Pairwise local quantum uncertainty in superpositions of Dicke states

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Abstract

Local quantum uncertainty is defined as the minimum amount of uncertainty in measuring a local observable for a bipartite state. It provides a measure for quantifying quantum correlations in bipartite quantum system and has operational significance in quantum metrology. In this work, we derive explicitly expressions for local quantum uncertainty for X two-qubit states which are of paramount importance in various field of quantum information. As illustration, we consider two-qubit states extracted from the multi-qubit Dicke states.

I.B. Coulamy, J.H. Warnes, M.S. Sarandy and A. Saguia, Phys. Lett. A **380** (2016) 1724.

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1 Introduction

Quantum correlations in multipartite systems are a fundamental resource in various protocols of quantum information processing [1, 2, 3, 4]. In this respect, the characterization the degree of quantumness of correlations between the different parts of a composite system is highly desirable. During the last two decades, several quantifiers were investigated in the literature (for a recent review see [5]). The literature contains different proposals measures of quantum correlations. The most familiar ones are the concepts of concurrence, entanglement of formation, quantum discord and its different geometric versions [6, 7, 8, 9, 10, 11]. The interest in quantum discord lies in the existence of nonclassical correlations even in separable states [10, 11] which are entirely captured by entanglement.This explains the the particular interest and the impressive efforts dedicated to the significance and the computation of quantum discord in different quantum systems. However, the derivation of the explicit expression of quantum discord of an arbitrary quantum system is very challenging in general. Quantum discord based on von Neumann entropy can be computed only for a restricted set of two-qubit systems.

An alternative way to overcome this problem consists in utilizing geometric methods to quantify the distance between a bipartite state and its closed one encompassing only classical correlations [12, 13, 14] (see also [15, 16, 17, 18]). Two variants of geometric quantum discord were introduced in terms of Schatten p-norm: trace norm ($p = 1$) and Hilbert-Schmidt norm ($p = 2$). However, the geometric quantum discord based on Hilbert-Schmidt norm is contractive under local operations by the unmeasured party and therefore can be a good indicator of quantum correlations [19, 20]. The explicit derivation of discord-like correlations for X states by means of trace discord was reported in [21]. But, one should recognize the computability of this quantifier is drastically difficult especially for higher dimensional quantum systems (qudits).

In contrast with the entropic and geometric quantifiers, an alternative reliable and computable discord-like measure was reported in [22]. This quantifier employs the concept of quantum uncertainty on local observables by mean of the formalism of the skew information, introduced in Ref. [23], to determine the uncertainty in the measurement of an observable. More precisely, the local quantum uncertainty is given by the minimum of the skew information over all possible local observables acting on one party of a bipartite system. This minimization can be analytically worked out in deriving the quantum correlations for any qubit-qudit bipartite system [22]. It is remarkable that the local quantum uncertainty is related to quantum Fisher information [24, 25, 26] which is the key ingredient in estimating precisions in quantum metrology protocols [22].

In this paper, we give the explicit analytical expressions of local quantum uncertainty for a generic family of two qubit X states. This completes the the recently obtained results for Bell diagonal, Werner and isotropic states [27, 28]. As illustration we consider the pairwise quantum correlation in multipartite symmetric qubit states.

The paper is structured as follows. In the first section, we give a brief review of the concept of local quantum uncertainty. The explicit anlaytical for of this measure of discord-like quantum correlations is derived for an arbitrary two-qubit X states. As by product, we recover the local quantum uncertainty for Bell diagonal states [27] and for orthogonally invariant two-qubit states derived in [28]. To illustrate our purpose, we consider collective n qubit systems possessing parity and exchange symmetries. In particular, we consider the pairwise quantum discord for two qubit states in Dicke states. We also derive the pairwise local quantum in balanced superpositions of spin coherent states for which we consider two partitioning schemes. The first bipartition lies on the factorization property of SU(2) coherent states. In this picture a j-spin coherent state factorizes as a product of $2j$ identical qubit states ($\frac{1}{2}$-spin coherent states). The second scheme is obtained by a trace procedure over the degree of freedom of $2j-2$ qubits. A special focus is dedicated to even and odd spin coherent states. Concluding remarks close this paper.

2 Local quantum uncertainty in two qubit X states

For a general two-qubit X-state, the quantification of local quantum uncertainty is available for some subsets of three parameters states [35, 27, 28]. An extension to five real parameters is considered in this section. We provide a method to compute the quantum correlations for a general two-qubit X- state which depends on seven real parameters. This class includes the maximally entangled Bell states, Werner states [34] which include both separable and nonseparable states, as well as others.

2.1 Local quantum uncertainty: definition

In quantum mechanics, the uncertainty of an observable H in a quantum state ρ is usually quantified by the variance as

$$\mathcal{V}(\rho, H) = \mathrm{Tr}(\rho H^2) - (\mathrm{Tr}\rho H)^2.$$

For pure states, the variance is of purely quantum nature. But, for mixed states, it comprises both classical and quantum contributions. The discrimination between classical and quantum parts is of paramount importance in quantum information theory. In this sense, to deal only with the quantum part of the variance, one employs the the formalism of skew information defined as [23, 24]

$$\mathcal{I}(\rho, K) = \mathrm{Tr}(\rho H^2) - \mathrm{Tr}(\sqrt{\rho} H \sqrt{\rho} H)$$

It expresses the information contained in the state ρ that is unaccessible by measuring the observable H. The skew information vanishes only and only when ρ and H commute. The difference $\mathcal{C}(\rho, K) = \mathcal{V}(\rho, K) - \mathcal{I}(\rho, K)$ has the meaning of classical mixing uncertainty. The disentanglement of the variance into classical and quantum parts is behind the relevance of the skew information in quantifying non classical correlations. Indeed, when the state $\rho = \rho_{12}$ describes a two-qubit system and $H = H_1 \otimes \mathbb{I}_2$ is a local observable acting only on the first qubit, the lower bound of the skew information leads to nonclassical correlations of the discord type [1113]. In fact, quantum discord quantifies the amount of information in a bipartite system which accessed by performing local measurements on one part of the global system. In this sense, The local quantum uncertainty is defined by the minimization of the skew information over local observables with fixed non-degenerate spectrum [22]

$$\mathcal{U}(\rho_{12}) \equiv \min_{H_1} \mathcal{I}(\rho_{12}, H_1 \otimes \mathbb{I}_2), \tag{1}$$

The properties, reliability and computability of this discord-like quantifier were reported in [22]. Indeed, the local quantum uncertainty vanishes for the so-called classical-quantum states of the form $\rho_{12} = \sum_i p_i |i\rangle_1 \langle i| \otimes \rho_2$, where

$\{|i\rangle\}$ is an orthonormal basis. Furthermore, this measure possess the invariance property under local unitary transformations and does not increase under local quantum transformations on the unmeasured subsystem. In this sense, the local quantum uncertainty provides a reliable discord-like measure. The explicit calculation of this quantum correlations indicator was reported in [22] for a $2 \times d$ bipartite system (qubit-qudit system). In particular, for a two qubit system (spin-$\frac{1}{2}$ particles), The local quantum uncertainty writes [22]

$$\mathcal{U}(\rho_{12}) = 1 - \max(\lambda_1, \lambda_2, \lambda_3), \tag{2}$$

where λ_i $(i = 1, 2, 3)$ denote the eigenvalues of the 3×3 matrix W whose matrix elements are given by

$$\omega_{ij} \equiv \mathrm{Tr}(\sqrt{\rho_{12}}\ \sigma_i \otimes \sigma_0\ \sqrt{\rho_{12}}\ \sigma_j \otimes \sigma_0), \tag{3}$$

where σ_0 stands for the identity matrix $\mathbb{I}$ and $i, j = 1, 2, 3$. The matrices σ_i $(i = 1, 2, 3)$ are the usual Pauli matrices. The optimization in the equation (2) simplifies when the state posses ceratin symmetries. This has been done for some symmetric class of states [22, 27, 28]. In this paper, we take a step further and focus on the local quantum uncertainty for the family of X states which include various types of quantum states usually used in investigating entanglement and quantum correlations in various condensed matter models such ones describing spin collective systems.

