Quantum discord and quantum metrology for bipartite quantum system

Abstract

We discuss the role of quantum correlations in quantum metrology. We consider a two-qubit state family which describes various quantum system. We derive the explicite expression of quantum Fisher information

1 Introduction

Quantum information processing has emerged as a rich, exciting field due to both its potential applications in cryptography [1] and computational speed-up [2, 3], and its value role in designing quantum systems that can be used to study fundamental physics in previously inaccessible regimes. The successes of this context is achieved by identify the key particulars of quantum phenomenon "coherence" [4]. This phenomenon is a necessary ingredient for any quantum behavior, emerging from wave-like probability distributions of measurement outcomes. Yet, subtler quantum features emerge when multipartite systems are considered. Quantum entanglement in multipartite quantum systems, comprising two or more parts, constitutes a key concept to distinguish between quantum and classical correlations also subsequently to understand quantum classical boundary. Besides its fundamental aspects, entanglement is intensively studied, and it is usually considered the most promising resource to provide speed-up to information protocols [5].

The states of any multipartite quantum system can be classified as being pure and mixed quantum states: the pure states are divided in entangled and classically correlated states, while mixed states can be categorized in entangled and separable states. Recently several studies have been carried on the main goal: if nonentangled states can enjoy some form of quantumness [6]. In generale, almost all separable states of multipartite systems are inherently quantum; they show a kind of quantum correlations called "discord", which is not observable when the state of the system is described by a classical probability distribution. In addition, these states are known as "Discordant states" , they responsible on the characterizing a non classical correlation via local quantum Uncertainty [7, 8]. Therefore, the quantum discord is an indicator of quantumness of correlations in a composite system, usually revealed via the state disturbance induced by local measurements [9, 10]. Recent results suggested that discord might enable quantum advantages in specific computation or communication settings. In addition, various purposes were introduced in this respect[11, 12].

Quantum metrology is another topic which can be studied in the presence of quantum correlation. In particular, the quantum correlation measure based on quantum Fisher information enables us to gain a deeper insight on how quantum correlations are instrumental in setting metrological precision [13, 14]. Several operational interpretations in this context have been proposed in the recent years [15, 16, 17, 18]. More specifically, a general quantitative equivalence between discord-type quantum correlations and the guaranteed precision in estimating technologically relevant parameters (such as phase) is established theoretically, and observed experimentally [19, 20].

In general, how to precisely measure the values of physical quantities; such as the phases of light in interferometers, magnetic strength, gravity and so on, is always an important topic in physics. Metrology have found various applications in this order. It is concerned with the largest possible precision achievable in various parameter estimation tasks and frame measurement schemes to achieve that precision. Quantum Fisher information (QFI) plays central role in quantum metrology [21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32] and its inverse depicts the lower bound on the variance of the estimator $\widehat{\theta}$ for the parameter θ due to the Cramr-Rao theorem [33]

$$\mathrm{Var}_\rho(\widehat{\theta}) \geq 1/[\nu \mathrm{F}(\rho, H)], \tag{1}$$

where $\nu \gg 1$ is the number of repetitions of the experiment, $\mathrm{Var}(\cdot)$ describes the variance and F is the quantum Fisher information (QFI) [34, 35], this recent quantity evaluates the precision of the state ρ to the phase shift assuming that the best measurement strategy is performed. In addition, for single parameter estimation, the best estimator statistical saturates the quantum Cramr-Rao bound:

$$\mathrm{Var}_{\rho}(\widehat{\theta}_{best}) = 1/[\nu \mathrm{F}(\rho, H)], \tag{2}$$

Our whole analysis is centralized on interpretation of the role discord in the context of quantum metrology, This approach can be seen by quantify quantum correlations discord-like in terms of QFI. In fact, discord has been studied on its own because of some interesting properties: it can be created by local operations and classical communications (LOCC) and it is intrinsically robust under noisy dynamics [11].

