Quantum interferometric power and the role of nonclassical correlations in quantum metrology for a special class of two-qubit states

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Abstract

We analyze the effects of quantum correlations, such as entanglement and discord, on the parameter precision in an interferometric configuration. We consider a parameterized family of two-qubit states possessing exchange and parity symmetries for which the analytical expression of the quantum discord based on von Neumann entropy is given. The local quantum Fisher information for the estimation of local unitary transformation is derived. We also derive explicitly the interferometric quantum power of the probe states. Our study corroborates the recent series of investigations focusing on the role of quantum correlations other than entanglement on the efficiency of parameter estimation.

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1 Introduction

Quantum metrology is an emerging field in quantum information science [1, 2, 3, 4]. It exploits the quantum mechanical laws to enhance the precision in estimating the values physical quantities such as phase, frequency, or magnetic fields. During the last decade, different metrology protocols have been theoretically investigated and experimentally implemented to gain precision in estimating various parameters [5]-[18] (see also the references quoted in [4]). This special interest is mainly motivated by the fact that in quantum metrology the parameter estimation goes beyond the classical limit and in some cases tends to the Heisenberg limit imposed by the laws of quantum mechanics. The quantum metrology originates from the theory of quantum estimation [19, 20, 21]. The key ingredient in quantum metrology is the quantum Fisher information [22] and its inverse depicts the lower bound in statistical estimation of an unknown parameter according to Cramér-Rao theorem [19, 20, 23]. The quantum metrology offers the possibility to surpass the restrictions imposed by classical laws of physics and to enhance the sensitivity of parameter estimation. Indeed, the estimation of a parameter encoded in a unitary transformation (e.g. a phase shift) of the probe states involving n non-entangled qubits, the precision scales as $\frac{1}{\sqrt{n\nu}}$ where ν is the number of repeated measurements which ameliorates the classical scaling given by $\frac{1}{\sqrt{\nu}}$. In the presence of n entangled qubits the optimal scaling rewrites $\frac{1}{n\sqrt{\nu}}$ enhancing the standard scaling limit by a factor of $\sqrt{n}$ [2, 3, 4]. In this sense, it is natural to ask if the quantum correlation other than entanglement can ameliorate the precision in metrology protocols. This issue was recently addressed in [24, 25, 26, 27]. In fact, to understand the role of quantum correlation beyond entanglement in a black-box quantum metrology task, a quantum correlation quantifier in term of quantum Fisher information was recently introduced [24] (see also [26]). This quantifier is termed quantum interferometric power which is a discord-type measure of quantum correlations and quantifies the precision in interferometric phase estimation. Non classical correlations that include entanglement but quantify the quantum correlations of separable states have been the subject of numerous studies. The most familiar one is the quantum discord introduced in 2001 [28, 29] to describe the quantum correlations which are not limited to entanglement. Entanglement and quantum discord are equivalent for pure states but they are fundamentally different for mixed states. The exact expression of quantum discord involves an optimization procedure which is very challenging especially for two-qubit states with rank higher than two.

In a given estimation protocol, one should first prepare the input state (i.e. ρ) which has to be sensitive to the parameter variations. The second step consists in encoding of the information about the unknown parameter (i.e. θ). This encoding can be realized by be a unitary evolution (i.e. $\rho \longrightarrow \rho_\theta$). The final part of this protocol concerns the measurement of an appropriate observable ((i.e. H)) in the output state (i.e. ρ_θ). In this work, we shall examine the role of quantum correlations in quantum phase estimation where the generic probe states belong to the following class of two qubit density matrices whose entries are specified in terms of two real parameters. They are defined as

$$\rho = \begin{pmatrix} c_1 & 0 & 0 & \sqrt{c_1 c_2} \\ 0 & \frac{1}{2}(1-c_1-c_2) & \frac{1}{2}(1-c_1-c_2) & 0 \\ 0 & \frac{1}{2}(1-c_1-c_2) & \frac{1}{2}(1-c_1-c_2) & 0 \\ \sqrt{c_1 c_2} & 0 & 0 & c_2 \end{pmatrix} \tag{1}$$

in the computational basis $\mathcal{B} = \{|00\rangle, 01\rangle, 10\rangle, 11\rangle\}$. The parameters c_1 and c_2 satisfy the conditions $0 \leq c_1 \leq 1$, $0 \leq c_2 \leq 1$ and $0 \leq c_1 + c_2 \leq 1$. We have taken all entries positives. This class of states arise in various collective spin models as well as bipartite quantum systems prepared in balanced superpositions of coherent states. Indeed, the bipartite density matrix extracted from the states of a symmetric multi-qubit systems take the form (1) as for instance the superpositions of Dicke states [30, 31] and even and odd spin coherent states [32]. They also arise in investigating the pairwise quantum correlation in balanced superpositions of multipartite Glauber coherent states which interpolates continuously between the generalized Greenberger-Horne-Zeilinger and Werner states [33, 34, 35, 36]. We notice also that we deliberately chosen two-qubit rank-2 states to employ the method developed in [37] to get the exact expressions for the quantum discord based on the von Neumann entropy.

This paper is organized as follows. In Section 2, quantum Fisher information is derived for the two-qubit states (1) when the dynamics of the first qubit is governed by a local Hamiltonian. We determine the minimum of the quantum Fisher information for some particular form of the local Hamiltonian. This analysis is helpful in deriving the quantum interferometric power in Section 3. Indeed, the quantum interferometric power is defined by minimizing the quantum Fisher information over all local Hamiltonians. We give the explicit form of this discord-like quantifier and we show the role of quantum correlations in enhancing the precision. In other words, less amount of quantum Fisher information implies more quantum correlations and vice-versa. This indicates that one gain better sensitivity by employing probe states with significant amount of quantum correlations. Also, we compare the quantum interferometric parameter with the entropy based quantum discord. In section 4, tighter bounds on the phase precision in the presence of quantum correlation are investigated in the spirit of the recent results obtained in [24, 26]. Concluding remarks close this paper.