2.2 Local quantum uncertainty for X states

In the computational basis of the Hilbert space associated with a two qubit system, the X density matrices have non-zero entries only along the diagonal and anti-diagonal and therefore they are parameterized by seven real parameters [29, 30]. The corresponding symmetry is fully characterized by the $su(2) \times su(2) \times u(1)$ subalgebra of the full $su(4)$ algebra describing an arbitrary two-qubit system [31]. The X states have already found applications in several investigations of concurrence, entanglement of formation, quantum discord [32, 33]. The density matrix for a two-qubit X state writes as

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{4}$$

in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The entries are subjected to the normalization property ($\mathrm{Tr}\rho = 1$), the positivity condition ($\rho_{11}\rho_{44} \geq |\rho_{14}|^2$ and $\rho_{22}\rho_{33} \geq |\rho_{23}|^2$) and the complex conjugation requirement ($\rho_{14} = \overline{\rho_{14}}$ and $\rho_{23} = \overline{\rho_{32}}$). The phase factors $e^{i\theta_{14}} = \frac{\rho_{14}}{|\rho_{14}|}$ and $e^{i\theta_{23}} = \frac{\rho_{23}}{|\rho_{23}|}$ of the off diagonal elements can be removed using the local unitary transformations

$$|0\rangle_1 \to \exp\Big(-\frac{i}{2}(\theta_{14} + \theta_{23})\Big)|0\rangle_1 \quad |0\rangle_2 \to \exp\Big(-\frac{i}{2}(\theta_{14} - \theta_{23})\Big)|0\rangle_2.$$

Hence, the anti-diagonal entries of the density matrix can be made positive. Hereafter, we assume that the elements of the density matrix are non negative. The eigenvalues of the density matrix ρ write

$$\lambda_1 = \frac{1}{2}t_1 + \frac{1}{2}\sqrt{t_1^2 - 4d_1}, \quad \lambda_2 = \frac{1}{2}t_2 + \frac{1}{2}\sqrt{t_2^2 - 4d_2}, \quad \lambda_3 = \frac{1}{2}t_2 - \frac{1}{2}\sqrt{t_2^2 - 4d_2}, \quad \lambda_4 = \frac{1}{2}t_1 - \frac{1}{2}\sqrt{t_1^2 - 4d_1}$$

with $t_1 = \rho_{11} + \rho_{44}$, $t_2 = \rho_{22} + \rho_{33}$, $d_1 = \rho_{11}\rho_{44} - \rho_{14}\rho_{41} = \rho_{11}\rho_{44} - \rho_{14}^2$, and $d_2 = \rho_{22}\rho_{33} - \rho_{32}\rho_{23} = \rho_{22}\rho_{33} - \rho_{32}^2$. The Fano-Bloch decomposition of the state ρ writes as

$$\rho = \frac{1}{4}\sum_{\alpha,\beta} R_{\alpha\beta}\sigma_\alpha \otimes \sigma_\beta \tag{5}$$

where the correlation matrix $R_{\alpha\beta}$ are given by $R_{\alpha\beta} = \mathrm{Tr}(\rho\ \sigma_\alpha \otimes \sigma_\beta)$ with $\alpha, \beta = 0, 1, 2, 3$. Explicitly, they write

$$R_{03} = 1 - 2\rho_{22} - 2\rho_{44}, \quad R_{30} = 1 - 2\rho_{33} - 2\rho_{44}, \quad R_{11} = 2\ (\rho_{32} + \rho_{41}),$$

$$R_{22} = 2\ (\rho_{32} - \rho_{41}), \quad R_{00} = \rho_{11} + \rho_{22} + \rho_{33} + \rho_{44} = 1, \quad R_{33} = 1 - 2\rho_{22} - 2\rho_{33}.$$

For simultaneously non vanishing t_1 and t_2, the square root of the density matrix ρ writes, in the computational basis, as

$$\sqrt{\rho} = \begin{pmatrix} \frac{\rho_{11}+\sqrt{d_1}}{\sqrt{t_1+2\sqrt{d_1}}} & 0 & 0 & \frac{\rho_{14}}{\sqrt{t_1+2\sqrt{d_1}}} \\ 0 & \frac{\rho_{22}+\sqrt{d_2}}{\sqrt{t_2+2\sqrt{d_2}}} & \frac{\rho_{23}}{\sqrt{t_2+2\sqrt{d_2}}} & 0 \\ 0 & \frac{\rho_{32}}{\sqrt{t_2+2\sqrt{d_2}}} & \frac{\rho_{33}+\sqrt{d_2}}{\sqrt{t_2+2\sqrt{d_2}}} & 0 \\ \frac{\rho_{41}}{\sqrt{t_1+2\sqrt{d_1}}} & 0 & 0 & \frac{\rho_{44}+\sqrt{d_1}}{\sqrt{t_1+2\sqrt{d_1}}} \end{pmatrix}. \tag{6}$$

and the associated eigenvalues $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$ are given by $\sqrt{\lambda_3}$ and $\sqrt{\lambda_4}$ are given by

$$\sqrt{\lambda_1} = \frac{1}{2}\sqrt{t_1 + 2\sqrt{d_1}} + \frac{1}{2}\sqrt{t_1 - 2\sqrt{d_1}}, \qquad \sqrt{\lambda_2} = \frac{1}{2}\sqrt{t_2 + 2\sqrt{d_2}} + \frac{1}{2}\sqrt{t_2 - 2\sqrt{d_2}}$$

$$\sqrt{\lambda_3} = \frac{1}{2}\sqrt{t_2 + 2\sqrt{d_2}} - \frac{1}{2}\sqrt{t_2 - 2\sqrt{d_2}}, \qquad \sqrt{\lambda_4} = \frac{1}{2}\sqrt{t_1 + 2\sqrt{d_1}} - \frac{1}{2}\sqrt{t_1 - 2\sqrt{d_1}}.$$

The Fano-Bloch representation of the matrix (6) writes as

$$\sqrt{\rho} = \frac{1}{4}\sum_{\alpha,\beta} \mathcal{R}_{\alpha\beta}\sigma_\alpha \otimes \sigma_\beta$$

with $\mathcal{R}_{\alpha\beta} = \mathrm{Tr}(\sqrt{\rho}\ \sigma_\alpha \otimes \sigma_\beta)$. The non vanishing matrix correlation elements $\mathcal{R}_{\alpha\beta}$ are explicitly given by

$$\mathcal{R}_{00} = \sqrt{t_1 + 2\sqrt{d_1}} + \sqrt{t_2 + 2\sqrt{d_2}} \qquad \mathcal{R}_{03} = \frac{1}{2}\frac{R_{30} + R_{03}}{\sqrt{t_1 + 2\sqrt{d_1}}} - \frac{1}{2}\frac{R_{30} - R_{03}}{\sqrt{t_2 + 2\sqrt{d_2}}}$$

$$\mathcal{R}_{30} = \frac{1}{2}\frac{R_{30} + R_{03}}{\sqrt{t_1 + 2\sqrt{d_1}}} + \frac{1}{2}\frac{R_{30} - R_{03}}{\sqrt{t_2 + 2\sqrt{d_2}}} \qquad \mathcal{R}_{11} = \frac{1}{2}\frac{R_{11} + R_{22}}{\sqrt{t_2 + 2\sqrt{d_2}}} + \frac{1}{2}\frac{R_{11} - R_{22}}{\sqrt{t_1 + 2\sqrt{d_1}}}$$

$$\mathcal{R}_{22} = \frac{1}{2}\frac{R_{11} + R_{22}}{\sqrt{t_2 + 2\sqrt{d_2}}} - \frac{1}{2}\frac{R_{11} - R_{22}}{\sqrt{t_1 + 2\sqrt{d_1}}} \qquad \mathcal{R}_{33} = \sqrt{t_1 + 2\sqrt{d_1}} - \sqrt{t_2 + 2\sqrt{d_2}}$$