In this paper, we discuss role of discord-type quantum correlations in metrology. We also give the description of local Quantum Uncertainty for bipartite systems which can translated into an improved sensitivity in parameter estimation. We obtain the special kind of quantum discord called " Quantum Interferometric Power". To be specific, we show that the Local Quantum Uncertainty and the Quantum Interferometric Power are parent discord-like measures which quantify the minimum amount of precision in interferometric phase estimation. We also discuss the role of bipartite discordant state on the measurement of the phase parameter. In fact, we analyze the effects of the quantum correlations on the efficiency of phase estimation by using different kinds of locales observables.

This paper is organized as follows. In Section 2 we present the basic of standard metrology task,i.e. an interferometric phase estimation protocol, where building up discord in the impute state guarantees non vanishing precision for any Hamiltonian generating the phase shift. In Section 3 we remind the basis of the Fisher information by introducing the definition of the QFI in terms of the estimator parameter. Section 4 is devoted to present a simple numerical example to clarify the role of discord in phase estimation. We conclude in Section 5 with some remarks and comments.

2 Quantum discord in quantum metrology: Definition

Quantum mechanics predicts that measurements of incompatible observables carry a minimum uncertainty which is independent of technical deficiencies of the measurement apparatus or incomplete knowledge of the state of the system. Nothing yet seems to prevent a single physical quantity, such as phase, from being measured with arbitrary precision. In a recent work [7], it has been shown that an intrinsic quantum uncertainty on a single observable $K^A = K_A \otimes \mathbb{I}_{\mathbb{B}}$ is ineludible in a number of physical situations when revealed on local observables of a bipartite system, such uncertainty defines an entire class of bona fide measures of nonclassical correlations of discord-type. All situations in this concepts requires a specific quantum effect which is provably measured via the function " Wigner-Yanase skew information ". This function quantify the quantum uncertainty on a single observable, it's defined by

$$\begin{aligned} J(\rho, K) &= -\tfrac{1}{2} Tr[\sqrt{\rho}, K]^2 \\ &= Tr[\rho K^2 - \sqrt{\rho} K \sqrt{\rho} K] \end{aligned} \tag{3}$$

The Wigner-Yanase skew information is a nonnegative, convex function of the commutator $[\rho, K^A]$. Generally, the Wigner-Yanase theorem imposes a limitation on the measurement of observables in the presence of a conserved quantity, and the notion of Wigner-Yanase skew information quantifies the amount of information on the values of observables not commuting with the conserved quantity [36, 37, 38].

In the following, we focus on a general bipartite state $\rho = \rho_{\mathrm{AB}}$ and on local observable K^A on subsystem A. In fact, The minimum skew information on a single local observable defines the local quantum uncertainty (LQU) as:

$$\mathcal{U}^A(\rho_{AB}) = \min_{k^A} J(\rho_{AB}, K^A). \tag{4}$$

Accordingly, the quantum discord which was originally introduced in the information-theoretic context is also defined by the minimum decrease of the mutual information $J(\rho_{AB})$ (after a local measurement identified by the Kraus operators $\{M_i^A\}$); as

$$\mathcal{D}^A(\rho_{AB}) = J(\rho_{AB}) - \max_{M_i^A} J(\rho M_i^A(A)B) \tag{5}$$

where $J(\rho_{AB})$ denotes the total correlation present in bipartite system AB, the second terms of the above equation defines the classical correlation which depends on Measurement M_i^A.

Therefore, $\mathcal{U}_A(\rho_{AB})$ satisfies all the properties reliable discord-like measures: it vanish iff ρ is classically correlated, $\rho_{AB} = \sum_i \mathrm{p_i}|i\rangle\langle i|_A \otimes \tau_{iB}$; it is invariant under local unitary; it nonincreasing under local operations on B; it reduces to an entanglement monotone for pure states.

The role of quantum discord in quantum metrology was first investigated by Modi et al [8], and important contributions in this order were made recently in [39, 40]. In the scenario considered in [40], an experimenter (Alice), must prepare the input state ρ_A without any prior information about the Hamiltonian H_A. It's assumed that the phase direction is unveiled at the output state. Thus, Alice is still allowed to carry out the most informative measurement and build the best possible estimator. As the sensitivity of the probe is given by the amount of coherence with respect to the eigenbasis of the Hamiltonian, there is no input which guarantees any degree of precision in the estimation.