2 Local quantum Fisher information

As mentioned in the introduction, the process of estimating the value of an unknown parameter consists in three different steps: (*i*) the preparation of the probe state (input-state), (*ii*) the interaction of the initialized state with the system (target) encoding the physical quantity to be estimated and finally (*iii*) the measure of the state (output-state) resulting from the interaction of the input-state and the system. In the situations where the Hamiltonian governing the dynamics of the probe state is known one can determine the value of the unknown parameter. In quantum metrology, the interferometric configuration constitutes one of the most interesting scenarios which are widely used in phase estimation (see for instance [4] and [38]) . We consider the two-qubit states of (1) as probe states and we assume that the dynamics of the first qubit is governed by the local phase shift transformation $e^{-i\theta H} \equiv e^{-i\theta\ H_1 \otimes \mathbb{I}}$ where H_1 is a local Hamiltonian acting on the qubit 1 and $\mathbb{I}$ is the 2×2 identity matrix. Thus, the output states write

$$\rho_\theta = e^{-i\theta H} \rho e^{+i\theta H}.$$

From the measurement of the observable H in the output states, the parameter θ can be estimated through an (unbiased) estimator $\hat{\theta}$. The quantum mechanics imposes the fundamental limit of the variance of the estimator $\hat{\theta}$. This is given by the quantum Cramér-Rao bound:

$$\text{var } \hat{\theta} \geq \frac{1}{\nu F(\rho, H)}$$

where ν is the number of times the estimation protocol is repeated and $F(\rho, H)$ is the quantum Fisher information, a measure which is widely utilized in different fields of quantum physics. For the parameter dependent states ρ_θ , the quantum Fisher information is defined by

$$F(\rho_\theta) \equiv F(\rho, H) = \text{Tr}(\rho_\theta L_\theta^2) \tag{2}$$

where L is the symmetric logarithmic derivative determined by the equation

$$\partial_\theta \rho = \frac{1}{2}(\rho_\theta L_\theta + L_\theta \rho_\theta). \tag{3}$$

It is clear that the spectral decomposition of the density matrix and its derivative with respect to the parameter θ provides us with the expression of the quantum Fisher information. The eigenvalues of the density matrices ρ (1) write

$$\lambda_1 = c_1 + c_2, \quad \lambda_2 = 1 - (c_1 + c_2), \quad \lambda_3 = 0 \quad , \lambda_4 = 0, \tag{4}$$

and the corresponding eigenstates are respectively given by

$$|\psi_1\rangle = \sqrt{\tfrac{c_1}{c_1+c_2}}(1,0,0,\sqrt{\tfrac{c_2}{c_1}}), \quad |\psi_2\rangle = \tfrac{\sqrt{2}}{2}(0,1,1,0), \quad |\psi_3\rangle = \tfrac{\sqrt{2}}{2}(0,1,-1,0), \quad |\psi_4\rangle = -\sqrt{\tfrac{c_2}{c_1+c_2}}(1,0,0,\sqrt{\tfrac{c_1}{c_2}}). \tag{5}$$

The explicit expression of quantum Fisher information was derived in [**?**] for density matrices with arbitrary ranks. For the states under consideration, it is simple to check that the quantum Fisher information takes the form

$$F(\rho, H) = \sum_{i=1}^{2} \lambda_i F(|\psi\rangle_i, H) - 8 \sum_{i\neq j}^{2} \frac{\lambda_i \lambda_j}{\lambda_i + \lambda_j} |\langle \psi_i | H | \psi_j \rangle|^2 \tag{6}$$

and the quantity $F(|\psi\rangle_i, H)$ is simply given in term of the variance of the operator H on the state $|\psi\rangle_i$ as

$$F(|\psi\rangle_i, H) = 4(\Delta H)^2_{|\psi\rangle_i}, \tag{7}$$

where the variance of the Hamiltonian H is given by $(\Delta H)^2_{|\psi\rangle_i} = \langle\psi_i|H^2|\psi_i\rangle - |\langle\psi_i|H|\psi_i\rangle|^2$. It is clear that the quantum Fisher information involves only the nonzero eigenvalues of the density matrix ρ and the corresponding eigenstates. We assumed that the dynamics of the probe state is governed by the local unitary transformation acting on the first qubit while leaving the second qubit unchanged. The general form of the local Hamiltonian acting on the first sub-system is given by

$$H_1 = \vec{r}.\vec{\sigma} := r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3 \tag{8}$$

where $\vec{r} = (\sin\alpha\ \cos\beta, \sin\alpha\ \sin\beta, \cos\alpha)$ and $\vec{\sigma}$ are the components of $\vec{\sigma}$ are the the usual Pauli matrices $(\sigma_1 = |0\rangle\langle 1| + |1\rangle\langle 0|,\ \sigma_2 = i(|1\rangle\langle 0| - |0\rangle\langle 1|),\ \sigma_3 = |0\rangle\langle 0| - |1\rangle\langle 1|)$. We note that the traceless Hamiltonian H with non degenerate spectrum is the maximal informative observable. From (6), one verifies that the quantum Fisher information takes the form

$$F(\rho, H) = 4 - 4\cos^2\alpha \frac{(c_1-c_2)^2}{c_1+c_2} - 8\sin^2\alpha \Big(1-(c_1+c_2)\Big)\Big((c_1+c_2) + 2\sqrt{c_1c_2}\cos 2\beta\Big) \tag{9}$$

In the following, we shall discuss the cases of three particular forms of the local Hamiltonian (8) governing the dynamics of the first qubit: (i) $H_1 = \sigma_1$, $H_1 = \sigma_2$ and $H_3 = \sigma_3$ and we then derive the bounds to precision in each scenario to determine the probe state in the family (1) which guarantees a minimum estimation efficiency. Clearly, in this figure we assume the prior knowledge of the Hamiltonian H.

For a unitary evolution along the x-direction (i.e. $H_1 = \sigma_1$) in (8)), the expression (9) becomes

$$F(\rho, \sigma_1) = 4 - 8[1-(c_1+c_2)][\sqrt{c_2}+\sqrt{c_1}]^2 \tag{10}$$

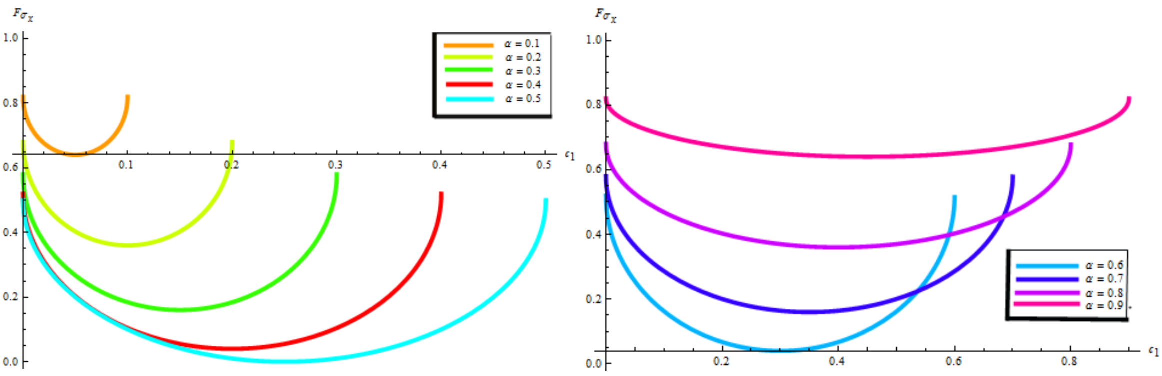

Figure 1. The quantum Fisher information $F(\rho, \sigma_1)$ along the x-direction versus c_1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