To determine a closed form of the matrix elements ω_{ij} (3), we use the following relations of the Pauli matrices

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \quad \mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij} \quad \mathrm{Tr}(\sigma_i\sigma_j\sigma_k\sigma_l) = 2(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

one shows that the matrix W, associated with the X density matrix (4) is diagonal and the diagonal elements write

$$\omega_{ii} = \frac{1}{4}\Big[\sum_\beta \Big(\mathcal{R}_{0\beta}^2 - \sum_k \mathcal{R}_{k\beta}^2\Big)\Big] + \frac{1}{2}\sum_\beta \mathcal{R}_{i\beta}^2 \tag{7}$$

where $i, k = 1, 2, 3$ and $\beta = 0, 1, 2, 3$. They can be cast in the following form

$$\omega_{ii} = \frac{1}{4}\eta^{\alpha\beta}(\mathcal{R}\mathcal{R}^t)_{\alpha\beta} + \frac{1}{2}(\mathcal{R}\mathcal{R}^t)_{ii} \tag{8}$$

where the summation over repeated indices is understood, the subscript t stands for transposition transformation, η is the diagonal matrix $\eta = (1, -1, -1, -1)$. They involve only the non vanishing Fano-Bloch components of the square root of the density matrix ρ_{12}. Using the expressions of the matrix elements $\mathcal{R}_{\alpha\beta}$, the eigenvalues ω_{ii} $(i = 1, 2, 3)$ can be also expanded in terms of the correlation matrix elements $R_{\alpha\beta}$ of the state ρ_{12} as

$$\omega_{11} = \frac{1}{4}\left[4\left(\sqrt{\lambda_1} + \sqrt{\lambda_4}\right)\left(\sqrt{\lambda_2} + \sqrt{\lambda_3}\right) + \frac{(R_{11}^2 - R_{22}^2) + (R_{03}^2 - R_{30}^2)}{(\sqrt{\lambda_1} + \sqrt{\lambda_4})(\sqrt{\lambda_2} + \sqrt{\lambda_3})}\right] \quad (9)$$

$$\omega_{22} = \frac{1}{4}\left[4\left(\sqrt{\lambda_1} + \sqrt{\lambda_4}\right)\left(\sqrt{\lambda_2} + \sqrt{\lambda_3}\right) + \frac{(R_{22}^2 - R_{11}^2) + (R_{03}^2 - R_{30}^2)}{(\sqrt{\lambda_1} + \sqrt{\lambda_4})(\sqrt{\lambda_2} + \sqrt{\lambda_3})}\right] \quad (10)$$

$$\begin{aligned} \omega_{33} &= \frac{1}{2}\left[\left(\sqrt{\lambda_1} + \sqrt{\lambda_4}\right)^2 + \left(\sqrt{\lambda_2} + \sqrt{\lambda_3}\right)^2\right] + \frac{1}{8}\left[\frac{(R_{03} + R_{30})^2 - (R_{11} - R_{22})^2}{\left(\sqrt{\lambda_1} + \sqrt{\lambda_4}\right)^2}\right] \\ &+ \frac{1}{8}\left[\frac{(R_{03} - R_{30})^2 - (R_{11} + R_{22})^2}{\left(\sqrt{\lambda_2} + \sqrt{\lambda_3}\right)^2}\right], \end{aligned} \quad (11)$$

where the quantities t_i and d_i $(i = 1, 2)$ are also expressed as

$$t_1 = \frac{1}{2}(R_{00} + R_{33}), \ t_2 = \frac{1}{2}(R_{00} - R_{33})$$

$$d_1 = \frac{1}{16}\left[(R_{00} + R_{33})^2 - (R_{30} + R_{03})^2 - (R_{11} - R_{22})^2\right], d_2 = \frac{1}{16}\left[(R_{00} - R_{33})^2 - (R_{30} - R_{03})^2 - (R_{11} + R_{22})^2\right],$$

in terms of the components $R_{\alpha\beta}$. We observe that for the X states (39) with positive entries, R_{11} is always lager than R_{22}. It follows that $\omega_{11} \geq \omega_{22}$ and only two distinct situations have to separately treated, that is $\omega_{11} \geq \omega_{33}$ and $\omega_{11} \geq \omega_{33}$ and the local quantum uncertainty for the states (39) simply as

$$\mathcal{U}(\rho_{12}) = 1 - \max(\omega_{11}, \omega_{33}). \quad (12)$$

2.3 Particular cases

We have already considered X states with non vanishing entries. For this class of two-qubit states, the density matrices split in two 2×2 bloc matrices corresponding to decoupling Hilbert subspaces $(1-4)$ and $(2-3)$. Now, we consider the situations where $t_1 = 0$ or $t_2 = 0$ corresponding respectively to 2×2 sub-block matrices $(1 - 4)$ and $(2 - 3)$. Indeed, when $t_1 = 0$, the trace condition imposes $t_2 = 1$ and vice-versa. Note that t_1 vanishes if and only if $\rho_{11} = \rho_{44} = 0$ and the positivity condition of the density matrix ρ (4) implies $\rho_{14} = \rho_{41} = 0$. In this case, we have $d_1 = 0$. Similarly, $t_2 = 0$ implies $\rho_{22} = \rho_{33} = 0$, $\rho_{23} = \rho_{32} = 0$ and $d_2 = 0$.

2.4 Block matrices (2-3)

In this case the correlation matrix elements of the matrix $\sqrt{\rho}$ write simply as

$$\mathcal{R}_{00} = -\mathcal{R}_{33} = \sqrt{1 + 2\sqrt{\rho_{22}\rho_{33} - \rho_{23}^2}}, \quad \mathcal{R}_{11} = \mathcal{R}_{22} = \frac{2\rho_{23}}{\sqrt{1 + 2\sqrt{\rho_{22}\rho_{33} - \rho_{23}^2}}}, \quad \mathcal{R}_{30} = -\mathcal{R}_{03} = \frac{\rho_{22} - \rho_{33}}{\sqrt{1 + 2\sqrt{\rho_{22}\rho_{33} - \rho_{23}^2}}}$$

Therefore, for $t_1 = 0$ the equation (8) gives $\omega_{11} = 0$, $\omega_{22} = 0$ and

$$\omega_{33} = \frac{1}{2}\left[1 + 2\sqrt{\rho_{22}\rho_{33} - \rho_{23}^2}\right] + \frac{1}{2}\left[\frac{(\rho_{22} - \rho_{33})^2 - 4\rho_{23}^2}{1 + 2\sqrt{\rho_{22}\rho_{33} - \rho_{23}^2}}\right] \quad (13)$$

in terms of the matrix elements of the density ρ_{12}.

2.5 Block matrices (1-4)

Similarly, in the special case where $t_2 = 0$ (or equivalently $\rho_{22} = \rho_{33} = \rho_{23} = \rho_{32} = 0$), the Fano-Bloch elements of the matrix $\sqrt{\rho}$ are given by

$$\mathcal{R}_{00} = \mathcal{R}_{33} = \sqrt{1 + 2\sqrt{\rho_{11}\rho_{44} - \rho_{14}^2}}, \quad \mathcal{R}_{11} = -\mathcal{R}_{22} = \frac{2\rho_{14}}{\sqrt{1 + 2\sqrt{\rho_{11}\rho_{44} - \rho_{14}^2}}}, \quad \mathcal{R}_{03} = \mathcal{R}_{30} = \frac{\rho_{11} - \rho_{44}}{\sqrt{1 + 2\sqrt{\rho_{11}\rho_{44} - \rho_{14}^2}}}$$

It follows that for the X states with $t_2 = 0$, one gets $\omega_{11} = 0$, $\omega_{22} = 0$ and

$$\omega_{33} = \frac{1}{2}\left[1 + 2\sqrt{\rho_{11}\rho_{44} - \rho_{14}^2}\right] + \frac{1}{2}\left[\frac{(\rho_{11} - \rho_{44})^2 - 4\rho_{14}^2}{1 + 2\sqrt{\rho_{11}\rho_{44} - \rho_{14}^2}}\right] \tag{14}$$

in terms of the non vanishing density matrix elements. In such two special situations, the local quantum uncertainty is simply given by $1 - \omega_{33}$.