However, the situation is different if Alice collaborates with a second player Bob and implements a two-arm interferometer to perform the estimation; Alice and Bob share a bipartite state ρ_{AB} undergoing a local unitary evolution $U_A = e^{-i\theta H_A}$ on the subsystem of Alice with a non-degenerate Hamiltonian H_A. The final state $U_A \rho_{AB} U_A^\dagger$ is then used to estimate the unknown parameter θ, this parameter can always be estimated with nonzero precision for any states ρ_{AB} of bipartite quantum system type-Discordant. This result is useful for our propose. In fact, we give an example of a bipartite quantum state to ensure the success of the estimation regardless of the phase direction. Interestingly, it has been also shown in [40], that this result is investigated by introducing a new quantifier of quantum correlations: interferometric power. The interferometric power is able to capture the worst-case precision of the procedure, and conclude that the presence of discord in a quantum state guarantees its usefulness for quantum metrology. Experiment supporting these theoretical results has also been reported in [40].

Our goal is to quantify quantum correlation discord-like in terms of QFI, and for that we define the minimum of QFI by

$$\mathcal{P}(\rho_{AB}; H_A) = \frac{1}{4} \min_{H_A} F(\rho_{AB}; H_A), \tag{6}$$

where this minimum is intended over all Hamiltonians $\{H_A\}$, and we inserted a normalization factor $\frac{1}{4}$ for convenience. We shall refer to (6) as the interferometric power (QIP) of the input state ρ_{AB}, since it naturally quantifies the guaranteed precision that such a state allows in an interferometric configuration [?]. Following the properties of QFI [41], the (QIP) acquires several interesting properties: it is non negative; invariant under local unitary transformations and non increasing under local operations on B; and in general $\mathcal{P}(H_A) \neq \mathcal{P}(H_B)$ except for symmetric quantum states (systems of pure states). All these important properties indulge us to avow that the $\mathcal{P}(\rho_{AB}; H_A)$ is the measure of discord-like quantum correlation.

3 Measure of precision via the quantum Fisher information process

The quantum correlation measure based on quantum Fisher information enables us to gain a deeper insight on how quantum correlations are instrumental in setting metrological precision [42, 8, 40] . The quantum Fisher information is not just limited in the field of quantum metrology. It has been widely applied in other aspects of quantum physics [43, 44] , like quantum information and open quantum systems. Thus, it is necessary and meaningful to study the quantum Fisher information as well as its properties and dynamical behaviors under various forms. In a recent study [45], it has been found that quantum Fisher information can be expressed for density matrices with arbitrary rank, and it can be reduce to the form of the convex roof of variance [46]. This result is useful for our purpose.
In this report, we give a general expression of quantum Fisher information for a non-full rank density matrix. For this end, we denote the spectral decomposition of the density matrix ρ_θ which still non-full rank as

$$\rho_\theta = \sum_{i=1}^{M} \lambda_i |\psi_i\rangle\langle\psi_i|, \tag{7}$$

Here, M is the rank of the density matrix ρ_θ, which equals to the dimension of the support of ρ_θ, λ_i and $|\psi_i\rangle$ are the ith eigenvalue and eigenstate of the density matrix, respectively.
In this representation, the quantum Fisher information F is defined as below [46]

$$F = tr(\rho_\theta L^2) \tag{8}$$

where L is the so-called symmetric logarithmic derivative operator and determined by

$$\partial_\theta \rho_\theta = \frac{1}{2}(L\rho_\theta + \rho_\theta L) \tag{9}$$

Using this representation, the expression of QFI for a non-full rank density matrix [47]

$$F = \sum_{i=1}^{M} \frac{(\partial_\theta \lambda_i)^2}{\lambda_i} + \sum_{i=1}^{M} 4\lambda_i \langle \partial_\theta \psi_i | \partial_\theta \psi_i \rangle - \sum_{i,k=1}^{M} \frac{8\lambda_i \lambda_k}{\lambda_i + \lambda_k} |\langle \psi_i | \partial_\theta \psi_k \rangle|^2 \tag{10}$$

