume the prior knowledge of the Hamiltonian H.

For a unitary evolution along the x-direction (i.e. $H_1 = \sigma_1$) in (8)), the local quantum Fisher information (9) becomes

$$F(\rho, \sigma_1) = 4 - 8[1 - (c_1 + c_2)][\sqrt{c_2} + \sqrt{c_1}]^2 \tag{10}$$

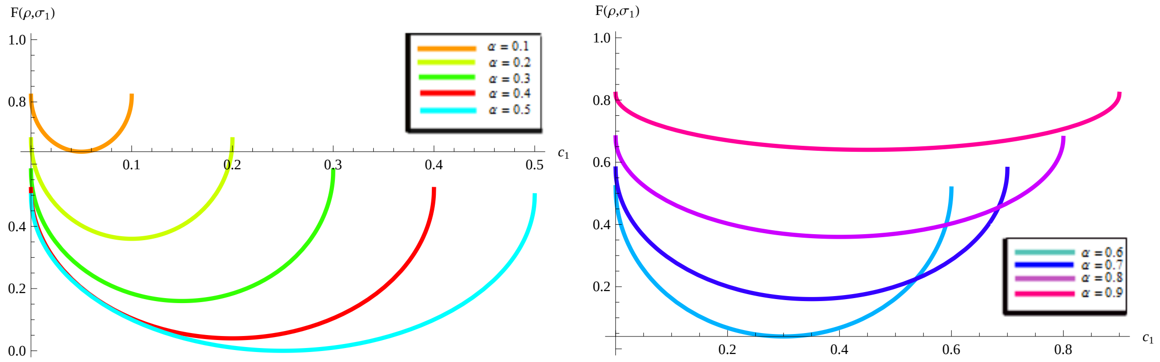

Figure 1. The quantum Fisher information $F(\rho, \sigma_1)$ along the x-direction versus c_1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

The behavior of the quantum Fisher information along the x-direction $F(\rho, \sigma_1)$, as function of the parameters c_1, is represented in the figure 1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$ where the quantity $\alpha = c_1 + c_2$ takes the particular values $\alpha = 0.1, 0.2, \cdots, 0.9$. As it can be inferred from this figure, the quantum Fisher information $F(\rho, \sigma_1)$ in the states (1) reaches its minimal value for states with $c_1 = c_2 = \frac{\alpha}{2}$. They are given by

$$\rho\bigg(c_1 = \frac{\alpha}{2}, c_2 = \frac{\alpha}{2}\bigg) = \alpha\ |\psi'\rangle\langle\psi'| + (1 - \alpha)\ |\psi\rangle\langle\psi| \tag{11}$$

where $|\psi\rangle$ and $|\psi'\rangle$ are states of Bell type:

$$|\psi\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad , \quad |\psi'\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{12}$$

On the other hand, the maximal value of $F(\rho, \sigma_1)$ is obtained in the states with $(c_1 = 0, c_2 = \alpha)$ or $(c_1 = \alpha, c_2 = 0)$. This value maximal is obtained in the states

$$\rho(c_1 = 0, c_2 = \alpha) = \alpha|11\rangle\langle 11| + (1 - \alpha)|\psi\rangle\langle\psi| \tag{13}$$

and

$$\rho(c_1 = \alpha, c_2 = 0) = \alpha|00\rangle\langle 00| + (1 - \alpha)|\psi\rangle\langle\psi|. \tag{14}$$

Hence for $\alpha \leq \frac{1}{2}$, the states encompassing high amount of quantum Fisher information are those with small values of the α. This situation is completely different in the interval $[\frac{1}{2}, 1]$ where the quantum Fisher information increases as the parameter α increases. For instance, for $c_1 = 0.45$, more precision is guaranteed in the states with $\alpha = 0.9$. For the subset of states of type (1) characterized by a fixed value of α ($\alpha \geq \frac{1}{2}$), the quantum Fisher information $F(\rho, \sigma_1)$ is maximal for $(c_1 = 0, c_2 = \alpha)$ or $(c_1 = \alpha, c_2 = 0)$ and the minimal value is reached for $c_1 = c_2 = \frac{\alpha}{2}$. We note that the local quantum Fisher plotted in the figure 1 (as well as in the figures 2 and 3) are normalized by the multiplicative factor $\frac{1}{4}$.

The second scheme we consider concerns the situation when the input state undergoes along the y-direction (i.e, $H_1 = \sigma_2$). In this case, the quantum Fisher information (9) reads as

$$F(\rho, \sigma_2) = 4 - 8[1 - (c_1 + c_2)][\sqrt{c_2} - \sqrt{c_1}]^2 \tag{15}$$

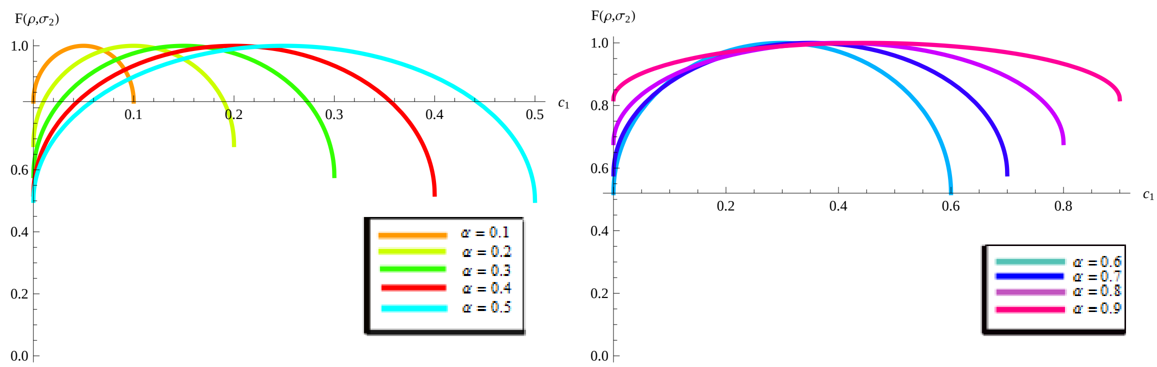

Figure 2. The quantum Fisher information $F(\rho, \sigma_2)$ along the y-direction versus c_1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