2.6 Special subsets of two qubit X states

To exemplify the results of the previous section, we focus now on some special class of two-qubit states for which the analysis becomes particulary. We shall consider three special X states: (i) Werner states, (ii) Bell-diagonal states and (iii) orthogonal invariant two-qubit states. These three types of two-qubit states are X states with correlation elements verifying (i) $R_{11} = R_{22} = R_{33}$ and $R_{30} = R_{03} = 0$, (ii) $R_{11} \neq R_{22} \neq R_{33}$ and $R_{30} = R_{03} = 0$, (iii) $R_{ii} \neq R_{i+1\ i+1} = R_{i+2\ i+2}$ and $R_{30} = R_{03} = 0$ with $i = 1, 2, 3$ (mod 3).

2.6.1 Werner states

The two-qubit Werner states given by [34]

$$\rho_{\rm W} = \frac{1-f}{3}\ \sigma_0 \otimes \sigma_0 + \frac{4f-1}{3}|\psi^-\rangle\langle\psi^-| \tag{15}$$

are the mixtures of maximally chaotic state and the maximally entangled state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ and f is the fidelity which characterizes the overlap between Bell state and isotropic state ($0 \leq f \leq 1$). The Concurrence of Werner states writes $\mathcal{C}(\rho_{\rm W}) = \max(0, 2f - 1)$ and they are separable for $f \leq \frac{1}{2}$ and entangled for $\frac{1}{2} \leq f \leq 1$. In the Fano-Bloch representation, the states $\rho_{\rm W}$ write

$$\rho_{\rm W} = \frac{1}{4}\left(\sigma_0 \otimes \sigma_0 + \frac{1-4f}{3}\sum_{i=1}^{3}\sigma_i \otimes \sigma_i\right). \tag{16}$$

Using the results (9), (10) and (11), one gets

$$\omega_{11} = \omega_{22} = \omega_{33} = \frac{2}{3}(1-f) + \frac{2}{\sqrt{3}}\sqrt{f(1-f)} \tag{17}$$

and the local quantum uncertainty is simply given by

$$\mathcal{U}(\rho_{\rm W}) = 1 - \frac{2}{3}(1-f) + \frac{2}{\sqrt{3}}\sqrt{f(1-f)} \tag{18}$$

which coincides with the result derived [35].

2.6.2 Two qubit Bell states

The subset of X states which are diagonal in the Bell basis are parameterized by three parameters. The corresponding density matrices are of the form

$$\rho_{\mathrm{B}} = \frac{1}{4}(\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i) \tag{19}$$

Using the results (9), (10) and (11), the eigenvalues of the matrix ω (cf. equation (3)) write ω_{11}, ω_{22} and ω_{33} rewrites also as

$$\omega_{11} = \frac{1}{2}\left(\sqrt{(1-c_1)^2 - (c_2+c_3)^2} + \sqrt{(1+c_1)^2 - (c_2-c_3)^2}\right) \tag{20}$$

$$\omega_{22} = \frac{1}{2}\left(\sqrt{(1-c_2)^2 - (c_3+c_1)^2} + \sqrt{(1+c_2)^2 - (c_3-c_1)^2}\right) \tag{21}$$

$$\omega_{33} = \frac{1}{2}\left(\sqrt{(1-c_3)^2 - (c_1+c_2)^2} + \sqrt{(1+c_3)^2 - (c_1-c_2)^2}\right) \tag{22}$$

in terms of the correlation elements c_1, c_2 and c_3 which coincides with the result derived in [27].

2.6.3 Orthogonal invariant two-qubit states

Any two qubit state invariant under the operation $\mathcal{O} \otimes \mathcal{O}$ (with $\mathcal{O}$ an arbitrary orthogonal matrix) can be expanded in terms of the three generators $\mathcal{I}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ as [28]

$$\rho_{\mathrm{O}} = a\mathcal{I} + b\mathcal{F}_1 + c\mathcal{F}_2 \tag{23}$$

where the real parameters a, b and c are positive and satisfy $4a+2b+2c = 1$ (trace condition), $\mathcal{I}$ is the identity and the operators $\mathcal{F}_1$ and $\mathcal{F}_2$

$$\mathcal{F}_1 = \sum_{ij} |ij\rangle\langle ji| \quad \mathcal{F}_2 = \sum_{ij} |ii\rangle\langle jj|$$

in the computational basis. The density matrix (23) is X shaped

$$\rho_{\mathrm{O}} = \begin{pmatrix} a+b+c & 0 & 0 & c \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ c & 0 & 0 & a+b+c \end{pmatrix}. \tag{24}$$

and using the results (9), (10) and (11), one verifies

$$\omega_{11} = \omega_{33} = 2\left(\sqrt{(a+b)(a+b+2c)} + \sqrt{a^2-b^2}\right) \quad \omega_{22} = 2\left(\sqrt{(a-b)(a+b+2c)} + (a+b)\right)$$

and one recovers the results obtained in [28].

3 local quantum uncertainty in symmetric multi-qubit systems

The multi-qubit symmetric states were shown relevant for different purposes in quantum information science [36, 37, 38, 39, 40, 41, 42, 43]. In this paper, we shall mainly focus on multipartite Dicke states and an ensemble of n spin-1/2 prepared even and odd spin coherent states. For this end, we consider n identical qubits. The corresponding Hilbert space the n tensored copies of $\mathcal{H} = \text{span}\{|0\rangle, |1\rangle\}$ $\mathcal{H}_n := \mathcal{H}^{\otimes n}$. In particular, multipartite states in $\mathcal{H}_n$ possessing the exchange symmetry are especially interesting from experimental as well as mathematical point of views. The Majorana [44] or Dicke [45] representation are the standard descriptions of an arbitrary symmetric n-qubit state. In the Majorana picture, a symmetric multi-qubit state is given by (up to a normalization factor)

$$|\psi_s\rangle = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} |\eta_{\sigma(1)}, \ldots, \eta_{\sigma(n)}\rangle, \tag{25}$$

where each single qubit state writes as $|\eta_i\rangle \equiv (1 + \eta_i \bar{\eta}_i)^{-\frac{1}{2}}(|0\rangle + \eta_i|1\rangle)$ $(i = 1, \ldots, n)$ with the sum is over the elements of the permutation group $\mathcal{S}_n$ of n objects. Alternatively, any symmetric n-qubit states can be expressed in Dicke representation using the symmetric Dicke states defined by [45]

$$|n, k\rangle = \sqrt{\frac{k!(n-k)!}{n!}} \sum_{\sigma \in \mathcal{S}_n} |\underbrace{0, \ldots, 0}_{n-k}, \underbrace{1, \ldots, 1}_{k}\rangle, \tag{26}$$

where k is the number of excitations excitations $(k = 0, 1, \cdots, n)$. The Dicke states generate an orthonormal basis of the symmetric Hilbert subspace of dimension $(n + 1)$.