which can intuitively interpreated as the " velocity" at which the density matrix moves for a given parameter value. This physical interpretation comes from the fact that the QFI is dependent on the parameterized density matrix ρ_θ and its first derivative $\partial_\theta \rho_\theta$.
The QIP for an arbitrary quantum of a bipartite system is obtained by applying the closed formula derived in the appendix of [ref6],

$$\mathcal{P}(\rho_{AB}) = \varsigma_{min}[M], \tag{11}$$

where $\varsigma_{min}[M]$ is the smallest eigenvalue of the 3×3 matrix M of elements

$$M_{m,n} = \frac{1}{2} \sum_{i,j:\lambda_i+\lambda_j \neq 0} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j} \langle \psi_i | \sigma_{mA} \otimes \mathbb{I}_B | \psi_j \rangle \langle \psi_j | \sigma_{nA} \otimes \mathbb{I}_B | \psi_i \rangle \tag{12}$$

with $\{\lambda_i, \psi_i\}$ being respectively the eigenvalues and eigenvectors of ρ_{AB}. This renders $\mathcal{P}(\rho_{AB})$ an operational and computable indicator of general non classica correlations for pratical purposes. Subsequently, we consider a family of two qubit density matrices (2-rank density matrix) whose entries are specified in terms of two real parametrs. They are defined as

$$\rho^Q = \begin{pmatrix} c_1 & 0 & 0 & \sqrt{c_1 c_2} \\ 0 & \frac{1}{2}(1-c_1-c_2) & \frac{1}{2}(1-c_1-c_2) & 0 \\ 0 & \frac{1}{2}(1-c_1-c_2) & \frac{1}{2}(1-c_1-c_2) & 0 \\ \sqrt{c_1 c_2} & 0 & 0 & c_2 \end{pmatrix} \tag{13}$$

in the computational basis $\mathcal{B} = \{|00\rangle, 01\rangle, 10\rangle, 11\rangle\}$. The parameters c_1 and c_2 satisfy the conditions $0 \leq c_1 \leq 1$, $0 \leq c_2 \leq 1$ and $0 \leq c_1 + c_2 \leq 1$. We have taken all entires positives. To see how to calculate the QFI, we take the state (13) as an example, This type of states include maximally entangled Bell states and Werner states [48]. Here, the non-vanishing eigenvalues in terms of the parameters c_1 and c_2, are respectively given by

$$\lambda_+ = c_1 + c_2, \quad \lambda_- = 1 - (c_1 + c_2), \quad \lambda_{++} = 0 \quad, \lambda_{--} = 0. \tag{14}$$

In addition, the corresponding eigenstates to λ_+ and λ_- writes

$$|\psi_+\rangle = \sqrt{\tfrac{c_1}{c_1+c_2}}(1,0,0,\sqrt{\tfrac{c_2}{c_1}}), \quad |\psi_-\rangle = \tfrac{\sqrt{2}}{2}(0,1,1,0), \quad |\psi_{++}\rangle = \tfrac{\sqrt{2}}{2}(0,1,-1,0), \quad |\psi_{--}\rangle = -\sqrt{\tfrac{c_2}{c_1+c_2}}(1,0,0,\sqrt{\tfrac{c_1}{c_2}}). \tag{15}$$

In this distribution, the quantum Fisher information for the states type (13) can be expressed by

$$F(\rho^Q) = 4\lambda_+\{\langle \partial_\theta \psi_+ | \partial_\theta \psi_+ \rangle - |\langle \psi_+ | \partial_\theta \psi_+ \rangle|^2\} + 4\lambda_-\{\langle \partial_\theta \psi_- | \partial_\theta \psi_- \rangle - |\langle \psi_- | \partial_\theta \psi_- \rangle|^2\} - 16 \frac{\lambda_+ \lambda_-}{\lambda_+ + \lambda_-} |\langle \psi_+ | \partial_\theta \psi_- \rangle|^2 \tag{16}$$

