**Remarques: les numeros des references sont les memes que dans la version corrigee sauf Laurat et GSAgarawal dont il faut inverser l'ordre.**

**Remarques: Tu lis tres tres attentivement cette partie. Tu apportes tes corrections et Tu me fais des propositions**

# 1 Gaussian quantum steering in two mechanical mode states

## 1.1 Gaussian steering measure

A rigorous formulation of a reliable and computable quantifier of steering for continuous variable systems was developed in [?]. For the two-mechanical state with the covariance matrix $V(t)$ (**??**), the Gaussian steerability $\mathcal{G}^{A\to B}$ (by Gaussian measurement on Alice's side) is quantified by

$$\mathcal{G}^{A\to B} = \max\left[0, \frac{1}{2}\ln\frac{\det V_1(t)}{4\det V(t)}\right] = \max\left[0, \ -\ln 2\left(v_{33}(t) - \frac{(v_{13}(t))^2}{v_{11}(t)}\right)\right], \tag{1}$$

where $v_{11}(t), v_{33}(t)$ and $v_{13}(t)$ are given by Eqs. [(**??**)-(**??**)]. This quantity vanishes for non-steerable states by Gaussian measurements [?]. By exchanging the roles of Alice and Bob (or the two mechanical modes), the the Gaussian steerability $\mathcal{G}^{B\to A}$ writes

$$\mathcal{G}^{B\to A} = \max\left[0, \frac{1}{2}\ln\frac{\det V_2(t)}{4\det V(t)}\right] = \max\left[0, \ -\ln 2\left(v_{11}(t) - \frac{(v_{13}(t))^2}{v_{33}(t)}\right)\right], \tag{2}$$

It is clear that the analytical expressions of $\mathcal{G}^{A\to B}$ and $\mathcal{G}^{B\to A}$, in terms of the parameters of the optomechanical system, are too cumbersome and will not be reported here. In general, the quantum steerability is asymmetric (i.e. $\mathcal{G}^{A\to B} \neq \mathcal{G}^{B\to A}$). This asymmetry can be characterized by the Gaussian steering difference $\mathcal{G}^{\Delta}_{AB}$ defined by [?]

$$\mathcal{G}^{\Delta}_{AB} = \left|\mathcal{G}^{A\to B} - \mathcal{G}^{B\to A}\right|. \tag{3}$$

In view of the inherent asymmetry of Gaussian steering measure, the steerability between Alice and Bob can be classified as follows: $(i)$ $\mathcal{G}^{A\to B} = \mathcal{G}^{B\to A} = 0$ refereed as no-way steering, $(ii)$ $\mathcal{G}^{A\to B} > 0$ and $\mathcal{G}^{B\to A} = 0$ or $\mathcal{G}^{A\to B} = 0$ and $\mathcal{G}^{B\to A} > 0$ refereed as one-way steering (a quantum state may be steerable from $A$ to $B$ but not from $B$ to $A$) and finally $(iii)$ $\mathcal{G}^{A\to B} > 0$ and $\mathcal{G}^{B\to A} > 0$ refereed as two-way steering. On the other hand, it has been been proven that the Gaussian steering is always upper bounded by the Gaussian Rényi-2 entanglement [?]. For the two mechanical mode states, specified by the covariance matrix $V(t)$ (**??**), the Gaussian Rényi-2 entanglement measure $\mathcal{E}_2$ is given by [?] (see also [?])

$$\mathcal{E}_2 = \frac{1}{2}\ln\left[h(s,d,g)\right], \tag{4}$$

with

$$h(s,d,g) = \begin{cases} 1 & \text{iff} \quad 4g \geqslant 4s-1, \\ \left[\frac{(4g+1)s-\sqrt{[(4g-1)^2-16d^2][s^2-d^2-g]}}{4(d^2+g)}\right]^2 & \text{iff} \quad 4|d|+1 \leq 4g < 4s-1, \end{cases} \tag{5}$$

