Remarques: les numeros des references sont les memes que dans la version corrigee. J'ai change l'ordre. Il faut retablir le bon ordre.

Remarques: Tu lis tres tres attentivement cette introduction. Il faut develloper la presentation des sections Sec2, Sec.3 car c'est la ou il faut dire ce qu'on veut faire et les resultats obtenus. Tu me fais des propositions

1 Introduction

Entanglement [2], Bell's non-locality [3] and quantum discord [4, 5] are three kinds of quantum correlations which have been extensively discussed in the literature. They coincide for pure states but are generally of difficult characterization in mixed states. In the mixed case an hierarchical structure emerges where the quantum steering arises as an intermediate form of quantum correlations between entanglement and Bell non-locality. The concept of quantum steering or Einstein-Podolsky-Rosen (EPR) steering [1] was introduced by Schrödinger [7] in generalizing the EPR paradox [6]. In simple words, this phenomenon dictates that, in a bipartite quantum system, measurement made by one party (Alice) can remotely alter (i.e. steer) the second party (Bob) at different location. The quantum steering defines the Alice's ability to change non locally the Bob's system state. Now, it is commonly accepted that the quantum steering can be interpreted as the entanglement certification when the measurements are performed by an un-trusted party [8]. This operational interpretation motivated several theoretical [14] and experimental [15] studies. One may quote for instance, the exploitation of quantum steering as an essential resource in quantum key distribution [16], secure quantum teleportation [17] and randomness generation [18]. For continuous-variables systems, an experimental criteria to detect quantum steering was proposed by Reid in [10] and the first experimental observation was reported in [11]. Recently, substantial experimental progress in detecting this kind of quantum correlations was accomplished for discrete as well as continuous variables [6, 12]. The issue concerning the detection of quantum steering under Gaussian measurements was investigated in [8]. More precisely, It has been proved that the violation of Reid criterion [10] constitutes a genuine indicator of quantum steering. This important result confirms the fact that quantum steering, which include the EPR paradox, characterizes the quantum correlations in two-mode Gaussian states more efficiently than the concept of non-separability. Another important point, discussed in [8], concerns the asymmetry of quantum steering and the existence of entangled states which are one way steerable (states that are steerable by Alice but not by Bob). This directional asymmetry was examined theoretically and experimentally in some recent works [13] showing the possible steerability only in one direction.

Motivated by the above mentioned achievements, especially quantum steering in two-mode Gaussian states, we examine Gaussian one-way steering in an optomechanical system fed by squeezed light. Specifically, we consider two spatially separated optomechanical Fabry-Perot cavities fed by broadband squeezed light. By adopting the resolved sideband regime approximation and the adiabatic elimination of the optical modes, we investigate the Gaussian steering and its asymmetry for two mixed mechanical modes. A special emphasis is also dedicated to mechanical modes exhibiting the steerability only in one direction. To characterize the quantum steering in the optomechanical system under consideration, we shall employ the formulation of the concept of steering in bipartite Gaussian states developed in [9].

Our interest in optomechanical systems is essentially motivated by the tremendous experimental and theoretical progress in this field of research in connection with quantum information physics. In fact, many significant achievements were realized such as for instance the test of quantum effects at macroscopic scales [19], the production of entangled states [20], ground state optical feedback cooling of the fundamental vibrational mode [21], the observation of quantum state transfer [22] and massive quantum superpositions or so-called Schrödinger cat states [23]. The main objective of these efforts is to pave the way for controlling optical-mechanical interactions at the quantum level to realize hybrid structures to encode the information. From this perspective, we think that the optomechanical system under consideration can be useful for experimental demonstration of quantum steering between two mechanical modes. We note that the first experimental achievement in this sense was realized for two Gaussian entangled optical fields (Z. Y. Ou, S. F. Pereira, H. J. Kimble, and K. C. Peng Phys. Rev. Lett. 68, 3663 Published 22 June 1992). On the other hand, it offers the experimental setup to demonstrate the one way steerability of two mechanical modes which, to the best of our knowledge, has not been considered before.