Changer la notation pour les axes par $F(\rho, \sigma_1)$

The behavior of the quantum Fisher information along the x-direction $F(\rho, \sigma_1)$, as function of the parameters c_1, is represented in the figure 1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$ where $\alpha = c_1 + c_2$ takes the special values $\alpha = 0.1, ..., 0.9$. As it can be inferred from this figure, the quantum Fisher information $F(\rho, \sigma_1)$ in state (1) reaches its minimal value for states with $c_1 = c_2 = \frac{\alpha}{2}$. They are given by

$$\rho\big(c_1 = \frac{\alpha}{2}, c_2 = \frac{\alpha}{2}\big) = \alpha\rho' + (1 - \alpha)\rho \tag{11}$$

where the states ρ and ρ' are respectively given by

$$\rho = |\psi\rangle\langle\psi|, \quad \rho' = |\psi'\rangle\langle\psi'|, \tag{12}$$

with

$$|\psi\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad , \quad |\psi'\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{13}$$

In the other hand, the maximal value of $F(\rho, \sigma_1)$ is obtained in the states with $(c_1 = 0, c_2 = \alpha)$ or $(c_1 = \alpha, c_2 = 0)$. This value maximal is obtained in the states

$$\rho(c_1 = 0, c_2 = \alpha) = \alpha|11\rangle\langle 11| + (1 - \alpha)|\psi\rangle\langle\psi|, \tag{14}$$

and

$$\rho(c_1 = \alpha, c_2 = 0) = \alpha|00\rangle\langle 00| + (1 - \alpha)|\psi\rangle\langle\psi|, \tag{15}$$

It is clear that for $\alpha \leq \frac{1}{2}$, the states encompassing high amount of quantum Fisher information are those with small values of the α. This situation is completely different for $\alpha \geq \frac{1}{2}$. In fact, the quantum Fisher information increases as the parameter α increases. For instance, for $c_1 = 0.45$, more precision is guaranteed in the states with $\alpha = 0.9$. For the subset of states of type (1) characterized by a fixed value of α ($\alpha \geq \frac{1}{2}$), the quantum Fisher information $F(\rho, \sigma_1)$ is maximal for $(c_1 = 0, c_2 = \alpha)$ or $(c_1 = \alpha, c_2 = 0)$ and the minimal value is reached for $c_1 = c_2 = \frac{\alpha}{2}$. We notice that the quantum Fisher plotted in the figure 1 as well as the figures below are normalized by the multiplicative factor $\frac{1}{4}$.

Now we consider the situation when the input state undergoes along the y-direction (i.e, $H_1 = \sigma_2$), the quantum Fisher information (9) reads as

$$F(\rho, \sigma_2) = 4 - 8[1 - (c_1 + c_2)][\sqrt{c_2} - \sqrt{c_1}]^2 \tag{16}$$

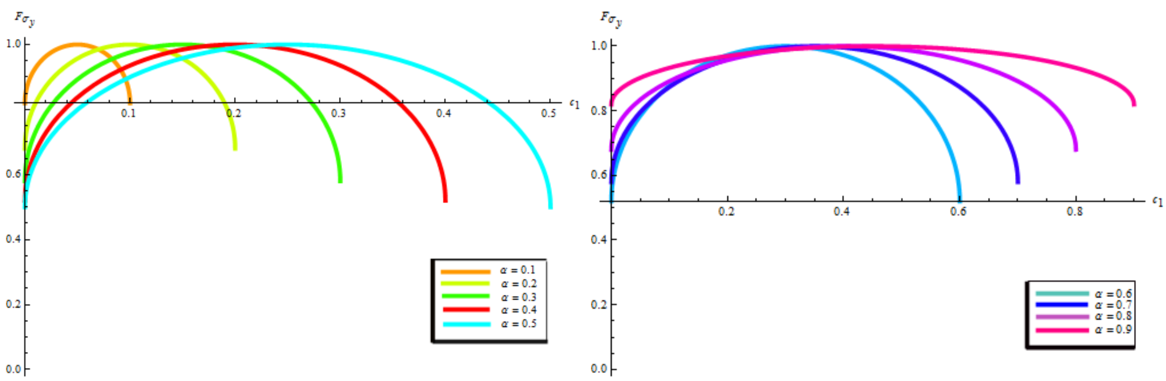

Figure 2. The quantum Fisher information $F(\rho, \sigma_2)$ along the y-direction versus c_1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

Changer la notation pour les axes par $F(\rho, \sigma_2)$

The figure 2 gives the quantum Fisher information $F(\rho, \sigma_2)$ along the y-direction as function of the parameter c_1 and c_2 for different values of $\alpha = c_1 + c_2$. For both cases $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$, the quantum Fisher information $F(\rho, \sigma_2)$ is maximal for the states satisfying $c_1 = c_2 = \frac{\alpha}{2}$ given by (11) and minimal for the states which correspond to the subset of states with ($c_1 = 0$, $c_2 = \alpha$) given by (14) or the states with ($c_1 = \alpha$, $c_2 = 0$) given by (15). It follows that probe states are governed by the local Hamiltonian σ_2, the best estimation is provided by the states given by the equation (11). Remark that for the local Hamiltonian σ_1, the best estimation is guaranteed by the probe states with ($c_1 = 0, c_2 = \alpha$) and ($c_1 = \alpha, c_2 = 0$) given respectively by (14) and (15).

For the dynamics in the the z-direction, the equation (9) reduces to

$$F(\rho, \sigma_3) = 4 - 4\frac{(c_1 - c_2)^2}{c_1 + c_2} \tag{17}$$

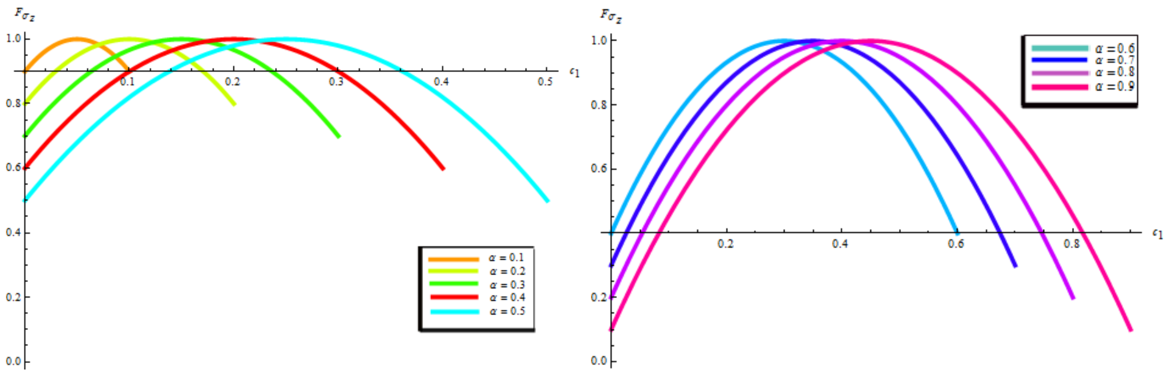

Figure 3. The quantum Fisher information $F(\rho, \sigma_3)$ along the z-direction versus c_1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

Changer la notation pour les axes par $F(\rho, \sigma_3)$

By comparing the results depicted in the figures 2 and 3, it is easily seen that the quantum Fisher information $F(\rho, \sigma_3)$ behaves like the quantum Fisher information $F(\rho, \sigma_2)$. Also, the expression $F(\rho, \sigma_3)$ is maximal for the states for the probe states with $c_1 = c_2 = \frac{\alpha}{2}$ (see equation (11)). It is simple to see from the equation (17) that the maximal value of the quantum Fisher information for any value of α is always equal to 4 (1 in the figure 3 because in the plot we divided $F(\rho, \sigma_3)$ by the factor 4). This indicates that, when the local Hamiltonian σ_3 governs the dynamics of the first qubit, the suitable probe states are given by (11) which are of Bell type.