Therefore, permutation invariance, in symmetric multi-qubit states, implies a restriction to $n+1$ dimensional subspace from the entire 2^n dimensional Hilbert space. The Dicke states (26) constitute a special subset of the symmetric multi-qubit states (25) corresponding to the situation where the first k-qubit are such that $\eta_i = 0$ for $i = 0, 1, \ldots k$ and the remaining qubits are in the states $|\eta_i = 1\rangle$ with $i = k + 1, \ldots, n$. The states (26) are the eigenstates of the collective spin operators J^2 and J_z defined as

$$J_\alpha = \frac{1}{2} \sum_{i=1}^{n} \sigma_{i\alpha} \quad \alpha = x, y, z$$

where the operators $\sigma_{i\alpha}$ stand for the spin $\frac{1}{2}$-Pauli operators. In this respect, the symmetric qubit states (26) are completely determined by the quantum angular momentum $j = 2n$ which may take integer or half integer values ($j = \frac{1}{2}, 1, \frac{3}{2}, \ldots$) specifying the irreducible representations classes of the group $SU(2)$. The $(2j + 1)$-dimensional Hilbert space is spanned by the irreducible tensorial set $\{|j, m\rangle, m = -j, -j + 1, \cdots, j - 1, j\} \equiv \{|n, k\rangle = |j = \frac{n}{2}, m + j\rangle, k = 0, 1, \cdots, n\}$ characterizing the spin-j representations of the group $SU(2)$.

3.1 Pairwise Local quantum uncertainty in Dicke states

Dicke states [55] were extensively investigated in connection with the development of quantum information science. They are the basic tool from which one can build various quantum states that are relevant in quantum information as for instance GHZ states, W states, $SU(2)$ coherent states and spin squeezed states [56, 57, 58, 59, 60].

As we shall consider the pairwise local quantum uncertainty in symmetric states of type (43), we need the reduced two-qubit density matrices extracted from the whole n particles system. The general form of bipartite density matrix writes as

$$\rho_{ij} = \begin{pmatrix} \rho_{11} & \rho_{21}^* & \rho_{31}^* & \rho_{41}^* \\ \rho_{21} & \rho_{22} & \rho_{32}^* & \rho_{42}^* \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{43}^* \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix} \tag{27}$$

in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ where the matrix elements are given by

$$\rho_{11} = \frac{1}{4}\left(1 + 2\langle\sigma_{i3} \otimes \sigma_{j0}\rangle + \langle\sigma_{i3} \otimes \sigma_{j3}\rangle\right) \qquad \rho_{44} = \frac{1}{4}\left(1 - 2\langle\sigma_{i3} \otimes \sigma_{j0}\rangle + \langle\sigma_{i3} \otimes \sigma_{j3}\rangle\right) \tag{28}$$

$$\rho_{21} = \rho_{31} = \frac{1}{2}(\langle\sigma_{i+} \otimes \sigma_{j0}\rangle + \langle\sigma_{i+} \otimes \sigma_{j3}\rangle) \qquad \rho_{42} = \rho_{43} = \frac{1}{2}(\langle\sigma_{i+} \otimes \sigma_{j0}\rangle - \langle\sigma_{i+} \otimes \sigma_{j3}\rangle) \tag{29}$$

$$\rho_{22} = \rho_{33} = \frac{1}{4}\left(1 - \langle\sigma_{i3}\sigma_{i+} \otimes \sigma_{j3}\rangle\right) \tag{30}$$

$$\rho_{32} = \langle\sigma_{i+} \otimes \sigma_{j-}\rangle \tag{31}$$

$$\rho_{41} = \frac{1}{4}(\langle\sigma_{i1} \otimes \sigma_{j1}\rangle - \langle\sigma_{i2} \otimes \sigma_{j2}\rangle + i2\langle\sigma_{i1} \otimes \sigma_{j2}\rangle). \tag{32}$$

For collective spin models, the pairwise reduced density matrix in the standard basis, $\{|\downarrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\uparrow\uparrow\rangle\}$ (with $\sigma_z|\uparrow\rangle = |\uparrow\rangle$ and $\sigma_z|\downarrow\rangle = -|\downarrow\rangle$) [46], can be derived in terms of the collective operators spin. Indeed, for states, with symmetry exchange, we have

$$\langle\sigma_{i\alpha} \otimes \sigma_{j0}\rangle = \frac{\langle J_\alpha\rangle}{n}, \qquad \langle\sigma_{i\alpha} \otimes \sigma_{j\alpha}\rangle = \frac{4\langle J_\alpha^2\rangle - n}{n(n-1)}$$

$$\langle\sigma_{i1} \otimes \sigma_{j2}\rangle = \frac{\langle J_1 J_2 + J_2 J_1\rangle}{n(n-1)}, \qquad \langle\sigma_{i+} \otimes \sigma_{j3}\rangle = \frac{4\langle J_+ J_3 + J_3 J_+\rangle}{n(n-1)}$$

where $\alpha = 1, 2, 3$. It follows that the explicit expressions for the elements of the reduced density matrix are given by

$$\rho_{11} = \frac{n^2 - 2n + 4\langle J_z^2\rangle + 4(n-1)\langle J_z\rangle}{4n(n-1)}, \quad \rho_{44} = \frac{n^2 - 2n + 4\langle J_z^2\rangle - 4(n-1)\langle J_z\rangle}{4n(n-1)} \tag{33}$$

$$\rho_{21} = \rho_{31} = \frac{(n-1)\langle J_+\rangle + \langle J_+ J_z + J_z J_+\rangle}{2n(n-1)}, \quad \rho_{42} = \rho_{43} = \frac{(n-1)\langle J_+\rangle \pm \langle J_+ J_z + J_z J_+\rangle}{2n(n-1)} \tag{34}$$

$$\rho_{22} = \rho_{33} = \rho_{23} = \rho_{32} = \frac{n^2 - 4\langle J_z^2\rangle}{4n(n-1)} = \frac{\langle J_x^2 + J_y^2\rangle - n/2}{n(n-1)}, \tag{35}$$

$$\rho_{41} = \frac{\langle J_+^2\rangle}{n(n-1)}, \tag{36}$$

For states with parity symmetry, the density matrix commutes with the operator $\sigma_3 \otimes \sigma_3$. This implies $\rho_{12} = \rho_{13} = \rho_{42} = \rho_{43} = 0$. In fact, for states with parity symmetry, we have $\langle J_1\rangle = \langle J_2\rangle = 0$ and $\langle J_1 J_3\rangle = \langle J_2 J_3\rangle = 0$. Hence the pairwise reduced density matrix is X shaped and writes as

$$\rho_{ij} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{41}^* \\ 0 & \rho_{22} & \rho_{22} & 0 \\ 0 & \rho_{22} & \rho_{22} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix} \tag{37}$$

We note that the local unitary transformation

$$|0\rangle_k \to \exp\left(\frac{i}{2}(\theta)\right)|0\rangle_k$$

eliminates the phase factors of the matrix element ρ_{41} with $\rho_{41} = |\rho_{41}|e^{i\theta}$ and $k = 1, 2$ labels the subsystems 1 and 2. It follows that non-zero elements of the correlation matrix take the simple form

$$R_{03} = R_{30} = \rho_{11} - \rho_{44} \quad R_{11} = 2(\rho_{22} + |\rho_{41}|) \quad R_{22} = 2(\rho_{22} - |\rho_{41}|) \quad R_{33} = 1 - 4\rho_{22} \tag{38}$$

The Dicke states are defined as the equal superposition of all basis states of n qubits having exactly k excitations (26). Nowadays it is commonly accepted that this family of symmetric states can be an useful resource in various quantum protocols for two main reasons. First, they can be generated experimentally (see [53]). indeed, the generation of Dicke states with trapped-ion qubits have been proposed [47]. On the other hand, quantum correlations in Dicke states are highly robust in presence of external decoherence effects and especially measurements on individual qubits [48]. From the equation (**??**), it is simply verified that the reduced density matrix ρ_{12}, describing two qubits extracted from the state (26), is given by

$$\rho_{12} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix} \tag{39}$$

with $\rho_{14} = \rho_{41} = 0$ and the non vanishing elements are

$$\rho_{11} = \frac{k(k-1))}{n(n-1)}, \ \rho_{44} = \frac{(n-k)(n-k-1)}{n(n-1)}, \ \rho_{22} = \rho_{23} = \rho_{32} = \rho_{33} = \frac{k(n-k)}{n(n-1)}. \tag{40}$$