The remainder of this paper is organized as follows. In Sec. 2, we present a detailed description of of the optomechanical system under investigation. We give the quantum Langevin equations governing the dynamics of the mechanical and optical modes. The required approximations to derive closed analytical expressions for the time-dependent covariance matrix of the mechanical fluctuations are also discussed. In Sec. 3, using the quantum steering formulation proposed in [9], we study the dynamics of Gaussian steering and its asymmetry for the two mechanical modes of the system taking into account the thermal and the squeezing light effects. The entanglement is quantified in terms of Rényi-2 entropy. In particular, to show that this quantifier can not detect the asymmetry of quantum correlations, we evaluated numerically the amount of entanglement and Gaussian steering. Finally, we close the paper with concluding remarks in in Sec. 4.

2 System and Hamiltonian

2.1 The model

The optomechanical system depicted in Fig.1 comprises two Fabry-Perot cavities where each cavity is composed by two mirrors. The first mirror is fixed and partially transmitting, while the second is movable and perfectly reflecting. The j^{th} cavity is pumped by coherent laser field with an input power $\wp_j$, a phase φ_j and a frequency ω_{L_j}. In addition, the two cavities are also pumped by two-mode squeezed light which can be for instance produced by spontaneous parametric down-conversion source (SPDC) [?]. The first (respectively, the second) squeezed mode is sent towards the first (respectively, second) cavity. The mirrors are represented by harmonic oscillators [?] with an effective mass m_{μ_j}, a mechanical damping rate γ_j and a frequency ω_{μ_j}. The starting point of all subsequent discussions will be the Hamiltonian governing the dynamics of optical and mechanical modes of the system. This Hamiltonian reads, in the rotating frame at the lasers frequencies, as [?]

$$H = \sum_{j=1}^{2} \left[\left(\omega_{c_j} - \omega_{L_j}\right) a_j^\dagger a_j + \omega_{\mu_j} b_j^\dagger b_j + g_j a_j^\dagger a_j (b_j^\dagger + b_j) + \varepsilon_j (e^{i\varphi_j} a_j^\dagger + e^{-i\varphi_j} a_j) \right]. \tag{1}$$

where $b_j, b_j^\dagger$ are the annihilation and creation operators associated with the mechanical mode describing the mirror j (for $j = 1, 2$). They satisfy the usual commutation relations $[b_j, b_k^\dagger] = \delta_{jk}$. As we shall mainly concerned in Sec. 3 with the quantum correlations between the mechanical modes, we will refer to the mode 1 as Alice and to the mode 2 as Bob. In equation (1), the objects a_j and $a_j^\dagger$ (for $j = 1, 2$) denote the annihilation and creation operators of the optical modes. They satisfy also the commutation rules $[a_j, a_k^\dagger] = \delta_{jk}$. The quantity g_j in the equation (1) is the optomechanical single-photon coupling rate between the j^{th} mechanical mode and the j^{th} optical mode. It is given by $g_j = \left(\omega_{c_j}/l_j\right) \sqrt{\hbar/m_{\mu_j}\omega_{\mu_j}}$ where l_j is the j^{th} cavity length. The coupling strength between the j^{th} external laser and its corresponding cavity field is defined by $\varepsilon_j = \sqrt{2\kappa_j \wp_j/\hbar\omega_{L_j}}$; κ_j being the energy decay rate of the j^{th} cavity.