To close this section, it is interesting to study the situation where the estimation protocol is blind in the sense that the one prepares the probe state without any prior knowledge of the Hamiltonian governing the dynamics of first subsystem. This issue is examined in what follows.

3 Nonclassical correlations and quantum interferometric power

To investigate the role of non classical correlation in improving the precision in quantum metrology protocols when the probe states are of type (1), we shall employ the discord-like measure introduced recently in [24] and termed as quantum interferometric power. We also compare this quantum correlations quantifier with the quantum discord based on von Neumann entropy derived in [28, 29].. The quantum interferometric power is a bona fide measure of discord-like correlations. It is defined by the minimum of the quantum Fischer information over all the possible spectrum local Hamiltonians [24]

$$\mathcal{P}(\rho) = \frac{1}{4} \min_{H_1} F(\rho, H_1), \tag{18}$$

where the minimization is performed over all Hamiltonians $\{H_1\}$ acting on the qubit 1. The quantum interferometric power is defined in term of quantum Fisher information and quantifies naturally the degree of precision that a bipartite state ρ provides to ensure the success of the estimation protocol regardless of the phase direction [24]. Obviously, the properties of quantum Fisher information [22] confers to quantum interferometric power many interesting properties: (i) non negative, (ii) invariant under local unitary transformations (iii) non increasing under local operations on the second qubit and (iv) asymmetric with respect to the two subsystems (except for symmetric quantum states). A closed analytical expression of the quantum interferometric power for an arbitrary bipartite quantum system was derived in [24]. Explicitly, it writes as

$$\mathcal{P}(\rho) = \min(\lambda_1, \lambda_2, \lambda_3), \tag{19}$$

where λ_1, λ_2 and λ_3 are the eigenvalues of the 3×3 matrix M whose elements are defined by

$$M_{ij} = \frac{1}{2} \sum_{k,l:\lambda_k+\lambda_l \neq 0} \frac{(\lambda_k - \lambda_l)^2}{\lambda_k + \lambda_l} \langle \psi_k | \sigma_i \otimes \mathbb{I} | \psi_l \rangle \langle \psi_l | \sigma_j \otimes \mathbb{I} | \psi_k \rangle \tag{20}$$

with λ_i and$|\psi_i\rangle$ being respectively the eigenvalues and the eigenvectors of density matrix ρ. When the probe states are of the form (1), the eigenvalues are given by (4) and their corresponding eigenstates are given by (5). Reporting (4) and (5) in the expression (20), it is simple to check that the matrix M is diagonal and the diagonal elements write

$$M_{11} = 1-2\big[1-c_1-c_2\big]\big[\sqrt{c_2}+\sqrt{c_1}\big]^2, \; M_{22} = 1-2\big[1-c_1-c_2\big]\big[\sqrt{c_2}-\sqrt{c_1}\big]^2, \; M_{33} = 4\frac{c_1 c_2}{c_1+c_2}+\big[1-c_1-c_2\big]. \tag{21}$$

Comparing the elements of the correlation matrix M and the quantum Fisher information given by the equations (10), (16) and (17), one has

$$M_{ii} = \frac{1}{4} F(\rho, \sigma_i), \qquad \text{for} \quad i = 1, 2, 3$$

and the quantum interferometric power (18) is the minimal amount of quantum information in the three spatial directions x, y and z, discussed in the previous section. It follows that the quantum interferometric power writes

$$\mathcal{P}(\rho) = \frac{1}{4} \min(F(\rho, \sigma_1), F(\rho, \sigma_2), F(\rho, \sigma_3)). \tag{22}$$

Using the expressions (21), one verifies that M_{22} is always greater than M_{11}. It follows that the smallest eigenvalues of the matrix M is either M_{11} or M_{33}. The difference $M_{11} - M_{33}$ is positive when the parameters c_1 and c_2 satisfy the condition

$$2(c_1 + c_2)^2 - (\sqrt{c_1} + \sqrt{c_2})^2 \geq 0. \tag{23}$$

Therefore for states with $\alpha \leq \frac{1}{2}$ ($\alpha = c_1 + c_2$), the quantity $M_{11} - M_{33}$ is non positive and the quantum interferometric power (22) writes as

$$\mathcal{P}(\rho) = 1 - 2[1 - (c_1 + c_2)][\sqrt{c_2} + \sqrt{c_1}]^2. \tag{24}$$

For states with $\alpha \geq \frac{1}{2}$, the condition(23) is satisfied for

$$0 \leq c_1 \leq \alpha_- \qquad \text{or} \qquad \alpha_+ \leq c_1 \leq \alpha \tag{25}$$

where the quantities $\alpha_\pm$ are defined by

$$\alpha_\pm = \frac{1}{2}\alpha \pm \sqrt{\alpha^3 - \alpha^4}. \tag{26}$$

In this case, the quantum interferometric power is given by

$$\mathcal{P}(\rho) = 1 - \frac{(c_1 - c_2)^2}{c_1 + c_2}, \tag{27}$$

Conversely for $\alpha_- \leq c_1 \leq \alpha_+$ the diffrence $M_{11} - M_{33}$ is negative and the quantum interferometric power reads as

$$\mathcal{P}(\rho) = 1 - 2\big[1 - (c_1 + c_2)\big]\big[\sqrt{c_2} + \sqrt{c_1}\big]^2, \tag{28}$$

The behavior of quantum interferometric power versus c_1 is presented in the figure 4 for different values of $\alpha = c_1 + c_2$.