From the correlation matrix elements (38), one obtains

$$R_{11} = R_{22} = \frac{2k(n-k))}{n(n-1)}, \ R_{33} = 1 - \frac{4k(n-k)}{n(n-1)}, \ R_{03} = R_{30} = \frac{2k-n}{n}. \tag{41}$$

Using the expressions (9), (10) and (11), one finds

$$\omega_{11} = \omega_{22} = \sqrt{\frac{2k(n-k)}{n(n-1)}}\left(\sqrt{k(k-1)} + \sqrt{(n-k)(n-k-1)}\right), \ \omega_{33} = 1 + 2\,\frac{k(k-n)}{n(n-1)}. \tag{42}$$

3.2 Pairwise local quantum uncertainty in even and odd spin coherent states

Any symmetric state $|\psi_s\rangle$ (25) can be expanded in terms of Dicke states (26) as follows

$$|\psi_s\rangle = \frac{1}{n!}\sum_{k=0}^{n} c_k \ |n,k\rangle, \tag{43}$$

where the c_k $(k = 0, \dots, n)$ stand for the complex expansion coefficients. In particular, when the qubit are all identical ($\eta_i = \eta$ for all qubits), it is simply verified that the coefficients c_k are given by

$$c_k = n!\sqrt{\frac{n!}{k!(n-k)!}}\frac{\eta^k}{(1+\eta\bar{\eta})^{\frac{n}{2}}} \tag{44}$$

and the symmetric multi-qubit states (25) write

$$|\psi_s\rangle := |n,\eta\rangle = (1+\eta\bar{\eta})^{-\frac{n}{2}}\sum_{k=0}^{n}\sqrt{\frac{n!}{k!(n-k)!}}\eta^k \ |n,k\rangle, \tag{45}$$

which are exactly the $j = \frac{n}{2}$-spin coherent states (see for instance [49]). In particular, the state $|n, \eta\rangle$ can be identified for $n = 1$ with spin-$\frac{1}{2}$ coherent state with $|0\rangle \equiv |\frac{1}{2}, -\frac{1}{2}\rangle$ and $|1\rangle \equiv |\frac{1}{2}, +\frac{1}{2}\rangle$). The standard $SU(2)$ coherent states are obtained by the action of an element of the coset space $SU(2)/U(1)$

$$D_j(\xi) = \exp(\xi J_+ - \xi^* J_-) \ , \tag{46}$$

on the extremal state $|j, -j\rangle$. This action gives the states

$$|j, \eta\rangle = D_j(\xi)|j, -j\rangle = \exp(\xi J_+ - \xi^* J_-)|j, -j\rangle = (1 + |\eta|^2)^{-j} \exp(\eta J_+)|j, -j\rangle \ , \tag{47}$$

where $\eta = (\xi/|\xi|) \tan |\xi|$. In the standard angular momentum basis $\{|j, m\rangle\}$, they write

$$|j, \eta\rangle = (1 + |\eta|^2)^{-j} \sum_{m=-j}^{j} \left[\frac{(2j)!}{(j + m)!(j - m)!}\right]^{1/2} \eta^{j+m}|j, m\rangle \ . \tag{48}$$

They satisfy the resolution to identity property

$$\int d\mu(j, \eta)|j, \eta\rangle\langle j, \eta| = I \ , \qquad d\mu(j, \eta) = \frac{2j + 1}{\pi} \frac{d^2\eta}{(1 + |\eta|^2)^2} \ . \tag{49}$$

The spin coherent states are not orthogonal to each other:

$$\langle j, \eta_1|j, \eta_2\rangle = (1 + |\eta_1|^2)^{-j}(1 + |\eta_2|^2)^{-j}(1 + \eta_1^*\eta_2)^{2j} \ . \tag{50}$$

The resolution to identity makes possible to expand an arbitrary state in terms of the coherent states $|j, \eta\rangle$. In the special case $j = \frac{1}{2}$, the spin coherent states (48) reduce to

$$|\eta\rangle = \frac{1}{\sqrt{1 + \bar{\eta}\eta}}| \downarrow\rangle + \frac{\eta}{\sqrt{1 + \bar{\eta}\eta}}| \uparrow\rangle. \tag{51}$$

Here and in the following $|\eta\rangle$ is short for the spin-$\frac{1}{2}$ coherent state $|\frac{1}{2}, \eta\rangle$ with $| \uparrow\rangle \equiv |\frac{1}{2}, \frac{1}{2}\rangle$ and $| \downarrow\rangle \equiv |\frac{1}{2}, -\frac{1}{2}\rangle$). It is important to notice that the tensorial product of two $SU(2)$ coherent states $|j_1, \eta\rangle$ and $|j_2, \eta\rangle$ produces a spin-$(j_1 + j_2)$ coherent state labeled by the same variable:

$$|j_1, \eta\rangle \otimes |j_2, \eta\rangle = (D_{j_1} \otimes D_{j_2}) (|j_1, j_1\rangle \otimes |j_2, j_2\rangle) = D_{j_1+j_2} |j_1 + j_2, j_1 + j_2\rangle \ = \ |j_1 + j_2, \eta\rangle. \tag{52}$$

Only coherent states possess this remarkable property. It allows to write any spin-j coherent states as a $2j$ tensorial product of spin-$\frac{1}{2}$ coherent states:

$$|j, \eta\rangle = (|\eta\rangle)^{\otimes 2j} = \left(\frac{1}{\sqrt{1 + \bar{\eta}\eta}}| \downarrow\rangle + \frac{\eta}{\sqrt{1 + \bar{\eta}\eta}}| \uparrow\rangle\right)^{\otimes 2j} = (1 + \bar{\eta}\eta)^{-j} \sum_{m=-j}^{+j} \begin{pmatrix} 2j \\ j + m \end{pmatrix}^{\frac{1}{2}} \eta^{j+m}|j, m\rangle,$$

reflecting that a spin-j coherent state may be viewed as a multipartite state containing $2j$ qubits.

The even and odd spin coherent states are defined by

$$|j, \eta, m\rangle = \mathcal{N}_m(|j, \eta\rangle + e^{im\pi}|j, -\eta\rangle) \tag{53}$$

where the integer $m \in \mathbb{Z}$ takes the values $m = 0 \pmod 2$ and $m = 1 \pmod 2$. The normalization factor $\mathcal{N}_m$ is

$$\mathcal{N}_m = \left[2 + 2p^{2j} \cos m\pi\right]^{-1/2}$$

where p denotes the overlap between the states $|\eta\rangle$ and $|-\eta\rangle$. It is given by

$$p = \langle\eta| - \eta\rangle = \frac{1-\bar{\eta}\eta}{1+\bar{\eta}\eta}. \tag{54}$$

For $j = \frac{1}{2}$, the even and odd coherent states coincide with $|\uparrow\rangle$ and $|\downarrow\rangle$. They can be identified with basis states for a logical qubit as $|0\rangle \rightarrow |\uparrow\rangle$ and $|1\rangle \rightarrow |\downarrow\rangle$. In this manner, the states $|j,\eta,m\rangle$ can be viewed as multipartite fermionic coherent states:

$$|j,\eta,m\rangle = \mathcal{N}_m((|\eta\rangle)^{\otimes 2j} + e^{im\pi}(|-\eta\rangle)^{\otimes 2j}). \tag{55}$$

The decomposition property (52) provides us with a picture where even and odd spin coherent states can be considered as comprising multipartite spin subsystems. This is our main motivation to investigate the quantum correlations present in a single spin coherent state. This issue is discussed in what follows.