2.2 Quantum Langevin equation

In the Heisenberg picture, the nonlinear quantum Langevin equations for optical and mechanical modes are given by

$$\begin{aligned} \partial_t b_j &= -\left(\gamma_j/2 + i\omega_{\mu_j}\right) b_j - i g_j a_j^\dagger a_j + \sqrt{\gamma_j} b_j^{in}, &(2)\\ \partial_t a_j &= -\left(\kappa_j/2 - i\Delta_j\right) a_j - i g_j a_j (b_j^\dagger + b_j) - i\varepsilon_j e^{i\varphi_j} + \sqrt{\kappa_j} a_j^{in}, &(3)\end{aligned}$$

where $\Delta_j = \omega_{L_j} - \omega_{c_j}$ for) is the j^{th} laser detuning [?] with $j = 1, 2$. In equation (3) b_j^{in} is the j^{th} random Brownian operator with zero mean value ($\langle b_j^{in}\rangle = 0$) which describes the noise induced by the vacuum fluctuations of the continuum of modes outside the cavity. We assume that the mechanical baths are Markovian so that the noise operators b_j^{in} have the following nonzero time-domain correlation functions [?, ?]

$$\begin{aligned} \langle b_j^{in\dagger}(t) b_j^{in}(t')\rangle &= n_{\mathrm{th},j}\delta(t-t'), &(4)\\ \langle b_j^{in}(t) b_j^{in\dagger}(t')\rangle &= (n_{\mathrm{th},j}+1)\delta(t-t'), &(5)\end{aligned}$$

where $n_{\mathrm{th},j} = \left[\exp(\hbar\omega_{\mu_j}/k_B T_j) - 1\right]^{-1}$ is the mean thermal photons number, T_j is the temperature of the j^{th} mirror environment and k_B is the Boltzmann constant. Another kind of noise affecting the system is the j^{th} input squeezed vacuum noise operator a_j^{in} with zero mean value. They have the following non zero correlation properties [?]

$$\begin{aligned}
\langle a_j^{in\dagger}(t)a_j^{in}(t')\rangle &= N\delta(t-t') \quad \text{for} \quad j=1,2, && (6)\\
\langle a_j^{in}(t)a_j^{in\dagger}(t')\rangle &= (N+1)\delta(t-t') \text{ for} \quad j=1,2, && (7)\\
\langle a_j^{in}(t)a_k^{in}(t')\rangle &= Me^{-i\omega_\mu(t+t')}\delta(t-t') \quad \text{for} \quad j\neq k=1,2, && (8)\\
\langle a_j^{in\dagger}(t)a_k^{in\dagger}(t')\rangle &= Me^{i\omega_\mu(t+t')}\delta(t-t') \text{ for} \quad j\neq k=1,2, && (9)
\end{aligned}$$

where $N=\sinh^2 r$, $M=\sinh r\cosh r$; r being the squeezing parameter (we have assumed that $\omega_{\mu_1}=\omega_{\mu_2}=\omega_\mu$).

2.3 Linearization of quantum Langevin equations

Due to the nonlinear nature of the radiation pressure, the exact solution coupled nonlinear quantum Langevin equations (2)-(3) is in general very challenging. To overcome this difficulty, we adopt the linearization scheme discussed in [?, ?]. In this scheme, the optical and mechanical operators a_j and b_j are decomposed as the sum of their mean value of the steady state plus fluctuation with zero mean value so that $\mathcal{O}_j=\langle\mathcal{O}_j\rangle+\delta\mathcal{O}_j=\mathcal{O}_{js}+\delta\mathcal{O}_j$ where $\mathcal{O}_j\equiv a_j, b_j$. The mean values b_{js} and a_{js} are obtained by solving the equations (2) and (3) in the steady state

$$\langle a_j\rangle = a_{js} = \frac{-i\varepsilon_j e^{i\varphi_j}}{\kappa_j/2-i\Delta'_j} \quad \text{and} \quad \langle b_j\rangle = b_{js} = \frac{-ig_j\left|a_{js}\right|^2}{\gamma_j/2+i\omega_{\mu_j}} \tag{10}$$

where $\Delta'_j=\Delta_j - g_j(b^*_{js}+b_{js})$ is the j^{th} effective cavity detuning including the radiation pressure effects [?, ?]. To simplify further our purpose, we assume that the double-cavity system is intensely driven ($|a_{js}|\gg 1$, for $j=1,2$). This assumption can be realized considering lasers with a large input power $\wp_j$ [?]. Therefore, the contributions arising from the nonlinear terms $\delta a_j^\dagger\delta a_j$, $\delta a_j\delta b_j$ and $\delta a_j\delta b_j^\dagger$ can be ignored. As result, one gets the following linearized Langevin equations