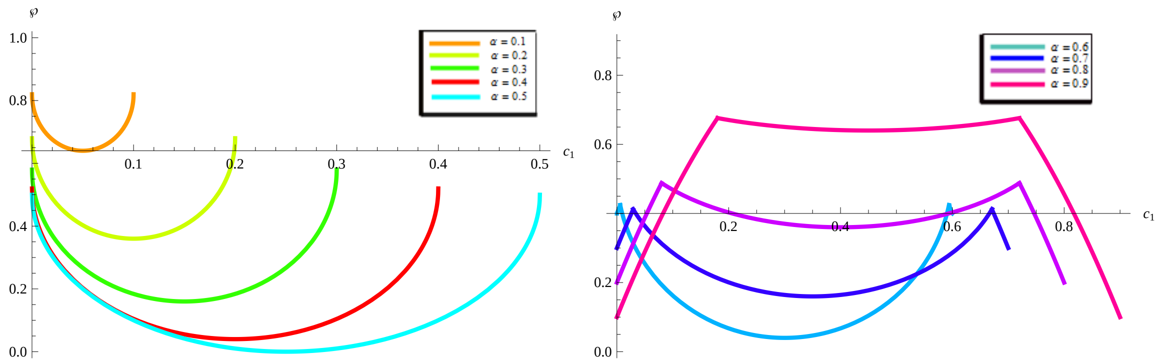

Figure 4. The quantum interferometric power $\mathcal{P}(\rho)$ as function of the parameter c_1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

Changer la notation pour les axes par $\mathcal{P}(\rho)$

The quantum interferometric power is depicted in figure 4. Note that for $\alpha \leq \frac{1}{2}$, the quantum interferometric power coincides (up to the scale factor $\frac{1}{4}$) with the local quantum Fisher information $F(\rho, \sigma_1)$ when the dynamics of the first qubit is governed by the local Hamiltonian σ_1. This can be also seen from the numerical results reported in the figures 3 and 4 for $\alpha \leq \frac{1}{2}$. According to the analysis presented in the previous section, it is clear that the quantum correlations enhance the degree of precision in the estimation of the phase parameter θ. Therefore, for a fixed value of the parameter α, the states (14) and (15) contain the maximal amount of quantum correlations and they offer the best estimation efficiency.

For states with $\alpha \geq \frac{1}{2}$, the situation becomes significantly different. First, we note the quantum interferometric power exhibits a sudden double change when $c_1 = \alpha_-$ and $c_1 = \alpha_+$ (α_- ($\alpha_\pm$ are given by (26)). This behavior is similar to the sudden change of geometric quantum discord based on Schatten p-norms (trace norm ($p = 1$)and Hilbert-Schmidt norm ($p = 2$)) Hilbert-Schmidt or trace norm which were widely investigated in the literature, especially in connection with quantum phase transitions [45, 42, 43, 44]. Three different phases characterize the behavior of quantum interferometric power: (i) $0 \leq c_1 \leq \alpha_-$ where $\mathcal{P}(\rho) = \frac{1}{4}F(\rho, \sigma_3)$ (ii) $\alpha_- \leq c_1 \leq \alpha_+$ where $\mathcal{P}(\rho) = \frac{1}{4}F(\rho, \sigma_1)$ and (iii) $\alpha_+ \leq c_1 \leq \alpha$ where $\mathcal{P}(\rho) = \frac{1}{4}F(\rho, \sigma_3)$. The minimal value of the quantum interferometric power $\mathcal{P}$ is obtained in the intermediate zone ($\alpha_- \leq c_1 \leq \alpha_+$) for the states with ($c_1 = c_2 = \frac{\alpha}{2}$) (11). It is also remarkable that this double sudden change in the behavior of the quantum interferometric power occurs when the states (1) contain the maximal amount of quantum correlation.

4 von Neumann entropy based quantum discord

In this section we compare the discord-like quantum interferometric power with the entropic quantum discord originally introduced in the information-theoretic context [28, 29]. It coincides with entanglement for pure states and goes beyond entanglement for mixed ones. Quantum discord is defined as the difference between total correlation and classical correlation in a bipartite state . The evaluation of quantum discord involves an optimization procedure for the conditional entropy over all local generalized measurement. This optimization is in general very challenging and this is the reason why the explicit analytical expressions of quantum discord is known only for a restricted class of two-qubit quantum states [39, 40] and including two-qubit rank-2 states [37]. For two-qubit density matrices of rank two, the connection between the quantum discord and the entanglement of formation, which is described by the KoashiWinter theorem [41], provides the algorithm to derive explicitly the quantum discord. The entropic quantum discord writes [28, 29]

$$\mathcal{D}(\rho_{12}) = S(\rho_1) + \tilde{S}_{\min} - S(\rho_{12}). \tag{29}$$

where $S(\rho) = -\mathrm{Tr}\rho \log \rho$ is the von-Neumann entropy, ρ_1 is the reduced density matrix of the first qubit and $\tilde{S}_{min}$ is the minimum of the conditional entropy over all positive operator-valued measures. For two-qubit rank-2 states, this entropy can be analytically derived by employing the the KoashiWinter theorem. In this sense, the lines of reasoning developed in [37] (see also [32] where similar notations are adopted) one first purifies the the states (1) as follows

$$\rho_{12} = \lambda_1 |\psi_1\rangle\langle\psi_1| + \lambda_2 |\psi_2\rangle\langle|\psi_2|, \tag{30}$$

where the eigenvalues and the corresponding eigenvectors are given by (4) and (5) respectively, by attaching a third a third qubit $\{|0\rangle_3, |1\rangle_3\}$ to obtain the following pure three-qubit state

$$|\psi\rangle_{123} = \sqrt{\lambda_1}|\psi\rangle_1 \otimes |0\rangle_3 + \sqrt{\lambda_2}|\psi\rangle_2 \otimes |1\rangle_3.$$

When a positive operator valued measure (POVM) measurement is performed on the qubit 1 , the Koashi-Winter theorem establishes a relationship between between the minimum of the conditional entropy $\tilde{S}_{min}$ and the entanglement of formation $E(\rho_{23})$ of the system described by $\rho_{23} = \mathrm{Tr}_1 \rho_{123}$. This result reads as

$$\tilde{S}_{min} = E(\rho_{23}) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - |C(\rho_{23})|^2}\right) \tag{31}$$

where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function and $C(\rho_{23})$ is the Wootters concurrence [46] which writes for the density matrix ρ_{23} as

$$|C(\rho_{23})|^2 = 2[1 - c_1 - c_2](\sqrt{c_1} - \sqrt{c_2})^2. \tag{32}$$

Setting $c_1 + c_2 = \alpha$ and reporting (32) in the equation (31), the explicit expression of quantum discord for the density $\rho \equiv \rho_{12}$ express as

$$\mathcal{D}(\rho) = H\left(c_1 + \frac{1-\alpha}{2}\right) + H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - 2(1-\alpha)(\alpha - 2\sqrt{c_1(\alpha - c_1)})}\right) - H(\alpha) \tag{33}$$

in terms of the parameters α and c_1.