Figure 1: Plot of the Gaussian steering $\mathcal{G}^{A\to B}$ (green solid line), $\mathcal{G}^{B\to A}$ (red solid line), the steering asymmetry $\mathcal{G}_{AB}^{\Delta}$ (blue dashed line) and entanglement $\mathcal{E}_2$ (yellow solid line) between the two mechanical modes $A$ and $B$ as a function of the scaled time $\gamma t$ for $\mathcal{C}_1 = 15$, $\mathcal{C}_2 = 35$ and $r = 1$. The mean thermal photons numbers $n_{\text{th},1}$ and $n_{\text{th},2}$ are fixed as: panel (a) $n_{\text{th},1} = 0.5$ and $n_{\text{th},2} = 1$, panel (b) $n_{\text{th},1} = 1$ and $n_{\text{th},2} = 0.5$, panel (c) $n_{\text{th},1} = 1$ and $n_{\text{th},2} = 1.2$ and finally panel (d) $n_{\text{th},1} = 1$ and $n_{\text{th},2} = 1.5$. This figure confirms the Gaussian steering is upper bounded by the Gaussian Rényi-2 entanglement and as expected the steerable states are always entangled but the reverse is not necessarily true.

where $s = \frac{1}{2}(v_{11}(t) + v_{33}(t))$, $d = \frac{1}{2}(v_{11}(t) - v_{33}(t))$ and $g = \left(v_{11}(t)v_{33}(t) - v_{13}^2(t)\right)$.

The expressions of $\mathcal{G}^{A\to B}$, $\mathcal{G}^{B\to A}$, $\mathcal{G}_{AB}^{\Delta}$ and $\mathcal{E}_2$ involve the covariance matrix elements which are expressed in terms of the squeezing parameter $r$, the $j^{th}$ optomechanical cooperativity $\mathcal{C}_j$ and the $j^{th}$ mean thermal photons number $n_{\text{th},j}$ (**??**). We shall consider the case where $n_{\text{th},1} \neq n_{\text{th},2}$ and $\mathcal{C}_1 \neq \mathcal{C}_2$ so that the system is not symmetric by swapping the first and the second mode. This condition is crucial to ensure the Gaussian steering asymmetry. An extensive numerical analysis of the parameters characterizing the optomechanical system reveals that the set of parameters which give significant results of Gaussian steering are very close to that used in the optomechanical experiments reported in [**?**]. Namely, the mass of the movable mirrors is $\mu_{1,2} = 145$ ng and oscillate at frequency $\omega_{\mu_{1,2}} = 2\pi \times 947 \times 10^3$ Hz with a mechanical damping rate $\gamma_{1,2} = 2\pi \times 140$ Hz. The two cavities have length $L_{1,2} = 25$ mm, wave length $\lambda_{1,2} = 1064$ nm, decay rate $\kappa_{1,2} = 2\pi \times 215 \times 10^3$ Hz, frequency $\omega_{c_{1,2}} = 2\pi \times 5.26 \times 10^{14}$ Hz and pumped by laser fields of frequency $\omega_{L_{1,2}} = 2\pi \times 2.82 \times 10^{14}$ Hz. For the powers of the coherent laser sources, we take $\wp_1 = 5$ mW and $\wp_2 = 11$ mW [**?**]. Using the expression of the dimensionless optomechanical cooperativity $\mathcal{C}_j$ given by (**??**), one gets $\mathcal{C}_1 \simeq 35$ and $\mathcal{C}_2 \simeq 15$.