$$\begin{aligned}
\delta\dot{b}_j &= -\left(\gamma_j/2+i\omega_{\mu_j}\right)\delta b_j + G_j\left(\delta a_j-\delta a_j^\dagger\right)+\sqrt{\gamma_j}b_j^{in}, && (11)\\
\delta\dot{a}_j &= -\left(\kappa_j/2-i\Delta'_j\right)\delta a_j - G_j\left(\delta b_j^\dagger+\delta b_j\right)+\sqrt{\kappa_j}a_i^{in}, && (12)
\end{aligned}$$

where the parameter G_j, defined by $G_j=g_j\left|a_{js}\right|=g_j\sqrt{\bar{n}^j_{\text{cav}}}$, is the j^{th} light-enhanced optomechanical coupling for the linearized regime [?]. The quantity $\bar{n}^j_{\text{cav}}$ is the number of photons circulating inside the j^{th} cavity [?]. We notice that the Eqs. (11) and (12) have been obtained by setting $a_{js}=-i\left|a_{js}\right|$ or equivalently by taking the phase φ_j of the j^{th} input laser field equal to $\varphi_j=-\arctan(2\Delta'_j/\kappa_j)$. Introducing the operators $\delta\tilde{b}_j$ and $\delta\tilde{a}_j$ defined respectively by $\delta b_j=\delta\tilde{b}_j e^{-i\omega_\mu t}$ and $\delta a_j=\delta\tilde{a}_j e^{i\Delta'_j t}$, the equations (11) and (12) rewrite

$$\begin{aligned}
\delta\dot{\tilde{b}}_j &= -\frac{\gamma_j}{2}\delta\tilde{b}_j + G_j\left(\delta\tilde{a}_j e^{i\left(\Delta'_j+\omega_\mu\right)t}-\delta\tilde{a}_j^\dagger e^{-i\left(\Delta'_j-\omega_\mu\right)t}\right)+\sqrt{\gamma_j}\tilde{b}_j^{in}, && (13)\\
\delta\dot{\tilde{a}}_j &= -\frac{\kappa_j}{2}\delta\tilde{a}_j - G_j\left(\delta\tilde{b}_j e^{-i\left(\Delta'_j+\omega_\mu\right)t}+\delta\tilde{b}_j^\dagger e^{-i\left(\Delta'_j-\omega_\mu\right)t}\right)+\sqrt{\kappa_j}\tilde{a}_i^{in}. && (14)
\end{aligned}$$

Next, we assume that the two cavities are driven at *the red sideband* ($\Delta'_j = -\omega_\mu$ for $j = 1, 2$) which corresponds to quantum states transfer regime [?, ?]. We note also that, in the resolved-sideband regime where the mechanical frequency ω_μ of the movable mirror is larger than the j^{th} cavity decay rate κ_j ($\omega_\mu \gg \kappa_1,\, \kappa_2$), one can use the rotating wave approximation (RWA) [?, ?]. Therefore in a frame rotating with frequency $\pm 2\omega_\mu$, the equations (13) and (14) give

$$\begin{aligned} \delta\dot{\tilde{b}}_j &= -\frac{\gamma_j}{2}\delta\tilde{b}_j + G_j\delta\tilde{a}_j + \sqrt{\gamma_j}\tilde{b}_j^{in}, && (15)\\ \delta\dot{\tilde{a}}_j &= -\frac{\kappa_j}{2}\delta\tilde{a}_j - G_j\delta\tilde{b}_j + \sqrt{\kappa_j}\tilde{a}_j^{in}, && (16) \end{aligned}$$

when the the fast oscillating terms are neglected.