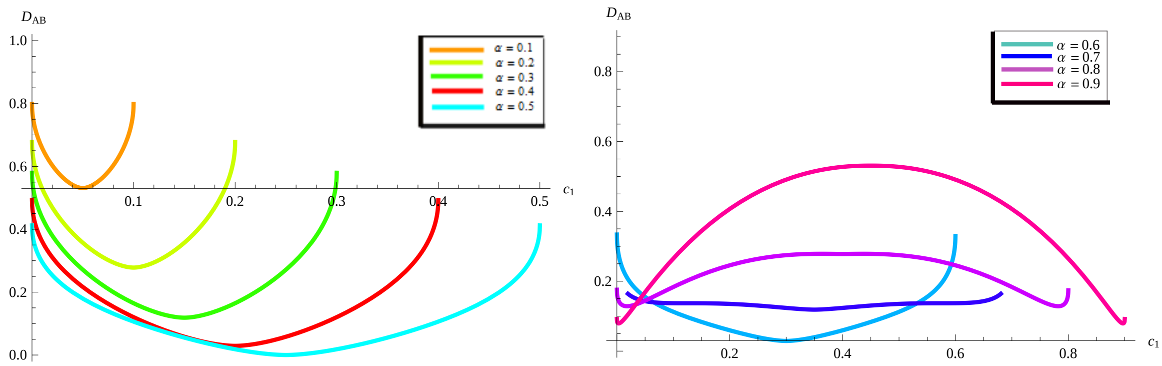

Figure 5. Quantum discord $\mathcal{D}(\rho)$ as function of the parameter c_1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

Changer la notation pour les axes par $\mathcal{D}(\rho)$

The comparison of the numerical calculations depicted in figures 4 and 5 show that quantum interferometric power and the entropic quantum discord present almost similar behavior except the double sudden change exhibited by the quantum correlations in figure 4 for $\alpha \geq \frac{1}{2}$. This confirms that the quantum interferometric power constitutes an appropriate to the tackle the issue of quantifying the quantum correlation. Also, in view of the technical difficulties arising in the analytical evaluation of quantum discord based on entropy, the discord-like quantum interferometric power provides a powerful way to quantify the quantum correlations in generic two-qubit states and more generally in bipartite systems of higher dimensional Hilbert spaces. Furthermore, the quantum interferometric power goes beyond entanglement. Indeed, for the states under consideration $\rho \equiv \rho_{12}$ (1), the Wootters concurrence is given by

$$\mathcal{C}_{12}(\rho) = |(\sqrt{c_1} + \sqrt{c_2})^2 - 1| \tag{34}$$

With $\alpha = c_1 + c_2$ and for $\alpha \leq \frac{1}{2}$ it rewrites

$$\mathcal{C}_{12}(\rho) = 1 - \alpha + 2\sqrt{c_1(\alpha - c_1)} \quad \text{with} \quad 0 \leq c_1 \leq \alpha, \tag{35}$$

and when $\alpha \geq \frac{1}{2}$, the concurrence is given by

$$\mathcal{C}_{12}(\rho) = (\sqrt{c_1} + \sqrt{\alpha - c_2})^2 - 1 \tag{36}$$

for

$$0 \leq c_1 \leq c_-, \quad c_+ \leq c_1 \leq \alpha \tag{37}$$

with $c_\pm = \frac{\alpha \pm \sqrt{2\alpha - 1}}{2}$ and the system is entangled. However, for $c_- \leq c_1 \leq c_+$, the concurrence is zero and the bipartite system is separable.

revoir expression de C_{12} en fonction de c_1: il y a des fautes.

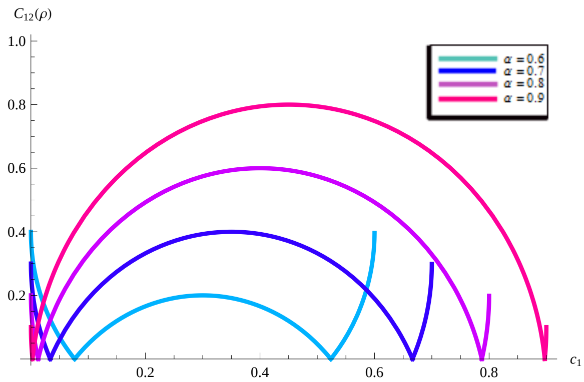

Figure 5. The concurrence $\mathcal{C}_{12}(\rho)$ as function of the parameter c_1 for $\alpha \geq \frac{1}{2}$.

It is important to stress that the quantum interferometric power $\mathcal{P}$ is nonzero except in the particular case $c_- = c_+ = 0.25$ or equivalently $\alpha = 0.5$ and $c_1 = c_2 = 0.5$ (see the equations (27) and (28)). This reflects that the quantum interferometric power is also a quantifier which characterizes the quantum correlations existing in separable states and in this respect goes beyond the notion of concurrence introduced by Wootters. On the other hand quantum interferometric power is directly related to quantum Fisher information and as by product this quantifier constitute an adequate tool to understand the role of quantum information in enhancing the parameter precision in quantum metrology protocols. Indeed, from the figure 4 for $\alpha \leq \frac{1}{2}$), the quantum interferometric power measured by reach the maximal values is for the states (14) and (15) respectively. Also, among the family of density matrices (1), the probe states which offer the efficient sensitivity are those with lower values of the parameter α ($\alpha = 0.1$ for instance). For two-qubit states with $\alpha \geq \frac{1}{2}$, the suitable probe states to ensure the best precision are those with $c_1 = \alpha_-$ or $c_1 = \alpha_+$ (α_- ($\alpha_\pm$ are given by (26)) and enhance more the precision estimation one should consider states with higher values of α. It is remarkable that the probe states ensuring the maximum precision to the estimation are exactly those encompassing the maximum amount of quantum correlations, i.e. with $c_1 = \alpha_-$ or $c_1 = \alpha_+$ where the sudden change of quantum interferometric power occurs.

5 Bounds of local local Fisher information

It is clear that even in the absence of entanglement, quantum correlations constitute a resource to enhance the parameter precision in the bipartite states ρ (1) for unitary parametrization process. Recently, bounds on the metrology precision in the presence of quantum correlation were derived in [26] by evaluating the upper and the lower bound of the local quantum Fisher information. In this sense, we denote by

$$\mathcal{F}_{\min}(\rho) = \frac{1}{4}\min_{\{H_1\}} \mathrm{F}(\rho, H_1) \qquad \mathcal{F}_{\max}(\rho) = \frac{1}{4}\max_{\{H_1\}} \mathrm{F}(\rho, H_A). \tag{38}$$

the minimal and the maximal amount of local quantum Fisher information over all local observables acting on the qubit 1. We notice that $\mathcal{F}_{\min}(\rho)$ is exactly the definition of quantum interferometric power defined by (18). To write down the expression of the upper and lower bounds, we optimize first the local quantum Fisher information (9) over the variables α and β parameterizing the orientation of the unit vector $\overrightarrow{r}$ (see equation (8)). By setting the derivatives to zero, one finds the solutions $\overrightarrow{r} = (1,0,0)$, $\overrightarrow{r} = (0,1,0)$ and $\overrightarrow{r} = (0,0,1)$ for which one obtains the quantum Fisher $F(\rho,\sigma_1)$, $F(\rho,\sigma_2)$ and $F(\rho,\sigma_3)$ given by the expressions (10), (16) and (17) respectively. Therefore, one gets

$$\mathcal{F}_{\min}(\rho) = \frac{1}{4}\min(F(\rho,\sigma_1), F(\rho,\sigma_2), F(\rho,\sigma_3)) \qquad \mathcal{F}_{\max}(\rho) = \frac{1}{4}\max(F(\rho,\sigma_1), F(\rho,\sigma_2), F(\rho,\sigma_3)). \tag{39}$$