The figure 2 gives the quantum Fisher information $F(\rho, \sigma_2)$ as function of the parameter c_1 and c_2 for different values of $\alpha = c_1 + c_2$. For both cases $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$, the quantum Fisher information $F(\rho, \sigma_2)$ is maximal for the states satisfying $c_1 = c_2 = \frac{\alpha}{2}$ given by (11) and it is minimal for the states with ($c_1 = 0$, $c_2 = \alpha$) given by (13) or the states with ($c_1 = \alpha$, $c_2 = 0$) given by (14). It follows that when the dynamics of probe states is governed by the local Hamiltonian $H_1 = \sigma_2$, the best estimation is provided by the states (11). We recall that for the local Hamiltonian $H_1 = \sigma_1$, the best estimation is guaranteed by the probe states with $(c_1 = 0, c_2 = \alpha)$ and $(c_1 = \alpha, c_2 = 0)$ given respectively by (13) and (14).
Finally, for $H_1 = \sigma_3$ the equation (9) gives

$$F(\rho, \sigma_3) = 4 - 4\frac{(c_1 - c_2)^2}{c_1 + c_2} \tag{16}$$

By comparing the results reported in the figures 2 and 3, it is easily seen that the local quantum Fisher information $F(\rho, \sigma_3)$ behaves like the quantum Fisher information $F(\rho, \sigma_2)$. Also, the expression $F(\rho, \sigma_3)$ is maximal for the probe states with $c_1 = c_2 = \frac{\alpha}{2}$ (see equation (11)). This indicates that, when the local Hamiltonian σ_3 governs the dynamics of the first qubit, the suitable probe states are given by (11) which are of Bell type.

The results obtained in this section are very enlightening especially when the estimation protocol is blind in

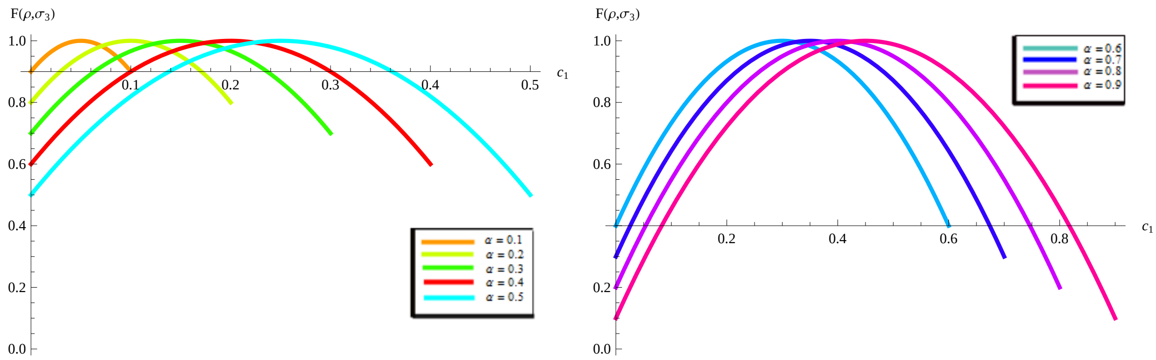

Figure 3. The quantum Fisher information $F(\rho, \sigma_3)$ along the z-direction versus c_1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

the sense that the one prepares the probe state without any prior knowledge of the Hamiltonian governing the dynamics of first subsystem. This issue is examined in what follows.

3 Nonclassical correlations and quantum interferometric power

To investigate the role of non classical correlation in improving the precision in quantum metrology protocols when the probe states are of type (1), we employ quantum interferometric power, introduced recently in [24], as quantifier of quantum correlations. We also compare this quantum correlations indicator with the quantum discord based on von Neumann entropy derived in [29, 30]. The quantum interferometric power is a bona fide measure of discord-like correlations [24]. It is defined by the minimum of the quantum Fischer information [24]

$$\mathcal{P}(\rho) = \frac{1}{4} \min_{H_1} F(\rho, H_1), \tag{17}$$

where the optimization is performed over all Hamiltonians $\{H_1\}$ acting on the qubit 1. Since, the quantum interferometric power is defined in term of quantum Fisher information, it is naturally related the degree of precision that a bipartite state ρ provides to ensure the success of the estimation protocol regardless of the phase direction [24]. The properties of quantum Fisher information [22] confers to quantum interferometric power many interesting properties: (i) non negative, (ii) invariant under local unitary transformations (iii) non increasing under local operations on the second qubit and (iv) asymmetric with respect to the two subsystems (except for symmetric quantum states). A closed analytical expression of the quantum interferometric power for an arbitrary bipartite quantum system was derived in [24]. Explicitly, it writes as

$$\mathcal{P}(\rho) = \lambda_M^{\min}, \tag{18}$$

where $\lambda_M^{\min}$ is the smallest eigenvalue of the 3×3 matrix M whose elements are defined by

$$M_{ij} = \frac{1}{2} \sum_{k,l:\lambda_k+\lambda_l \neq 0} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} \langle \psi_k | \sigma_i \otimes \mathbb{I} | \psi_l \rangle \langle \psi_l | \sigma_j \otimes \mathbb{I} | \psi_k \rangle \tag{19}$$

with λ_i and $|\psi_i\rangle$ being respectively the eigenvalues and the eigenvectors of density matrix ρ. When the probe states are of the form (1), the eigenvalues are given by (4) and the corresponding eigenstates are given by (5). Reporting (4) and (5) in the matrix elements (19), it is simple to check that off-diagonal element of the matrix