In the other hand, the quantum interferometric power (12) can be formally given by

$$\begin{aligned} M_{m,n} = & \tfrac{1}{2}\tfrac{(\lambda_+-\lambda_-)^2}{\lambda_++\lambda_-}\{\langle\psi_+|\sigma_m\otimes 1|\psi_-\rangle\langle\psi_-|\sigma_n\otimes 1|\psi_+\rangle + \langle\psi_-|\sigma_m\otimes 1|\psi_+\rangle\langle\psi_+|\sigma_n\otimes 1|\psi_-\rangle\} \\ + & \tfrac{1}{2}\tfrac{(\lambda_+-\lambda_{++})^2}{\lambda_++\lambda_{++}}\{\langle\psi_+|\sigma_m\otimes 1|\psi_{++}\rangle\langle\psi_{++}|\sigma_n\otimes 1|\psi_+\rangle + \langle\psi_{++}|\sigma_m\otimes 1|\psi_+\rangle\langle\psi_+|\sigma_n\otimes 1|\psi_{++}\rangle\} \\ + & \tfrac{1}{2}\tfrac{(\lambda_+-\lambda_{--})^2}{\lambda_++\lambda_{--}}\{\langle\psi_+|\sigma_m\otimes 1|\psi_{--}\rangle\langle\psi_{--}|\sigma_n\otimes 1|\psi_+\rangle + \langle\psi_{--}|\sigma_m\otimes 1|\psi_+\rangle\langle\psi_+|\sigma_n\otimes 1|\psi_{--}\rangle\} \\ + & \tfrac{1}{2}\tfrac{(\lambda_--\lambda_{++})^2}{\lambda_-+\lambda_{++}}\{\langle\psi_-|\sigma_m\otimes 1|\psi_{++}\rangle\langle\psi_{++}|\sigma_n\otimes 1|\psi_-\rangle + \langle\psi_{++}|\sigma_m\otimes 1|\psi_-\rangle\langle\psi_-|\sigma_n\otimes 1|\psi_{++}\rangle\} \\ + & \tfrac{1}{2}\tfrac{(\lambda_--\lambda_{--})^2}{\lambda_-+\lambda_{--}}\{\langle\psi_-|\sigma_m\otimes 1|\psi_{--}\rangle\langle\psi_{--}|\sigma_n\otimes 1|\psi_-\rangle + \langle\psi_{--}|\sigma_m\otimes 1|\psi_-\rangle\langle\psi_-|\sigma_n\otimes 1|\psi_{--}\rangle\} \end{aligned} \tag{17}$$

Thus, one verifies that the matrix M is reduces to

$$\begin{pmatrix} M_{11} & 0 & 0 \\ 0 & M_{22} & 0 \\ 0 & 0 & M_{33} \end{pmatrix}, \tag{18}$$

it is simple to verify that

$$\begin{array}{rcl} M_{11} & = & 1-2[1-(c_1+c_2)][\sqrt{c_2}+\sqrt{c_1}]^2, \\ M_{22} & = & 1-2[1-(c_1+c_2)][\sqrt{c_2}-\sqrt{c_1}]^2, \\ M_{33} & = & 4\frac{c_1c_2}{(c_1+c_2)}+\{1-(c_1+c_2)\}. \end{array} \tag{19}$$

To evaluate the quantum interferometric power defined in the equation (11), we first compare the coefficients of the matrix M (19). Remark that the M_{22} is always greater than $M33$. In this case, we discuse the smallest eigenvalue of the matrix M over the elements M_{11} and M_{33}, it is simple to verify that the difference $M_{11}-M_{33}$ is positive when the parameters c_1 and c_2 satisfy the following condition

$$2(c_1+c_2)^2-(\sqrt{c_1}+\sqrt{c_2})^2 \geq 0. \tag{20}$$

Otherwise, we have $M_{11} \leq M_{33}$.
Indeed, for the set of states $\alpha \leq \frac{1}{2}$, the differnce $M_{11}-M_{33}$ is non positive and the quantum interferometric power (17) writes

$$\mathcal{P} = M_{11}, \tag{21}$$

which rewrites explicitly as

$$\mathcal{P} = 1-2[1-(c_1+c_2)][\sqrt{c_2}+\sqrt{c_1}]^2. \tag{22}$$

On the other hand, for $\alpha \geq \frac{1}{2}$, the condition(20) is satisfied for

$$0 \leq c_1 \leq \alpha_- \quad , \quad \alpha_+ \leq c_1 \leq \alpha \tag{23}$$