This result can be alternatively derived using the method reported in [26]. Indeed, by optimizing over all local traceless Hamiltonians acting on the first qubit, the bounds of the local quantum Fisher information write

$$\mathcal{F}_{\min}(\rho) = 1 - \lambda_w^{\max}, \quad \mathcal{F}_{\max}(\rho) = 1 - \lambda_w^{\min}, \tag{40}$$

where $\lambda_w^{\max}$ and $\lambda_w^{\min}$ denote respectively the largest and the smallest values of the following quantities

$$W_{ii} = \sum_{m\neq n} \frac{2\lambda_m\lambda_n}{\lambda_m+\lambda_n}\langle m|\sigma_i\otimes \mathbb{I}|n\rangle\langle n|\sigma_i\otimes\mathbb{I}|n\rangle, \qquad i=1,2,3, \tag{41}$$

which can be written also as

$$W_{ii} = 1 - M_{ii} = 1 - \frac{1}{4}F(\rho,\sigma_i). \tag{42}$$

In this respect, using the relation (42) and the results (21), it is simple to see that, for the two-qubit states under consideration, the matrix W is diagonal and one gets

$$W_{11} = 2\big[1-(c_1+c_2)\big]\big[\sqrt{c_1}+\sqrt{c_2}\big]^2, \quad W_{22} = 2\big[1-(c_1+c_2)\big]\big[\sqrt{c_1}-\sqrt{c_2}\big]^2, \quad W_{33} = \tfrac{(c_1-c_2)^2}{c_1+c_2}. \tag{43}$$

Since W_{11} is always greater than W_{22}, one obtains

$$\lambda_w^{\max} = \max(W_{11}, W_{33}) \qquad \lambda_w^{\min} = \min(W_{22}, W_{33}).$$

As the inverse of quantum Fisher provides the lower bound on the error in statistical estimation of unknown parameter and since the local quantum Fisher information is bounded by below by $\mathcal{F}_{\min}(\rho)$ and above by $\mathcal{F}_{\max}(\rho)$, we introduce the following quantity

$$\delta\theta = \tfrac{1}{F_{\min}(\rho)} - \tfrac{1}{F_{\max}(\rho)}. \tag{44}$$

to examine the range of the possible variations of the inverse of local quantum information and the role of quantum correlations present in the probe states to decrease the difference between the upper and the lower values and how this affects the enhancement of the parameter estimation. We note that $F_{\min}(\rho)$ and the quantum interferometric power are proportional. Therefore, for $\alpha \leq \frac{1}{2}$, one obtains

$$F_{\min} = 1 - 2(1-\alpha)(\sqrt{c_1}+\sqrt{\alpha-c_1})^2 \tag{45}$$

For $\alpha \geq \frac{1}{2}$, when the parameter c_1 satisfies $\alpha_- \leq c_1 \leq \alpha_+$, $F_{\min}$ reads as (45); the quantities α_- and α_+ are defined by (26). Conversely, for $0 \leq c_1 \leq \alpha_-$ and $\alpha_+ \leq c_1 \leq \alpha$, one gets

$$F_{\min} = 1 - \frac{(2c_1-\alpha)^2}{\alpha} \tag{46}$$

Similarly, to write the expression of the upper bound of local quantum fisher information $F_{\max}$, one compares W_{22} and W_{33}. The inequality $W_{22} \geq W_{33}$ holds when the parameter c_1 satisfies the following condition

$$(\sqrt{c_1} - \sqrt{c_2})^2 - 2(c_1 + c_2)^2 \geq 0. \tag{47}$$

To examine this two parameters condition, we fix $\alpha = c_1 + c_2$ and we consider separately the situations where $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$. From (47), it is simply verified, for $\alpha \geq \frac{1}{2}$, that $\lambda_w^{min} = W_{33}$ and one has

$$F_{\max} = 1 - 2(1-\alpha)(\sqrt{c_1} - \sqrt{\alpha - c_1})^2. \tag{48}$$

For the situation where $\alpha \leq \frac{1}{2}$, one has $\lambda_w^{min} = W_{22}$ for

$$\alpha_- \leq c_1 \leq \alpha_+$$

where α_+ and α_- are given by (26). It follows that

$$F_{\max} = 1 - 2(1-\alpha)(\sqrt{c_1} - \sqrt{\alpha - c_1})^2, \tag{49}$$

for $\alpha_- \leq c_1 \leq \alpha_+$ and

$$F_{\max} = 1 - \frac{(2c_1 - \alpha)^2}{\alpha} \tag{50}$$

for $0 \leq c_1 \leq \alpha_-$ and $\alpha_+ \leq c_1 \leq \alpha$. Combining the results (45), (46), (48), (49), (50) and (50), one gets the explicit form of the difference $\delta\theta$. Indeed, for $\alpha \leq \frac{1}{2}$, one obtains

$$\delta\theta = 4\ \frac{F(\rho, \sigma_3) - F(\rho, \sigma_1)}{F(\rho, \sigma_1)F(\rho, \sigma_3)} \tag{51}$$

when $c_1 \in [0, \alpha_-] \cup [\alpha_+, \alpha]$ and

$$\delta\theta = 4\ \frac{F(\rho, \sigma_2) - F(\rho, \sigma_1)}{F(\rho, \sigma_1)F(\rho, \sigma_2)} \tag{52}$$

for $c_1 \in [\alpha_-, \alpha_+]$ where $F(\rho, \sigma_1)$, $F(\rho, \sigma_2)$ and $F(\rho, \sigma_3)$ are given by (10), (16) and (17), respectively. It is remarkable that the difference δ involves only the local quantum Fisher information obtained for the local observables σ_1, σ_2 and σ_3. Similarly, for $\alpha \geq \frac{1}{2}$, one finds

$$\delta\theta = 4\ \frac{F(\rho, \sigma_2) - F(\rho, \sigma_3)}{F(\rho, \sigma_2)F(\rho, \sigma_3)} \tag{53}$$

when $c_1 \in [0, \alpha_-] \cup [\alpha_+, \alpha]$ and

$$\delta\theta = 4\ \frac{F(\rho, \sigma_2) - F(\rho, \sigma_1)}{F(\rho, \sigma_1)F(\rho, \sigma_2)} \tag{54}$$

for $c_1 \in [\alpha_-, \alpha_+]$. The figure 6 gives for different values of $\alpha = c_1 + c_2$, the behavior of the difference $\delta\theta$ versus the versus the parameter c_1 labeling the probe states (1). For $\alpha \leq \frac{1}{2}$, the figure 6 shows that $\delta\theta$ is minimal for the states with $(c_1 = 0, c_2 = \alpha)$ and $(c_1 = \alpha, c_2 = 0)$. Conversely, it is maximal for states with $(c_1 = c_2 = \frac{\alpha}{2})$. The two qubit states with $(c_1 = 0, c_2 = \alpha)$ and $(c_1 = \alpha, c_2 = 0)$, respectively given by (14) and (15), are those presenting the maximal amount of quantum correlations as quantified by quantum interferometric power (see figure 4). This results elucidates the role of quantum correlation in enhancing the precision in quantum estimation protocols. Furthermore, the quantum interferometric power for the states with $(c_1 = c_2 = \frac{\alpha}{2})$ given by (11), is minimal (see figure 4). Thus the fact that the quantity $\delta\theta$ is maximal corroborate the crucial importance of the role of quantum correlations in improving the estimation of the parameter θ.