M are zero and the diagonal elements write as

$$M_{11} = 1-2\big[1-c_1-c_2\big]\big[\sqrt{c_2}+\sqrt{c_1}\big]^2, \ M_{22} = 1-2\big[1-c_1-c_2\big]\big[\sqrt{c_2}-\sqrt{c_1}\big]^2, \ M_{33} = 4\frac{c_1c_2}{c_1+c_2}+\big[1-c_1-c_2\big] \quad (20)$$

in terms of the parameters c_1 and c_2 labeling the probe states (1). Comparing the elements of the correlation matrix M and the quantum Fisher information given by the equations (10), (15) and (16), one has

$$M_{ii} = \frac{1}{4}F(\rho, \sigma_i), \qquad \text{for} \quad i = 1, 2, 3$$

and the quantum interferometric power (17) is the minimal amount of quantum information in the three spatial directions x, y and z discussed in the previous section. It follows that the quantum interferometric power writes

$$\mathcal{P}(\rho) = \frac{1}{4}\min(F(\rho, \sigma_1), F(\rho, \sigma_2), F(\rho, \sigma_3)). \quad (21)$$

From (20), it is simple to see that M_{22} is always greater than M_{11}. It follows that the smallest eigenvalues of the matrix M is either M_{11} or M_{33}. The difference $M_{11} - M_{33}$ is positive when the parameters c_1 and c_2 satisfy the condition

$$2(c_1 + c_2)^2 - (\sqrt{c_1} + \sqrt{c_2})^2 \geq 0. \quad (22)$$

Therefore for states with $\alpha \leq \frac{1}{2}$ ($\alpha = c_1 + c_2$), the quantity $M_{11} - M_{33}$ is non positive and the quantum interferometric power (21) writes as

$$\mathcal{P}(\rho) = 1 - 2[1 - (c_1 + c_2)][\sqrt{c_2} + \sqrt{c_1}]^2. \quad (23)$$

For states with $\alpha \geq \frac{1}{2}$, the condition (22) is satisfied for

$$0 \leq c_1 \leq \alpha_- \qquad \text{or} \qquad \alpha_+ \leq c_1 \leq \alpha \quad (24)$$

where the quantities $\alpha_\pm$ are defined by

$$\alpha_\pm = \frac{1}{2}\alpha \pm \sqrt{\alpha^3 - \alpha^4}. \quad (25)$$

In this case, the quantum interferometric power is given by

$$\mathcal{P}(\rho) = 1 - \frac{(c_1 - c_2)^2}{c_1 + c_2}, \quad (26)$$

Conversely for $\alpha_- \leq c_1 \leq \alpha_+$, the difference $M_{11} - M_{33}$ is negative and the quantum interferometric power reads as

$$\mathcal{P}(\rho) = 1 - 2\big[1 - (c_1 + c_2)\big]\big[\sqrt{c_2} + \sqrt{c_1}\big]^2. \quad (27)$$

The behavior of quantum interferometric power versus c_1 is presented in the figure 4 for different values of $\alpha = c_1 + c_2$.

The quantum interferometric power is depicted in figure 4. Note that for $\alpha \leq \frac{1}{2}$, the quantum interferometric power (up to the scale factor $\frac{1}{4}$) coincides with the local quantum Fisher information $F(\rho, \sigma_1)$ corresponding to the situation where the dynamics of the first qubit is governed by the local Hamiltonian σ_1. This can be also seen from the numerical results reported in the figures 1 and 4 for $\alpha \leq \frac{1}{2}$. According to the analysis presented in the previous section, it is clear that the quantum correlations enhance the degree of precision in the estimation

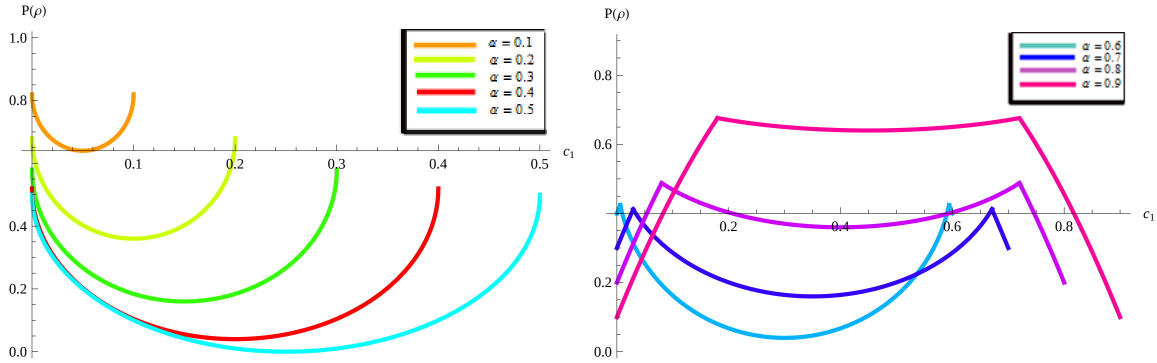

Figure 4. The quantum interferometric power $\mathcal{P}(\rho)$ as function of the parameter c_1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

of the phase parameter θ. Therefore, for a fixed value of the parameter α, the states (13) and (14) contain the maximal amount of quantum correlations and they offer the best estimation efficiency.