where

$$\alpha_\pm = \frac{1}{2}\alpha \pm \sqrt{\alpha^3-\alpha^4} \tag{24}$$

In this case, the quantum interferometric power is given by

$$\mathcal{P} = M_{33}, \tag{25}$$

Conversely, for $\alpha_- \leq c_1 \leq \alpha_+$ the diffrence $M_{11}-M_{33}$ is negative and the quantum interferometric power reads

$$\mathcal{P} = M_{11}, \tag{26}$$

The behavior of quantum interferometric power versus c_1 is given in the figure 1 for different values of $\alpha = c_1+c_2 (\alpha = 0.1, 0.2, ..., 0.9)$ and for two different cases: $\alpha \leq \frac{1}{2}$ and $\alpha. \geq \frac{1}{2}$.

Figure 1 with $0 \leq \alpha \leq 0.5$, gives the variation of quantum interferometric power for different values of α. It can be clearly seen in this figure that, the QIP in the states (13) reaches its maximum value 0.82 for $\alpha = 0.1$. This maximum value decreases as the parameter α increases. We can also seen that, the QIP is maximal for two important situations $((c_1 = 0, c_2 = \alpha)$,$(c_1 = \alpha, c_2 = 0))$ and minimal for the states with $(c_1 = c_2 = \frac{\alpha}{2})$. These situations are addressed in the following section.
The curve of the quantum interferometric power is completely different for with $\alpha \geq \frac{1}{2}$. In fact, the QIP changes suddenly when $c_1 = \alpha_-$ and $c_1 = \alpha_+$ (α_- and α_+ are given by the expressions (24)). This sudden change of QIP occurs when the states ρ^Q(13) have a maximum amount of quantum correlation. On the other hand, the behavior of Quantum Interferomletric Power presents three distinct phases: $0 \leq c_1 \leq \alpha_-$, $\alpha_- \leq c_1 \leq \alpha_+$ and $\alpha_+ \leq c_1 \leq \alpha$. The minimal value of the Quantum Interferometric Power $\mathcal{P}$ is obtained in the intermediate phase $(\alpha_- \leq c_1 \leq \alpha_+)$ for the states with $(c_1 = c_2 = \frac{\alpha}{2})$

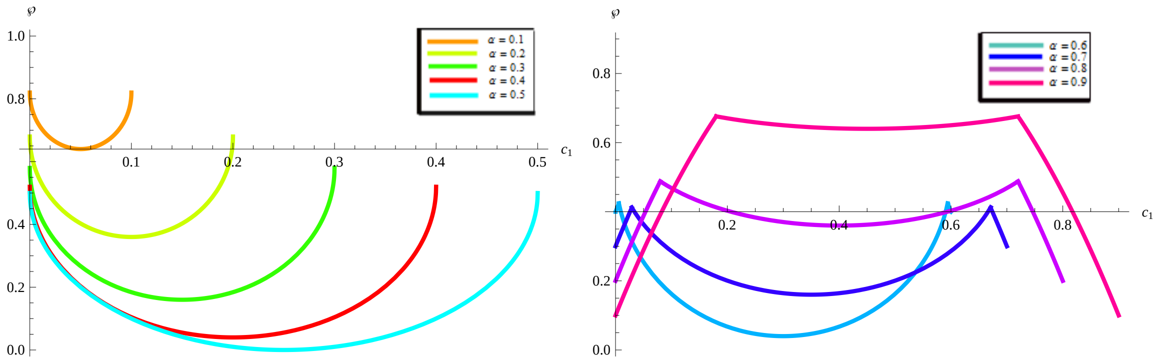

Figure 1. The quantum interferometric power $\mathcal{P}$ as function of the parameter c_1 for $\alpha \leq \frac{1}{2}(\alpha \geq \frac{1}{2})$ respectively.