2.4 The adiabatic elimination of the optical modes

Being interested only in the quantum correlations between mechanical modes, the ideal configuration is the adiabatic regime which corresponds to the situation where the mirrors have a large mechanical quality factor and weak effective optomechanical coupling ($\kappa_j \gg G_j,\, \gamma_j$) [?]. In this limiting configuration, by inserting the steady state solution of (16) into (15), one shows that the j^{th} mirror dynamics reduces to

$$\delta\dot{\tilde{b}}_j = -\frac{\Gamma_j}{2}\delta\tilde{b}_j + \sqrt{\gamma_j}\tilde{b}_j^{in} + \sqrt{\Gamma_{\mathrm{a}_j}}\tilde{a}_j^{in} = -\frac{\Gamma_j}{2}\delta\tilde{b}_j + \tilde{F}_j^{in}, \tag{17}$$

where $\Gamma_{\mathrm{a}_j} = 4G_j^2/\kappa_j$ is the effective relaxation rate induced by radiation pressure [?], $\Gamma_j = \Gamma_{\mathrm{a}_j} + \gamma_j$ and $\tilde{F}_j^{in} = \sqrt{\gamma_j}\tilde{b}_j^{in} + \sqrt{\Gamma_{\mathrm{a}_j}}\tilde{a}_j^{in}$. In terms of the quadratures

$$\delta\tilde{q}_j = (\delta\tilde{b}_j^\dagger + \delta\tilde{b}_j)/\sqrt{2}, \quad \delta\tilde{p}_j = i(\delta\tilde{b}_j^\dagger - \delta\tilde{b}_j)/\sqrt{2}, \tag{18}$$

$$\tilde{F}_{q_j}^{in} = (\tilde{F}_j^{in,\dagger} + \tilde{F}_j^{in})/\sqrt{2}, \qquad \tilde{F}_{p_j}^{in} = i(\tilde{F}_j^{in,\dagger} - \tilde{F}_j^{in})/\sqrt{2}, \tag{19}$$

the linear quantum Langevin equations (17) can be cast in matricial form [?]

$$\dot{u}(t) = Su(t) + n(t), \tag{20}$$

where $S = \mathrm{diag}(-\frac{\Gamma_1}{2}, -\frac{\Gamma_1}{2}, -\frac{\Gamma_2}{2}, -\frac{\Gamma_2}{2})$, $u(t)^{\mathrm{T}} = (\delta\tilde{q}_1, \delta\tilde{p}_1, \delta\tilde{q}_2, \delta\tilde{p}_2)$ and $n(t)^{\mathrm{T}} = (\tilde{F}_{q_1}^{in}, \tilde{F}_{p_1}^{in}, \tilde{F}_{q_2}^{in}, \tilde{F}_{p_2}^{in})$. Needless to say, the form of the matrix S guarantees the full stability of the system and in this case the use of the Routh-Hurwitz criterion [?] is not necessary. Thus, we end up with linear evolution equations for the mechanical modes with zero-mean Gaussian noises. We notice that the mechanical fluctuations in the stable regime will also evolve to an asymptotic zero-mean Gaussian state. It follows that the state of the system is completely described by the correlation matrix $V(t)$ whose elements are given by

$$V_{ii'}(t) = \frac{1}{2}(\langle u_i(t)u_{i'}(t) + u_{i'}(t)u_i(t)\rangle). \tag{21}$$

Using Eqs. (20) and (21), it is simple to check that the matrix $V(t)$ satisfies the following evolution equation [?]

$$\frac{d}{dt}V(t) = SV(t) + V(t)S^{\mathrm{T}} + D, \tag{22}$$

where D is the noise correlation matrix defined by $D_{kk'}\delta(t-t') = (\langle n_k(t)n_{k'}(t') + n_{k'}(t')n_k(t)\rangle)/2$. Using the correlation properties of the noise operators given by the set of equations (4)-(9), one shows that the matrix D takes the form

$$D = \begin{pmatrix} D_{11} & 0 & D_{13} & 0 \\ 0 & D_{22} & 0 & D_{24} \\ D_{13} & 0 & D_{33} & 0 \\ 0 & D_{24} & 0 & D_{44} \end{pmatrix}, \tag{23}$$