For states with $\alpha \geq \frac{1}{2}$, the situation becomes significantly different. First, we note the quantum interferometric power exhibits a sudden double change when $c_1 = \alpha_-$ and $c_1 = \alpha_+$ ($\alpha_\pm$ are given by (25). This behavior is similar to the sudden change of geometric quantum discord based on Hilbert-Schmidt distance or trace norm which were extensively investigated in the literature, especially in connection with quantum phase transitions [41, 42, 43, 44]. Three different phases characterize the behavior of quantum interferometric power: (i) $0 \leq c_1 \leq \alpha_-$ where $\mathcal{P}(\rho) = \frac{1}{4}F(\rho, \sigma_3)$ (ii) $\alpha_- \leq c_1 \leq \alpha_+$ where $\mathcal{P}(\rho) = \frac{1}{4}F(\rho, \sigma_1)$ and (iii) $\alpha_+ \leq c_1 \leq \alpha$ where $\mathcal{P}(\rho) = \frac{1}{4}F(\rho, \sigma_3)$. The minimal value of the quantum interferometric power $\mathcal{P}(\rho)$ is obtained in the intermediate zone ($\alpha_- \leq c_1 \leq \alpha_+$) for the states with ($c_1 = c_2 = \frac{\alpha}{2}$) (11). It is also remarkable that this double sudden change in the behavior of the quantum interferometric power occurs when the states (1) contain the maximal amount of quantum correlations.

It becomes clear the quantum interferometric power as a quantifier provides a tool to decide about the degree of quantumness in a bipartite system and plays a prominent role in understanding the role of quantum information in enhancing the parameter precision in quantum metrology protocols. Indeed, from the figure 4 for $\alpha \leq \frac{1}{2}$), the quantum interferometric power reaches the maximal values for the probe states (13) and (14) which offer the efficient sensitivity in the estimation of the phase θ. In particular, among the probe states with lower values of α are more suitable than those with α approaching the value 0.5. Similarly, from the figure 4 one observe that for two-qubit states with $\alpha \geq \frac{1}{2}$, the suitable probe states to ensure the best precision are those with $c_1 = \alpha_-$ or $c_1 = \alpha_+$ ($\alpha_\pm$ are given by (25) and the precision estimation can be enhanced by considering states with higher values α. It is remarkable that the probe states ensuring the maximum precision to the estimation are exactly those encompassing the maximum amount of quantum correlations, i.e. with $c_1 = \alpha_-$ or $c_1 = \alpha_+$ where the sudden change of quantum interferometric power occurs.

4 von Neumann entropy based quantum discord

In this section we compare the discord-like quantum interferometric power with the entropic quantum discord originally introduced in the information-theoretic context [29, 30]. It coincides with entanglement for pure states and goes beyond entanglement for mixed ones. Quantum discord is defined as the difference between total correlations and classical correlations in a bipartite state. The evaluation of quantum discord involves an optimization procedure for the conditional entropy over all local generalized measurement. This optimization is in general very challenging and this is the reason why the explicit analytical expressions of quantum discord is known only for a restricted class of two-qubit quantum states [45, 46] and two-qubit rank-2 states [38]. For two-qubit density matrices of rank two, the connection between the quantum discord and the entanglement of formation, which is described by the Koashi-Winter theorem [47], provides the algorithm to derive explicitly the quantum discord. The entropic quantum discord writes [29, 30]

$$\mathcal{D}(\rho_{12}) = S(\rho_1) + \tilde{S}_{\min} - S(\rho_{12}). \tag{28}$$

where $S(\rho) = -\text{Tr}\rho \log \rho$ is the von-Neumann entropy, ρ_1 is the reduced density matrix of the first qubit and $\tilde{S}_{min}$ is the minimum of the conditional entropy over all positive operator-valued measurements. For this end, we rewrites the two-qubit rank-2 states (1) as

$$\rho_{12} = \lambda_1 |\psi_1\rangle\langle\psi_1| + \lambda_2 |\psi_2\rangle\langle|\psi_2|, \tag{29}$$

where the eigenvalues and the corresponding eigenvectors are given by (4) and (5) respectively. The quantum discord can be analytically derived by employing the Koashi-Winter theorem. Indeed, along the line of reasoning developed in [38] (see also [33] where similar notations are adopted), one first purifies the states (29) by attaching a third qubit $\{|0\rangle_3, |1\rangle_3\}$. This purification produces the following pure three-qubit state

$$|\psi\rangle_{123} = \sqrt{\lambda_1}|\psi\rangle_1 \otimes |0\rangle_3 + \sqrt{\lambda_2}|\psi\rangle_2 \otimes |1\rangle_3.$$

When a positive operator valued measure (POVM) measurement is performed on the qubit 1, the Koashi-Winter theorem establishes a relationship between the minimum of the conditional entropy $\tilde{S}_{min}$ and the entanglement of formation $E(\rho_{23})$ of the system described by $\rho_{23} = \text{Tr}_1 \rho_{123}$. This result reads as

$$\tilde{S}_{min} = E(\rho_{23}) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - |C(\rho_{23})|^2}\right) \tag{30}$$

where $H(x) = -x \log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function and $C(\rho_{23})$ is the well known Wootters concurrence for the state ρ_{23} [48]. It is given by

$$|C(\rho_{23})|^2 = 2[1 - c_1 - c_2](\sqrt{c_1} - \sqrt{c_2})^2. \tag{31}$$

Setting $c_1 + c_2 = \alpha$ and reporting (31) in the equation (30), the explicit expression of quantum discord, quantifying the quantum correlations in the bipartite states $\rho \equiv \rho_{12}$, express as

$$\mathcal{D}(\rho) = H\left(c_1 + \frac{1-\alpha}{2}\right) + H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - 2(1-\alpha)(\alpha - 2\sqrt{c_1(\alpha - c_1)})}\right) - H(\alpha) \tag{32}$$

in terms of the parameters α and c_1 ranging from 0 to α.