## 1.2 Dynamics of Gaussian steering asymmetry

### 1.2.1 *Thermal effects*

Using the steering quantifier presented in the previous section, we shall first consider the steerability of the two-mechanical mode states when the squeezing parameter is $r = 1$. We note that two-mode squeezed states with a squeezing parameter ranging from $r = 0$ to $r = 2$ can be produced experimentally [?]. The time evolution of Gaussian steering in both ways, their difference and the Rényi-2 entanglement are depicted in Fig. ??. For the mean thermal photons numbers $n_{\mathrm{th},1}$ and $n_{\mathrm{th},2}$ we considered the following values: $n_{\mathrm{th},1} = 0.5$, $n_{\mathrm{th},2} = 1$ (panel (a)), $n_{\mathrm{th},1} = 1$, $n_{\mathrm{th},2} = 0.5$ (panel (b)), $n_{\mathrm{th},1} = 1$, $n_{\mathrm{th},2} = 1.2$ (panel (c)) and $n_{\mathrm{th},1} = 1$, $n_{\mathrm{th},2} = 1.5$ (panel (d)), which are almost of the same order of magnitude as those used in [?]. Initially, the two mechanical modes are un-entangled. The entanglement is induced by the transfer from the optical modes to the mechanical motion. This interesting aspect of quantum correlations transfer can, in principle, be extended lo to the situations where the mirrors are separated by long distances and might be of interest from applicative point of view. First, we remark that Fig. ?? shows that a minimal amount of entanglement is required for the apparition of Gaussian quantum steering. The results depicted in Fig. ?? show that the evolution of quantum steering can be classified in three different steering regimes: (i) no-way/one-way/two-way (panels (a) and (b)), (ii) no-way/one-way/two-way/one-way and (panel (c))(iii) no-way/one-way/two-way/one-way/no-way (panel (d)). We begin our analysis of these different regimes by noticing that the steering from $B \rightarrow A$ appears before the creation of the Gaussian steering in the opposite way ($B \rightarrow A$). For the regime (i), the behavior of quantum steering presented in the panel (a) and (b) shows that the steerability asymmetry is non zero over a long period of time when $n_{\mathrm{th},1} = 0.5$, $n_{\mathrm{th},2} = 1$ in contract with the situation where $n_{\mathrm{th},1} = 1$, $n_{\mathrm{th},2} = 0.5$ (panel (b)) for which the mechanical modes are steerable symmetrically in both states. The presence of important thermal effects leads to regime of type (ii). In fact, for $n_{\mathrm{th},1} = 1$, $n_{\mathrm{th},2} = 1.2$ (panel (c)), one observes that that the steering from $A \rightarrow B$ tends to disappear before one from $B \rightarrow A$ leading to a revival of the one-way steering phenomenon (see Fig. ?? (panel (c))). This behavior indicates the the robustness of the quantum steering from $B$ to $A$ and the resilience of the steering from $B \rightarrow A$ against thermal effects. The duration of the robustness of the one-way steering in the direction $B \rightarrow A$ is limited by the amount of the thermal fluctuations inside the cavities. Indeed, the panel (d) of Fig. ?? reveals the death of the one-way steering in the system for relatively high temperatures (or alternatively sensibly high thermal photons numbers) and the quantum steering disappears in both sides. The regime (iii) corresponds to a cyclic evolution of the quantum steering. Furthermore, we stress that the Gaussian steerability is strongly sensitive to the thermal effects in comparison with entanglement.

### 1.2.2 *The squeezing effect on the Gaussian steering*

Next, we fix the mean thermal photons number in each cavity as $n_{\mathrm{th},1} = n_{\mathrm{th},2} = 1$ to discuss the Gaussian steering and the entanglement dynamics by varying the squeezing parameter $r$. Fig. ??