4 Analytical expression of QFI for unitary parameterization process with different kinds of Hamiltonians

The estimation protocol splits into three steps: preparation of an input state, which has to be sensitive to the parameter variations, encoding of the information about the unknown parameter, which it is assumed to be a unitary evolution, measurement of an appropriate observable in the output state. In the present section, we are in the position to compute the phase uncertainty under different kinds of Hamiltonians, this quantity is determined by computing the quantum Fisher information.
To determine the explicit form of quantum fisher information in the representation $\{\lambda_\pm, |\psi_\pm\rangle\}$, we consider an estimation of the parameter θ introduced by the following unitary operation

$$U_\theta(H^A) = e^{iH^A\theta}, \tag{27}$$

where $H^A = H_A \otimes \mathbb{I}$. Here $\mathbb{I}$ is the 2×2 identity matrix and H_A is the local Hamiltonian which is generates a phase transformation defined as

$$H_A = \overrightarrow{r}.\overrightarrow{\sigma} = r_1\sigma_1 + r_2\sigma_2 + r_3\sigma_3 \tag{28}$$

where $|r| = 1$, i.e, $\overrightarrow{r} = \{\sin\alpha\ \cos\beta, \sin\alpha\ \sin\beta, \cos\alpha\}$ and $\overrightarrow{\sigma}$ are the Pauli matrices which reads as

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|, \quad \sigma_y = i(|1\rangle\langle 0| - |0\rangle\langle 1|) \quad \sigma_z = |0\rangle\langle 0| - |1\rangle\langle 1| \tag{29}$$

respectively. In this case, the QFI (16) reduces to

$$F(\rho^Q, H^A) = \lambda_+\{F(\rho^Q, H^A)_{|\psi\rangle_+}\} + \lambda_-\{F(\rho^Q, H^A)_{|\psi\rangle_-}\} - 16\frac{\lambda_+\lambda_-}{\lambda_+ + \lambda_-}|\langle\psi_+|H^A|\psi_-\rangle|^2 \tag{30}$$

In the mean time, $F(\rho^Q, H^A)_{|\psi\rangle_\pm}$ reduces to the form that is proportional to the variance of operator H^A on the eigenstates, i.e,

$$F(\rho^Q, H^A)_{|\psi\rangle_i} = 4(\Delta^2 H^A)_{|\psi\rangle_i}, \tag{31}$$

the subscript i is defined as $i = +(-)$ respectively and $(\Delta^2 H^A)_{|\psi\rangle_i} = \langle\psi_i|(H^A)^2|\psi_i\rangle - |\langle\psi_i|H^A|\psi_i\rangle|^2$ is the variance. It is obvious that QFI is only constituted by the nonzero eigenvalues and the corresponding eigenstates of the density matrix(13). Finaly, it can be expressed as

$$F(\rho^Q, H^A) = 4 - 4\cos^2\alpha\frac{(c_1 - c_2)^2}{(c_1 + c_2)} - 8\{1 - (c_1 + c_2)\}\{(c_1 + c_2)\sin^2\alpha + 2\sqrt{c_1 c_2}\sin^2\alpha\ \cos 2\beta\} \tag{32}$$

4.1 Analytical expression for direction σ_x

To compute the quantum Fisher information $F(\rho^Q, \sigma_x)$, which it is assumed a unitary evolution along the x-direction, i.e. $U_\theta(\sigma_x^A) = e^{i\sigma_x^A \theta}$. We concentrate on the expression of quantum Fisher information defined in (32) and using $\{\alpha = \frac{\pi}{2}, \beta = 0\}$. It is easy to see that the QFI takes the following form

$$F(\rho^Q, \sigma_x) = 4 - 8[1 - (c_1 + c_2)][\sqrt{c_2} + \sqrt{c_1}]^2 \tag{33}$$