The comparison of the numerical calculations depicted in figures 4 and 5 show that quantum interferometric power and the entropic quantum discord present almost similar behavior except the double sudden change exhibited by the quantum correlations in figure 4 for $\alpha \geq \frac{1}{2}$. This confirms that the quantum interferometric power constitutes an appropriate to the tackle the issue of quantifying the quantum correlation. Also, in view of

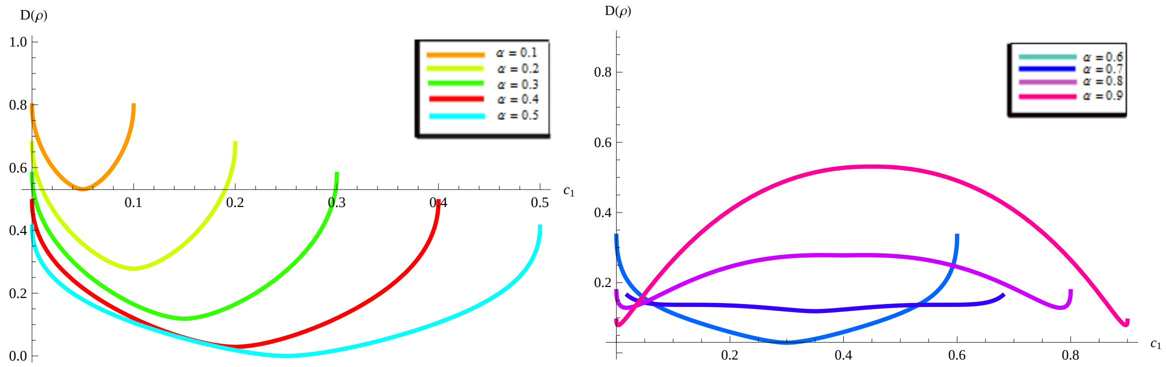

Figure 5. Quantum discord $\mathcal{D}(\rho)$ as function of the parameter c_1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

the technical difficulties arising in the analytical evaluation of quantum discord based on entropy, the discord-like quantum interferometric power provides a powerful way to quantify the quantum correlations in generic two-qubit states and more generally in bipartite systems of higher dimensional Hilbert spaces. Furthermore, the quantum interferometric power goes beyond entanglement. Indeed, for the states under consideration $\rho \equiv \rho_{12}$ (1), the Wootters concurrence is given by

$$\mathcal{C}_{12}(\rho) = |(\sqrt{c_1} + \sqrt{c_2})^2 - 1| \tag{33}$$

Setting $\alpha = c_1 + c_2$, the concurrence $\mathcal{C}_{12}(\rho)$ rewrites for $\alpha \leq \frac{1}{2}$ as

$$\mathcal{C}_{12}(\rho) = 1 - \alpha - 2\sqrt{c_1(\alpha - c_1)} \quad \text{with} \quad 0 \leq c_1 \leq \alpha. \tag{34}$$

In the case where $\alpha \geq \frac{1}{2}$, the concurrence is given by

$$\mathcal{C}_{12}(\rho) = 1 - \alpha - 2\sqrt{c_1(\alpha - c_1)} \quad \text{for} \quad 0 \leq c_1 \leq c_- \quad \text{and} \quad c_+ \leq c_1 \leq \alpha, \tag{35}$$

and

$$\mathcal{C}_{12}(\rho) = \alpha - 1 + 2\sqrt{c_1(\alpha - c_1)} \quad \text{for} \quad c_- \leq c_1 \leq c_+, \tag{36}$$

where $c_\pm = \frac{\alpha \pm \sqrt{2\alpha - 1}}{2}$. The states ρ are are separable for the couple of parameters $(c_1, c_2) = \left(\frac{\alpha + \sqrt{2\alpha-1}}{2}, \frac{\alpha - \sqrt{2\alpha-1}}{2}\right)$ and $(c_1, c_2) = \left(\frac{\alpha - \sqrt{2\alpha-1}}{2}, \frac{\alpha + \sqrt{2\alpha-1}}{2}\right)$ for each fixed value of α larger than $\frac{1}{2}$ (see figure 6). It is important to

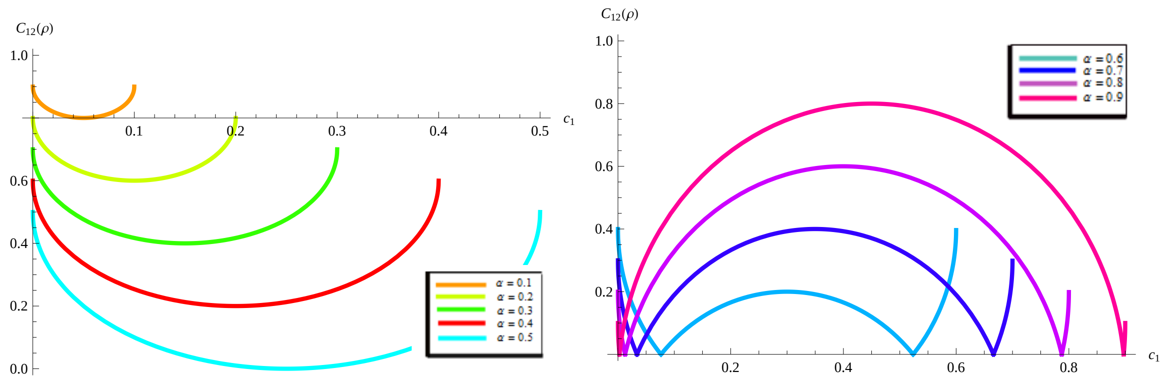

Figure 6. The concurrence $\mathcal{C}_{12}(\rho)$ as function of the parameter c_1 for different values of α.