3.2.1 Pure bipartite spin coherent states

Let us first consider the following balanced superposition of spin coherent states

$$|j,\eta,\theta\rangle = \mathcal{N}_\theta(|j,\eta\rangle + e^{i\theta}|j,-\eta\rangle) \tag{56}$$

where the normalization factor is given by $|\mathcal{N}_\theta|^{-2} = 2 + 2p^{2j}\cos\theta$. Using the factorization or the splitting property of spin coherent states (52), the states (56) can be also expressed as

$$|j,\eta,\theta\rangle = \mathcal{N}_\theta(|j_1,\eta\rangle \otimes |j_2,\eta\rangle + e^{i\theta}|j_1,-\eta\rangle \otimes |j_2,-\eta\rangle) \tag{57}$$

with $j = j_1 + j_2$. They can rewritten as a two qubit states in the basis

$$|j_i,\eta,0\rangle \longrightarrow |0\rangle_{j_i} \qquad |j_i,\eta,\pi\rangle \longrightarrow |1\rangle_{j_i}, \quad i = 1,2.$$

defined by means of odd and even spin coherent associated with the angular momenta j_1 and j_2. Indeed, for each subsystem, an orthogonal basis $\{|0\rangle_l, |1\rangle_l\}$, with $l = j_1$ or j_2, can be defined as

$$|0\rangle_l = \frac{|l,\eta\rangle + |l,-\eta\rangle}{\sqrt{2(1+p^{2l})}} \qquad |1\rangle_l = \frac{|l,\eta\rangle - |l,-\eta\rangle}{\sqrt{2(1-p^{2l})}}. \tag{58}$$

The bipartite density state $\rho = |j,\eta,m\rangle\langle j,\eta,m|$ is pure. The concurrence in this pure bipartite system writes

$$\mathcal{C}_{j_1,j_2}(\theta) = \frac{\sqrt{1-p^{4j_1}}\sqrt{1-p^{4j_2}}}{1+p^{2j}\cos\theta}. \tag{59}$$

Using the Schmidt decomposition, the state (57) can be written as

$$|j,\eta,\theta\rangle = \sqrt{\lambda_+}|+\rangle_1 \otimes |+\rangle_2 + \sqrt{\lambda_-}|-\rangle_1 \otimes |-\rangle_2 \tag{60}$$

where $\lambda_\pm$ denote the eigenvalues of the reduced density of the first subsystem $\rho_{j_1} = \mathrm{Tr}_{j_2}(\rho)$ obtained by tracing out the spin j_2 . They write as

$$\lambda_\pm = \frac{1}{2}\left(1 \pm \sqrt{1-\mathcal{C}^2}\right). \tag{61}$$

in terms of the concurrence $\mathcal{C} \equiv \mathcal{C}_{j_1,j_2}(\theta)$ given by (59). In the basis $\{|+\rangle_1 \otimes |+\rangle_2, |+\rangle_1 \otimes |-\rangle_2, |-\rangle_1 \otimes |+\rangle_2, |-\rangle_1 \otimes |-\rangle_2\}$, the density matrix $\rho_{j_1,j_2}(\theta) = |j,\eta,\theta\rangle\langle j,\eta,\theta|$ takes the form

$$\rho_{j_1,j_2}(\theta) = \begin{pmatrix} \lambda_+ & 0 & 0 & \sqrt{\lambda_+\lambda_-} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\lambda_+\lambda_-} & 0 & 0 & \lambda_- \end{pmatrix} \tag{62}$$

Using the result (7), one verifies that $\omega_{11} = 0$, $\omega_{22} = 0$ and $\omega_{33} = 1 - 4\lambda_+\lambda_-$. It follows that the local quantum uncertainty coincides with the squared concurrence (59)

$$\mathcal{U}(\rho_{j_1,j_2}(\theta)) = \mathcal{C}^2_{j_1,j_2}(\theta). \tag{63}$$

For $\theta = m\pi$ ($m \in \mathbb{Z}$), the logical qubits $|j,\eta,m=0\rangle$ and $|j,\eta,m=1\rangle$ coincide with even and odd spin coherent states. They behave like a multipartite state of Greenberger-Horne-Zeilinger (GHZ) type [50] in the asymptotic limit $p \to 0$. Indeed, in this limit, the states $|\eta\rangle$ and $|-\eta\rangle$ approach orthogonality and an orthogonal basis can be defined such that $|\mathbf{0}\rangle \equiv |\eta\rangle$ and $|\mathbf{1}\rangle \equiv |-\eta\rangle$. Thus, the state $|j,\eta,m\rangle$ becomes of GHZ-type

$$|j,\eta,m\rangle \sim |\mathrm{GHZ}\rangle_{2j} = \frac{1}{\sqrt{2}}(|\mathbf{0}\rangle \otimes |\mathbf{0}\rangle \otimes \cdots \otimes |\mathbf{0}\rangle + e^{im\pi}|\mathbf{1}\rangle \otimes |\mathbf{1}\rangle \otimes \cdots \otimes |\mathbf{1}\rangle) \tag{64}$$

which is maximally entangled and the bipartite local quantum uncertainty is $\mathcal{U}(\rho_{j_1,j_2}(\theta = m\pi)) = 1$.

Another interesting limiting case concerns the situation where $p^2 \to 1$ (or $\eta \to 0$). In this case the state $|j,\eta,m = 0 \pmod 2\rangle$ (55) reduces to ground state of a collection of $2j$ fermions

$$|j,0,0 \pmod 2\rangle \sim |\downarrow\rangle \otimes |\downarrow\rangle \otimes \cdots \otimes |\downarrow\rangle, \tag{65}$$

which is completely separable and

$$\mathcal{U}(\rho_{j_1,j_2}(\theta = m\pi) = 0.$$

The odd spin coherent state $|j,\eta,1 \pmod 2\rangle$ becomes a multipartite state of W type [51]

$$|j,0,1 \pmod 2\rangle \sim |\mathrm{W}\rangle_{2j} = \frac{1}{\sqrt{2j}}(|\uparrow\rangle \otimes |\downarrow\rangle \otimes \cdots \otimes |\downarrow\rangle + |\downarrow\rangle \otimes |\uparrow\rangle \otimes \ldots \otimes |\downarrow\rangle + \cdots + |\downarrow\rangle \otimes |\downarrow\rangle \otimes \cdots \otimes |\uparrow\rangle) . \tag{66}$$

and, in this limiting situation, the local quantum uncertainty is given by

$$\mathcal{U}(\rho_{j_1,j_2}(\theta = \pi)) = 4\frac{j_1 j_2}{j_1 + j_2}.$$

3.2.2 Mixed bipartite states

Now, we consider bipartite mixed density matrices ρ_{ij} obtained by a trace procedure consisting in removing all degrees of freedom of all qubits except the two qubits i and j. Since we consider quantum systems possessing the exchange symmetry, the trace procedure leads to $2j(2j-1)/2$ identical density matrices ρ_{12}. After some algebra, one gets

$$\rho_{12} = \mathcal{N}^2(|\eta,\eta\rangle\langle\eta,\eta| + |-\eta,-\eta\rangle\langle-\eta,-\eta| + e^{im\pi}q|-\eta,-\eta\rangle\langle\eta,\eta| + e^{-im\pi}q|\eta,\eta\rangle\langle-\eta,-\eta|). \tag{67}$$