where $D_{11} = D_{22} = \Gamma_{\mathrm{a}_1}(N+1/2) + \gamma_1(n_{\mathrm{th},1}+1/2)$, $D_{33} = D_{44} = \Gamma_{\mathrm{a}_2}(N+1/2) + \gamma_2(n_{\mathrm{th},2}+1/2)$ and $D_{13} = -D_{24} = M\sqrt{\Gamma_{\mathrm{a}_1}\Gamma_{\mathrm{a}_2}}$. The equation (22) is easily solvable and the solution writes as

$$V(t) = \begin{pmatrix} v_{11}(t) & 0 & v_{13}(t) & 0 \\ 0 & v_{22}(t) & 0 & v_{24}(t) \\ v_{13}(t) & 0 & v_{33}(t) & 0 \\ 0 & v_{24}(t) & 0 & v_{44}(t) \end{pmatrix} \equiv \begin{pmatrix} V_1(t) & V_3(t) \\ V_3^{\mathrm{T}}(t) & V_2(t) \end{pmatrix}, \tag{24}$$

with $V_1(t) = \mathrm{diag}(v_{11}(t), v_{22}(t))$, $V_2(t) = \mathrm{diag}(v_{33}(t), v_{44}(t))$ and $V_3(t) = \mathrm{diag}(v_{13}(t), v_{24}(t))$. Notice that $V(t)$ is a real, symmetric and positive definite matrix. The 2×2 matrices $V_1(t)$ and $V_2(t)$ represent the first and second mechanical modes respectively, while the information about the correlations between them is encoded in the sub-matrix $V_3(t)$. Considering identical damping rates ($\gamma_1 = \gamma_2 = \gamma$), the explicit expressions of the covariance matrix elements are given by

$$v_{11}(t) = v_{22}(t) = \frac{(2N+1)\mathcal{C}_1 + 2n_{\mathrm{th},1}+1}{2(\mathcal{C}_1+1)} + \frac{(-2N+1)\mathcal{C}_1 - 2n_{\mathrm{th},1}+1}{2(\mathcal{C}_1+1)} e^{-\gamma(\mathcal{C}_1+1)t}, \tag{25}$$

$$v_{33}(t) = v_{44}(t) = \frac{(2N+1)\mathcal{C}_2 + 2n_{\mathrm{th},2}+1}{2(\mathcal{C}_2+1)} + \frac{(-2N+1)\mathcal{C}_2 - 2n_{\mathrm{th},2}+1}{2(\mathcal{C}_2+1)} e^{-\gamma(\mathcal{C}_2+1)t}, \tag{26}$$

$$v_{13}(t) = -v_{24}(t) = \frac{2M\sqrt{\mathcal{C}_1\mathcal{C}_2}}{\mathcal{C}_1+\mathcal{C}_2+2}\left(1 - e^{-\frac{\gamma}{2}(\mathcal{C}_1+\mathcal{C}_2+2)t}\right), \tag{27}$$

in terms of the j^{th} optomechanical cooperativity $\mathcal{C}_j$ defined by [?]

$$\mathcal{C}_j = \Gamma_{\mathrm{a}_j}/\gamma = 4G_j^2/\gamma\kappa_j = \frac{8\omega_{c_j}^2}{\gamma m_{\mu_j}\omega_\mu\omega_{L_j} l_j^2}\frac{\wp_j}{\left[\left(\frac{\kappa_j}{2}\right)^2 + \omega_\mu^2\right]}. \tag{28}$$

Remak that when $r = 0$, the equations (??) and (27) give det $V_3(t) = 0$. Accordingly, without squeezed light, the two mechanical modes are separable and they are not steerable in any direction [?]. Hence, to detect the steerability in such configuration, the squeezing parameter must take non vanishing values. In fact, if $r \neq 0$, we have $\det V_3(t) < 0$ which is a necessary condition for two-mode Gaussian state to be entangled [?]. This reflects the crucial role of the squeezed light in the transfer quantum correlations from light to mechanical modes.