stress that for the family of states under consideration, the quantum interferometric power $\mathcal{P}$ is nonzero except in the particular case $c_- = c_+ = 0.25$ or equivalently $\alpha = 0.5$ and $c_1 = c_2 = 0.25$ (see the equations (26) and (27)). This indicates that the quantum interferometric power quantifies also the amount of the quantum correlations existing in separable states and in this respect goes beyond the entanglement.

5 Bounds of local Fisher information

Quantum correlations constitute a resource to enhance the parameter precision in the bipartite states ρ (1) for unitary parametrization process even in the absence of entanglement. Recently, it has been shown that the local quantum Fisher information is connected to the speed of quantum evolution when the distance between two arbitrary states is measured with the Bures distance [26]. Based on this relation, bounds on the metrology precision were derived by evaluating the upper and the lower bound of local quantum Fisher information. As by product, it has been found that under a local unitary evolution, the lower bound of speed gives the amount of quantum correlations existing in the bipartite state and coincides with the quantum interferometric power (17). On the other hand, the upper bound of the local quantum Fisher information is interpreted as the maximum achievable speed [26] (see also [27]). It follows that the error in estimating the phase θ can be bounded as [26]

$$\frac{1}{F_{\max}(\rho)} \leq \Delta\theta \leq \frac{1}{F_{\min}(\rho)} \tag{37}$$

where

$$F_{\min}(\rho) = \frac{1}{4}\min_{\{H_1\}} \mathrm{F}(\rho, H_1), \qquad F_{\max}(\rho) = \frac{1}{4}\max_{\{H_1\}} \mathrm{F}(\rho, H_1). \tag{38}$$

the minimal and the maximal amount of local quantum Fisher information over all local observables acting on the qubit 1. We notice that $F_{\min}(\rho)$ is exactly the definition of quantum interferometric power defined by (17). To write down the expressions of the upper and lower bounds, we optimize first the local quantum Fisher information (9) over the variables α and β parameterizing the orientation of the unit vector $\overrightarrow{r}$ (see equation (8)). By setting the derivatives to zero, one finds the solutions $\overrightarrow{r} = (1,0,0)$, $\overrightarrow{r} = (0,1,0)$ and $\overrightarrow{r} = (0,0,1)$ for which one obtains the quantum Fisher $F(\rho,\sigma_1)$, $F(\rho,\sigma_2)$ and $F(\rho,\sigma_3)$ given by the expressions (10), (15) and (16) respectively. Therefore, one gets

$$F_{\min}(\rho) = \frac{1}{4}\min(F(\rho,\sigma_1), F(\rho,\sigma_2), F(\rho,\sigma_3)) \qquad F_{\max}(\rho) = \frac{1}{4}\max(F(\rho,\sigma_1), F(\rho,\sigma_2), F(\rho,\sigma_3)). \tag{39}$$

This result can be alternatively derived using the method reported in [26]. Indeed, by optimizing over all local traceless Hamiltonians acting on the first qubit, the bounds of the local quantum Fisher information write

$$F_{\min}(\rho) = 1 - \lambda_W^{\max}, \quad F_{\max}(\rho) = 1 - \lambda_W^{\min}, \tag{40}$$

where $\lambda_W^{\max}$ and $\lambda_W^{\min}$ denote respectively the largest and the smallest values of the following quantities

$$W_{ii} = \sum_{m\neq n} \frac{2\lambda_m\lambda_n}{\lambda_m + \lambda_n} \langle m|\sigma_i \otimes \mathbb{I}|n\rangle\langle n|\sigma_i \otimes \mathbb{I}|n\rangle, \qquad i = 1,2,3, \tag{41}$$

involving the non vanishing eigenvalues of the density matrix ρ given by (4). They can be written also as

$$W_{ii} = 1 - M_{ii} = 1 - \frac{1}{4}F(\rho,\sigma_i), \tag{42}$$

where the matrix elements M_{ii} are defined by (19). Thus, using the relation (42) and the results (20), one gets

$$W_{11} = 2\big[1-(c_1+c_2)\big]\big[\sqrt{c_1}+\sqrt{c_2}\big]^2, \quad W_{22} = 2\big[1-(c_1+c_2)\big]\big[\sqrt{c_1}-\sqrt{c_2}\big]^2, \quad W_{33} = \tfrac{(c_1-c_2)^2}{c_1+c_2}. \tag{43}$$

Since W_{11} is always greater than W_{22}, one obtains

$$\lambda_W^{\max} = \max(W_{11}, W_{33}), \qquad \lambda_W^{\min} = \min(W_{22}, W_{33}).$$

In view of the delimiting relation (37), the error on the estimated parameter is in the error interval $\left[\frac{1}{F_{\max}(\rho)}, \frac{1}{F_{\min}(\rho)}\right]$. Thus, it is natural to ask on the role of quantum correlations in the states (1) in enlarging or reducing the width of the error interval. For this purpose, we define the width of the error interval as

$$\delta\theta = \tfrac{1}{F_{\min}(\rho)} - \tfrac{1}{F_{\max}(\rho)}. \tag{44}$$