In the particular case $\alpha = \frac{1}{2}$, the expressions of $F_{\min}$ and $F_{\min}$ reduce to

$$F_{\min} = 1 - \frac{1}{2}(\sqrt{2c_1} + \sqrt{1 - 2c_1})^2, \quad F_{\max} = 1 - \frac{1}{2}(\sqrt{2c_1} - \sqrt{1 - 2c_1})^2. \tag{55}$$

Thus, when $c_1 = c_2 = \frac{1}{4}$, the quantum interferometric power $\mathcal{P} \sim F_{\min}$ vanishes (see also the figure 4) and $F_{\max} = 1$. This explains the infinite behavior of $\delta\theta$ for $c_1 = c_2 = \frac{1}{4}$ (see figure 6). It is interesting the note that in this case the state ρ (1) is separable. Indeed, using the equation (34), one verifies that the concurrence is zero. This result shows that the separable states are not suitable for parameter estimation. The limiting case $\alpha = \alpha = \frac{1}{2}$ is very illustrative. In fact, the upper and lower local Fisher information satisfy the additivity relation $F_{\min} + F_{\max} = 1$ which implies $F_{\min}$ increases as $F_{\max}$ decreases and vice-versa. It follows that when the values of $F_{\min}$ and $F_{\max}$ approach each other, the difference δ decreases and in this case the probe states ρ exhibit increasing amount of quantum correlations. Similar results are obtained for other values of the parameter α. They are reported in the figure 6. The comparison of these results and ones depicted in figure 4, show that the quantity $\delta\theta$ varies inversely with the amount of quantum correlations encompassed in the probe states. In this respect, it can be used to decide about their suitability in enhancing the estimation of the parameter θ. Finally, we note that the sudden change of quantum interferometric power $\mathcal{P}$ occurring when $\alpha \geq \frac{1}{2}$ (figure 4) is responsible of the sudden change in the behavior of the quantity $\delta\theta$.

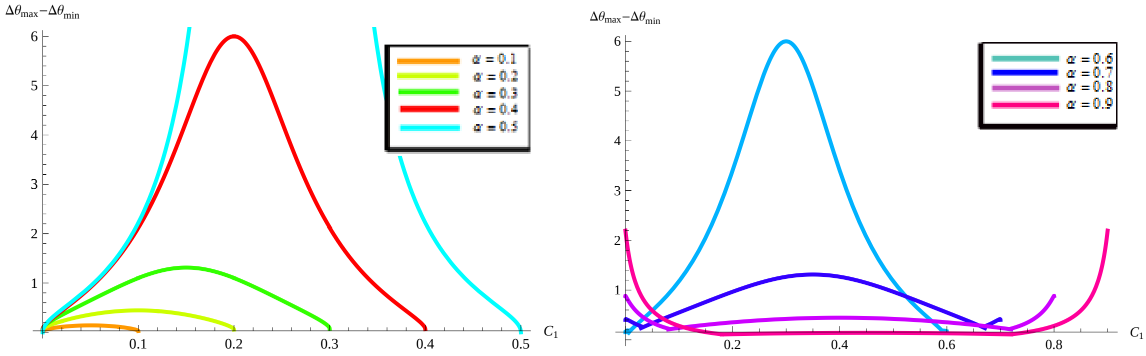

Figure 6 . $\delta\theta$ versus the parameter c_1 for different values of $\alpha = c_1 + c_2$.

Changer la notation pour les axes par $\delta\theta$

Using the bounds of local quantum Fisher information, we investigate the relation between the difference defined

$$\delta\mathcal{F} = F_{\max} - F_{\min} \tag{56}$$

and the quantum correlations existing in the generic class of states ρ (1). In this respect, combining the results (45), (46), (48), (49), (50) and (50), one gets the explicit form of the difference $\delta\mathcal{F}$. Indeed, for $\alpha \leq \frac{1}{2}$, one obtains

$$\delta\mathcal{F} = \frac{1}{\alpha}\left[\left(2\sqrt{c_1(\alpha - c_1)} + \alpha(1-\alpha)\right)^2 - \alpha^4\right] \tag{57}$$

when $c_1 \in [0, \alpha_-] \cup [\alpha_+, \alpha]$ and

$$\delta\mathcal{F} = 8\ (1-\alpha)\sqrt{c_1(\alpha - c_1)} \tag{58}$$

for $c_1 \in [\alpha_-, \alpha_+]$. Similarly, for $\alpha \geq \frac{1}{2}$, one finds

$$\delta\mathcal{F} = \frac{1}{\alpha}\left[\alpha^4 - \left(2\sqrt{c_1(\alpha - c_1)} - \alpha(1-\alpha)\right)^2\right] \tag{59}$$

when $c_1 \in [0, \alpha_-] \cup [\alpha_+, \alpha]$ and

$$\delta\mathcal{F} = 8\ (1-\alpha)\sqrt{c_1(\alpha - c_1)} \tag{60}$$

for $c_1 \in [\alpha_-, \alpha_+]$.

Tracer $\delta\mathcal{F}$ en fonction de c_1.

6 Concluding remarks

To summarize, we first derived the analytical expressions of local quantum Fisher information for some particular unitary parametrization processes which gives significant advantages in examining the role of quantum correlations in determining high-precision of the estimated parameter. We also investigated the amount of quantum correlations in a generic class of two-qubit states by using the concept of quantum interferometric power which a discord-like quantifier. The characterization of the quantum correlations in term of this new quantifier provides the the appropriate tool to examine the role of quantum correlations in quantum metrology. In this paper, we deliberately considered density matrices of rank two to give a comparison between the quantum interferometric power and and the quantum discord based on von Neumann entropy which can be easily derived for this kind of states using the Koashi-winter theorem. Indeed, the amount of quantum correlations depicted in figures 4 and 5 show clearly that the quantum interferometric power can be used as a good measure to reveal the quantum correlations in bipartite quantum states especially ones of higher rank. It must be noticed that the quantum interferometric power can be derived explicitly for quantum systems with higher dimensional Hilbert spaces [24, 26]. Based on this comparison, it becomes clear that the quantum interferometric power provides a nice geometrical tool in identifying , quantifying and characterizing quantum correlations for bipartite qudit systems and offers the way to overcome the mathematical difficulties encountered in deriving the explicit expressions of entropic quantum discord. The second important aspect investigated in this paper deals with the role of quantum correlations in quantum metrology. We explicitly derived the tight bounds of the error on the estimated parameter. The difference between the higher and the lower bounds is investigated in detail and in particular we find that this difference is smaller for two qubit states encompassing a large amount of quantum correlations and becomes larger for states presenting less quantum correlations. This indicates that the quantum correlations are efficient in boosting the the performance of information processing protocols.

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