displays the corresponding dynamics for $r = 0.1$ ( panel (a)), $r = 0.5$ (panel (b)), $r = 1$ (panel (c)), $r = 1.1$ (panel (d)) and $r = 1.7$ (in the inset). We first observe that the duration of entanglement generation decreases as the degree of squeezing increases. We recall that in the squeezed light is indispensable to generate entanglement. As can be inferred from Fig. **??**, the entanglement tends asymptotically to non zero values and exhibits a freezing behavior of the amount of quantum correlations between the two mechanical modes. The other interesting aspect is that the steerability detection's requires an optimal amount of light squeezing. This can be easily seen by comparing the results depicted in the panel (a) for $r = 0.1$ and in the panel (b) for $r = 0.5$. Indeed, for $r = 0.1$ the light squeezing is insufficient to produce steerability. Therefore, by increasing the squeezing parameter $r$, the quantum steering can be observed (as in the panels (a), (b) and (c)) but it must be stressed that from a certain degree of squeezing the quantum steering disappears completely (see the inset representing the case $r = 1.7$). Clearly, from a critical degree of squeezing, the squeezed light degrades the quantum steerability between the mechanical modes. This can be explained by the fact that the input thermal noise affecting each cavity becomes important and more aggressive, reducing the quantum correlations in the system. The plot reveals that the Gaussian steering occurs for $r = 0.5$ (panel (b)), $r = 1$ (panel (c)), $r = 1.1$ (panel (d)). In the cases where $r = 0.5$ and $r = 1$, the behavior of quantum steering is divided in three phases: no-way, one-way and two-way steering. We note also that, in these two situations, the amounts of quantum steering $\mathcal{G}^{A\to B}$, $\mathcal{G}^{B\to A}$ and the steering asymmetry $\mathcal{G}^{\Delta}_{AB}$ tend to constant values for long time evolution. Like entanglement, the Gaussian steering between the mechanical modes exhibits a freezing behavior. Comparing the one-way steering for $r = 0.5$ and $r = 1$, one observes that in the first case the state is steerable in the direction $A \to B$ and in the second case the steerability is in the opposite direction $B \to A$. The results reported in the panel (d), for $r = 1.1$, shows that the bipartite system is always steerable only in the direction $B \to A$ (i.e., $\mathcal{G}^{A\to B} = 0$). This result indicates also that the steerability from $A \to B$ disappears and this can be followed by the degradation of the steering of the in the opposite direction by increasing the degree of squeezing (see the inset). Finally to close this section, we mention that the results depicted in Figs. **??** and **??** show that the quantum Gaussian steering is always upper bounded by the Gaussian Rényi-2 entanglement $\mathcal{E}_2$. Moreover, the steering asymmetry $\mathcal{G}^{\Delta}_{AB}$ (see the blue dashed-lines in Figs. **??** and **??**) cannot exceed the value $\ln 2$, it is maximal when the state is non-steerable in one way ($\mathcal{G}^{A\to B} > 0$ and $\mathcal{G}^{B\to A} = 0$ or $\mathcal{G}^{A\to B} = 0$ and $\mathcal{G}^{B\to A} > 0$) and it decreases with increasing steerability in either way, which is consistent with the results found in [**?**].

Figure 2: Plot of the Gaussian steering $\mathcal{G}^{A\to B}$ (green solid line), $\mathcal{G}^{B\to A}$ (red solid line), the steering asymmetry $\mathcal{G}^{\Delta}_{AB}$ (blue dashed line) and entanglement $\mathcal{E}_2$ (yellow solid line) between the two mechanical modes $A$ and $B$ as a function of the scaled time $\gamma t$ for $\mathcal{C}_1 = 15$ and $\mathcal{C}_2 = 35$. We used $n_{\mathrm{th},1} = n_{\mathrm{th},2} = 1$ as values of the mean thermal photons numbers. The squeezing parameter $r$ is fixed as: panel (a) $r = 0.1$ ($r = 1.7$ in the inset), panel (b) $r = 0.5$, panel (c) $r = 1$ and finally panel (d) $r = 1.1$. Interestingly, panel (d) shows a situation where the states of the two mechanical modes are entangled (for $\gamma t > 0.05$), nevertheless they are steerable only in one direction (from $B \to A$), which reflects genuinely the asymmetry of quantum correlations. As shown also in panel (a) and the inset (à), entangled states are not necessarily steerable, whereas steerable states are always entangled as depicted in panel (b),(c) and (d).