We note that that the lower bound of local quantum Fisher information $F_{\min}(\rho)$ determines the quantum interferometric power analyzed in the previous section. Hence, for $\alpha \leq \frac{1}{2}$, the lower bound is given by

$$F_{\min}(\rho) = 1 - 2(1-\alpha)(\sqrt{c_1}+\sqrt{\alpha-c_1})^2. \tag{45}$$

Similarly, for $\alpha \geq \frac{1}{2}$ and when the parameter c_1 satisfies the condition $\alpha_- \leq c_1 \leq \alpha_+$, $F_{\min}(\rho)$ is given by the expression (45). The quantities α_- and α_+ are defined by (25). Conversely, when the parameter c_1 is such that $0 \leq c_1 \leq \alpha_-$ and $\alpha_+ \leq c_1 \leq \alpha$, one gets

$$F_{\min}(\rho) = 1 - \frac{(2c_1-\alpha)^2}{\alpha} \tag{46}$$

To determine the lower value of the error interval, one needs the expression of the upper bound of local quantum fisher information $F_{\max}(\rho)$. For this, one compares W_{22} and W_{33}. The inequality $W_{22} \geq W_{33}$ holds when the parameter c_1 satisfies the following condition

$$(\sqrt{c_1}-\sqrt{c_2})^2 - 2(c_1+c_2)^2 \geq 0. \tag{47}$$

For a fixed value of the parameter $\alpha = c_1+c_2$, this condition is examined by considering separately the situations where $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$. Thus, for $\alpha \geq \frac{1}{2}$ one verifies that the condition (47) is always verified and one has $\lambda_w^{min} = W_{22}$. It follows that for $\alpha \geq \frac{1}{2}$, the maximal amount of local quantum Fisher information is

$$F_{\max}(\rho) = 1 - 2(1-\alpha)(\sqrt{c_1}-\sqrt{\alpha-c_1})^2. \tag{48}$$

For the situation where $\alpha \leq \frac{1}{2}$, one has $\lambda_w^{min} = W_{22}$ for

$$\alpha_- \leq c_1 \leq \alpha_+$$

where α_+ and α_- are given by (25). Then, one obtains

$$F_{\max}(\rho) = 1 - 2(1-\alpha)(\sqrt{c_1}-\sqrt{\alpha-c_1})^2, \tag{49}$$

for $\alpha_- \leq c_1 \leq \alpha_+$ and

$$F_{\max}(\rho) = 1 - \frac{(2c_1-\alpha)^2}{\alpha} \tag{50}$$

for $0 \leq c_1 \leq \alpha_-$ and $\alpha_+ \leq c_1 \leq \alpha$. Combining the results (45), (46), (48), (49) and (50), one gets the analytical expression of the width of the error interval $\delta\theta$. Indeed, when $\alpha \leq \frac{1}{2}$, it writes as

$$\delta\theta = 4\,\frac{F(\rho,\sigma_3) - F(\rho,\sigma_1)}{F(\rho,\sigma_1)F(\rho,\sigma_3)}, \quad \text{for} \quad c_1 \in [0, \alpha_-] \cup [\alpha_+, \alpha], \tag{51}$$

and

$$\delta\theta = 4\,\frac{F(\rho,\sigma_2) - F(\rho,\sigma_1)}{F(\rho,\sigma_1)F(\rho,\sigma_2)}, \quad \text{for} \quad c_1 \in [\alpha_-, \alpha_+]. \tag{52}$$

Similarly, for $\alpha \geq \frac{1}{2}$, one finds

$$\delta\theta = 4\,\frac{F(\rho,\sigma_2) - F(\rho,\sigma_3)}{F(\rho,\sigma_2)F(\rho,\sigma_3)} \quad \text{for} \quad c_1 \in [0, \alpha_-] \cup [\alpha_+, \alpha] \tag{53}$$

and

$$\delta\theta = 4\,\frac{F(\rho,\sigma_2) - F(\rho,\sigma_1)}{F(\rho,\sigma_1)F(\rho,\sigma_2)} \quad \text{for} \quad c_1 \in [\alpha_-, \alpha_+]. \tag{54}$$

We stress that the expressions of the width of the error interval $\delta\theta$ for the states ρ involves only the local quantum Fisher $F(\rho,\sigma_1)$, $F(\rho,\sigma_2)$ and $F(\rho,\sigma_3)$ given respectively by (10), (15) and (16). The figure 6 gives the change of the width $\delta\theta$, versus the parameter c_1 labeling the probe states (1), for different values of $\alpha = c_1 + c_2$.

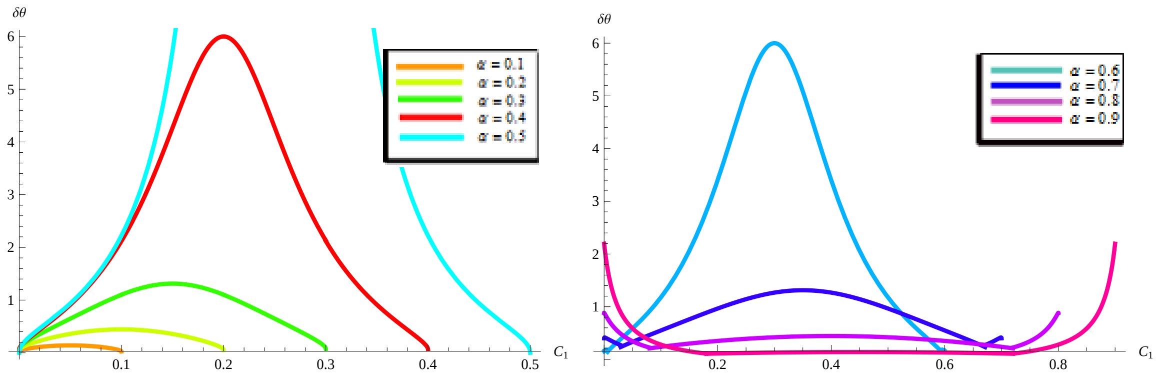

Figure 6 . $\delta\theta$ versus the parameter c_1 for different values of $\alpha = c_1 + c_2$.