The quantity q occurring in (67) is defined by

$$q = p^{2j-2}.$$

Setting $\eta = e^{i\phi}\sqrt{\frac{1-p}{1+p}}$, the density matrix takes the form

$$\rho_{12} = \frac{1}{4(1+p^{2j}\cos m\pi)} \begin{pmatrix} (1+p)^2(1+q\cos m\pi) & 0 & 0 & e^{-2i\phi}(1-p^2)(1+q\cos m\pi) \\ 0 & (1-p^2)(1-q\cos m\pi) & (1-p^2)(1-q\cos m\pi) & 0 \\ 0 & (1-p^2)(1-q\cos m\pi) & (1-p^2)(1-q\cos m\pi) & 0 \\ e^{2i\phi}(1-p^2)(1+q\cos m\pi) & 0 & 0 & (1-p)^2(1+q\cos m\pi) \end{pmatrix} \quad (68)$$

in the computational basis. The phase factor ϕ will be taken equal to zero. In other words, the phase factor can be removed by a loacal transformation and the local quantum uncertainty remains unchanged as we discussed here above. The bipartite mixed density ρ_{12} (68) writes in Fano-Bloch representation as

$$\rho_{12} = \sum_{\alpha\beta} R_{\alpha\beta}\sigma_\alpha \otimes \sigma_\beta \quad (69)$$

where the non vanishing matrix elements $R_{\alpha\beta}$ $(\alpha, \beta = 0, 1, 2, 3)$ are given by

$$R_{00} = 1, \quad R_{11} = \frac{1-p^2}{1+p^{2j}\cos m\pi}, \quad R_{22} = \frac{(p^2-1)\,p^{2j-2}\cos m\pi}{1+p^{2j}\cos m\pi},$$

$$R_{33} = \frac{p^2+p^{2j-2}\cos m\pi}{1+p^{2j}\cos m\pi}, \quad R_{03} = R_{30} = \frac{p+p^{2j-1}\cos m\pi}{1+p^{2j}\cos m\pi}.$$

From the equations (9), (10) and (11), one obtains

$$\omega_{11} = \sqrt{\frac{1-p^2}{1+p^2}}\,\frac{\sqrt{1-p^{4j-4}}}{1+p^{2j}\cos m\pi} \quad (70)$$

$$\omega_{22} = p^2\sqrt{\frac{1-p^2}{1+p^2}}\,\frac{\sqrt{1-p^{4j-4}}}{1+p^{2j}\cos m\pi} \quad (71)$$

$$\omega_{33} = \frac{2p^2}{1+p^2}\,\frac{1+p^{2j-2}\cos m\pi}{1+p^{2j}\cos m\pi} \quad (72)$$

Since $\omega_{22} \leq \omega_{11}$, we have $\omega_{\max} = \max(\omega_{11}, \omega_{22})$ and one verifies that

$$\mathrm{sign}(\omega_{11} - \omega_{33}) = \mathrm{sign}\Big(2(1-p^4) - (1+3p^4)(1+p^{2(j-1)}\cos m\pi)\Big)$$

Etudier le signe dans les $j = 1, 3/2, 2, 5/2, ...$ pour $m = 0$ et $m = 1$

Tracer LQU en fonction de p^2 pour $j = 1, 3/2, 2, 5/2, ...$

We note that for the special case $j = 1$, the state (56) is a pure state with $j_1 = j_2 = 1/2$. In this case, it is simple to verify that $\omega_{11} = \omega_{22} = 0$ and $\omega_{33} = \frac{4p^2}{(1+p^2)^2}$ for $m = 0$ and $\omega_{33} = 0$ for $m = 1$. Therefore, in this case the local quantum uncertainty writes

$$\mathcal{U}(\rho_{1/2,1/2}) = \frac{(1-p^2)^2}{(1+p^2)^2}$$

for $m = 0$ and

$$\mathcal{U}(\rho_{1/2,1/2}) = 1$$

for $m = 1$ in agreement with the result (63).

For $j > 1$ and $p \longrightarrow 0$, one obtains

$$\omega_{11} = 1, \omega_{22} = 0, \omega_{33} = 0$$

and the local quantum uncertainty is zero.

Similarly, for $j > 1$ and $p^2 \longrightarrow 1$, it is simple to check that the local quantum uncertainty vanishes for even spin coherent states ($m = 0$). In this limiting situation, the matrix elements (70), (71) and (72) become

$$\omega_{11} = \frac{j-2}{\sqrt{2}j}, \omega_{22} = \frac{j-2}{\sqrt{2}j}, \omega_{33} = \frac{j-1}{j}$$

for odd spin coherent states ($m = 1$). The local quantum uncertainty is then given by

$$\mathcal{U}(\rho_{12}(m=1)) \longrightarrow \frac{1}{j}.$$

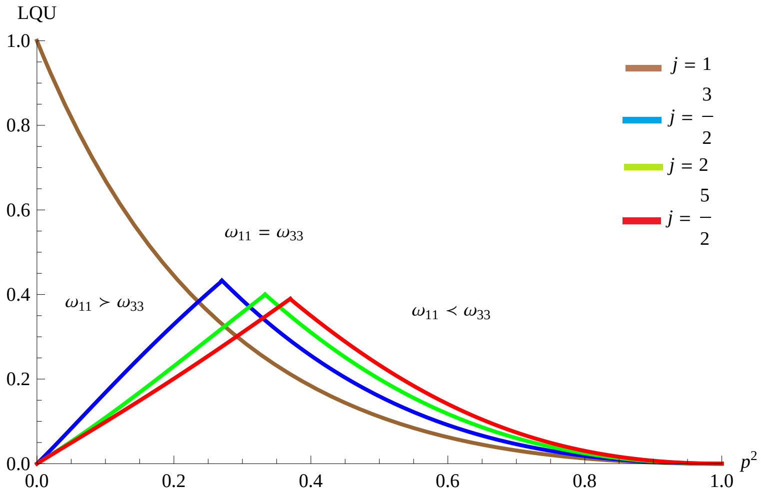

Figure 1. The local quantum uncertainty in symmetric states ($m = 0$) for different values of j.

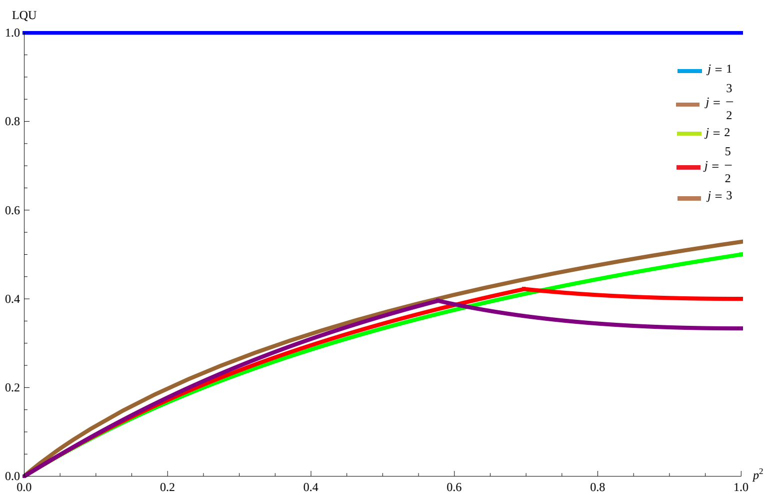

Figure 2. The local quantum uncertainty in antisymmetric states ($m = 1$) for different values of j.

4 Concluding remarks

In conclusion, we have derived the analytical expression of local quantum uncertainty for two-qubit in X-states. This quantum correlations quantifier provides an efficient and computable way to characterize the nature of correlations present in a multi-partite quantum system. Moreover, the analytical results obtained in this paper covers some special class of two-qubit states recently investigated in the literature. We quote Bell states and orthogonally invariant two-qubit system examined in [27] and [28], respectively. We also evaluated the pairwise local quantum uncertainty in multi-partite systems with exchange and parity symmetries. As illustration, we quantified the pairwise quantum correlations in balanced superpositions of Dicke states. Our interest in such states is mainly motivated by their relevance in various collective spin systems such as Dicke model [52] and Lipkin-Meshkov-Glick [54] model exhibiting quantum phase transition.

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