Concerning the states with $\alpha < \frac{1}{2}$, the figure 6 shows that the width of the error interval $\delta\theta$ vanishes for the states with $(c_1 = 0, c_2 = \alpha)$ and $(c_1 = \alpha, c_2 = 0)$ given respectively by (13) and (14). This means that the upper and the lower local quantum Fisher information coincide and the states (13) and (14) are the suitable states offering the best parameter estimation. Remarkably, these states are exactly those among the states of type (1) encompassing the maximal amount of quantum correlations (see figure 4 or/and figure 5).

Conversely, the width of the error interval $\delta\theta$ is maximal for states with $(c_1 = c_2 = \frac{\alpha}{2})$ which exhibit the minimal amount of quantum correlations. Clearly, the width of the error interval $\delta\theta$ reduces (resp. reduces) for states with more (resp. less) amount of quantum correlations. This results elucidates the role of quantum correlation in enhancing the precision in quantum estimation protocols. For the two-qubit sttaes with $\alpha > \frac{1}{2}$, the width of the error interval $\delta\theta$ is minimal (almost vanishing) for the states with $(c_1 = \alpha_-, c_2 = \alpha_+)$ and $(c_1 = \alpha_+, c_2 = \alpha_-)$ where $\alpha_\pm$ are given by (25). These states contain the maximal amount of quantum correlations and for which the quantum interferometric power exhibit a sudden double change (see figure 4). This corroborates the crucial role of quantum correlations in improving the estimation of the parameter θ.

In the particular case $\alpha = \frac{1}{2}$, the expressions of $F_{\min}$ and $F_{\min}$ reduce to

$$F_{\min} = 1 - \frac{1}{2}(\sqrt{2c_1} + \sqrt{1 - 2c_1})^2, \quad F_{\max} = 1 - \frac{1}{2}(\sqrt{2c_1} - \sqrt{1 - 2c_1})^2. \tag{55}$$

Thus, when $c_1 = c_2 = \frac{1}{4}$, the quantum interferometric power $\mathcal{P} \sim F_{\min}$ vanishes (see also the figure 4) and $F_{\max} = 1$. This explains the infinite behavior of $\delta\theta$ for $c_1 = c_2 = \frac{1}{4}$ (see figure 6). It is interesting the note that in this case the state ρ (1) is separable. Indeed, using the equation (33), one verifies that the concurrence is zero. This result shows that the separable states are not suitable for parameter estimation. The limiting case $\alpha = \frac{1}{2}$ is very illustrative. In fact, the upper and lower local Fisher information satisfy the additivity relation $F_{\min} + F_{\max} = 1$ which implies $F_{\min}$ increases as $F_{\max}$ decreases and vice-versa. It follows that when the values of $F_{\min}$ and $F_{\max}$ approach each other, the difference $\delta\theta$ decreases and in this case the probe states ρ exhibit increasing amount of quantum correlations.

6 Concluding remarks

For a special class of two-qubit states, we derived the analytical expressions of local quantum Fisher information for some particular unitary parametrization processes. A special attention were devoted to location unitary transformations corresponding to bit flip, phase flip and both that are generated respectively by the usual Pauli matrices σ_1, σ_3 and σ_2. This gives significant advantages in examining the role of quantum correlations in determining high-precision of the estimated parameter. We investigated the amount of quantum correlations in a generic class of two-qubit states by using the concept of quantum interferometric power which a discord-like quantifier. The characterization of the quantum correlations in term of this new quantifier provides the appropriate tool to examine the role of quantum correlations in quantum metrology. In this paper, we deliberately considered density matrices of rank two to give a comparison between the quantum interferometric power and the quantum discord based on von Neumann entropy which is easily derived using the Koashi-winter theorem. Indeed, the amount of quantum correlations depicted in figures 4 and 5 show clearly that the quantum interferometric power can be used as a good measure to reveal the quantum correlations in bipartite quantum states especially ones of higher rank for which the computation of the analytical expressions of the entropic quantum discord is very challenging. It must be noticed that the quantum interferometric power a can be derived explicitly for quantum systems with higher dimensional Hilbert spaces [24, 26]. In this sense, the quantum interferometric power provides a nice geometrical tool in identifying, quantifying and characterizing quantum correlations for bipartite quantum systems and offers the way to overcome the mathematical difficulties encountered in deriving the explicit expressions of entropic quantum discord. The second important aspect investigated in this paper concerns the role of quantum correlations in quantum metrology. We explicitly derived the tight bounds of the error on the estimated parameter in an interferometric metrology protocol. The difference between the higher and the lower bounds of the local quantum Fisher information is investigated in detail in relation with the parameter estimation error. In particular we found that this difference is smaller for two qubit states encompassing a large amount of quantum correlations and becomes larger for states presenting less quantum correlations. This indicates that the quantum correlations are efficient in boosting the performance of metrology protocols.

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