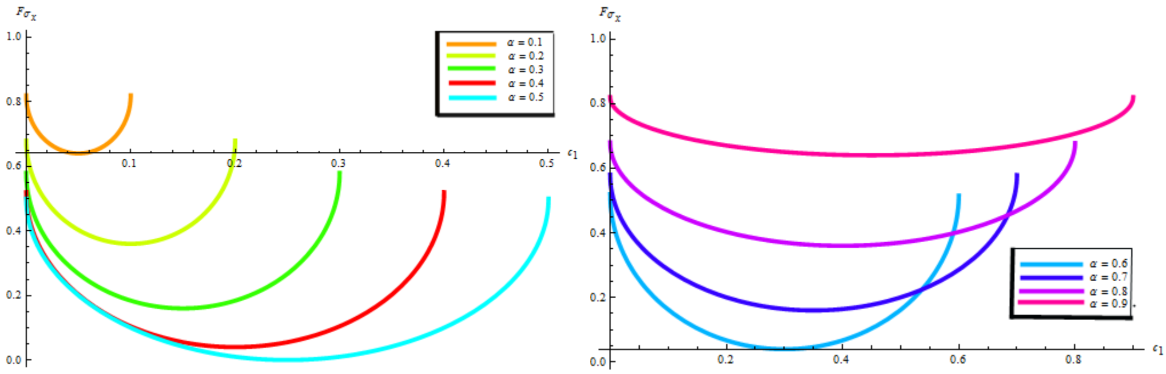

Figure 2. The quantum Fisher information $F(\rho^Q, \sigma_x)$ along the x-direction versus c_1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

The behavior of the quantum Fisher information along the x-direction $F(\rho^Q, \sigma_x)$, as function of the parameters c_1, is represented in the figure 2 for two cases($\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$). Figure 2 with $\alpha \leq \frac{1}{2}$, gives the variation of quantum correlation measures based on quantum Fisher information $F(\rho^Q, \sigma_x)$ for different values of $\alpha = c_1 + c_2$ ($\alpha = 0.1, ..., 0.9$). As it can be inferred from this figure, the quantum Fisher information $F(\rho^Q, \sigma_x)$ in the states (13) reaches its minimal value for $c_1 = \frac{\alpha}{2}$. These minimally discordant states are given by

$$\rho^Q(c_1 = \frac{\alpha}{2}, c_2 = \frac{\alpha}{2}) = \alpha\rho' + (1 - \alpha)\rho \tag{34}$$

where the states ρ and ρ' are respectively given by

$$\rho = |\psi\rangle\langle\psi|, \quad \rho' = |\psi'\rangle\langle\psi'|, \tag{35}$$

The corresponding eigenstates denoted by $|\psi\rangle$ and $|\psi'\rangle$ can be written as

$$|\psi\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad , \quad |\psi'\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \tag{36}$$

In the other hand, the maximal value of $F(\rho^Q, \sigma_x)$ is obtained in the states with $(c_1 = 0, c_2 = \alpha)$ or $(c_1 = \alpha, c_2 = 0)$. This value maximal correspond respectively to the states of the forms

$$\rho^Q(c_1 = 0, c_2 = \alpha) = \alpha|11\rangle\langle 11| + (1 - \alpha)|\psi\rangle\langle\psi|, \tag{37}$$

and

$$\rho^Q(c_1 = \alpha, c_2 = 0) = \alpha|00\rangle\langle 00| + (1 - \alpha)|\psi\rangle\langle\psi|, \tag{38}$$

The situation is completely different for $\alpha \geq \frac{1}{2}$. In fact, the quantum Fisher information increases as the parameter c_1 increases. For high values of $c_1(c_1 = 0.45$ for instance), more precision is provided in the states with $\alpha = 0.9$. Thus, the quantum Fisher information $F(\rho^Q, \sigma_x)$ is maximal for $c_1 = c_2 = \frac{\alpha}{2}$. Remark: In this figure, as well as in the other figures presented in this section, the expression obtained of QFIs are plotted normalized by a factor $\frac{1}{4}$.

4.2 Analytical expression for direction σ_y

Let us derive the explicit expression of the quantum Fisher information $F(\rho^Q, \sigma_y)$, we consider an unitary evolution along the y-direction, i.e, $U_\theta(\sigma_y^A) = e^{i\sigma_y^A \theta}$ $(\sigma_y^A = \sigma_y \otimes \mathbb{I})$. In this cases, we have $\{\alpha = \frac{\pi}{2}, \beta = \frac{\pi}{2}\}$ and the explicit expression of the quantum Fisher information $F(\rho^Q, \sigma_y)$ in the state (13) writes

$$F(\rho^Q, \sigma_y) = 4 - 8[1 - (c_1 + c_2)][\sqrt{c_2} - \sqrt{c_1}]^2 \tag{39}$$

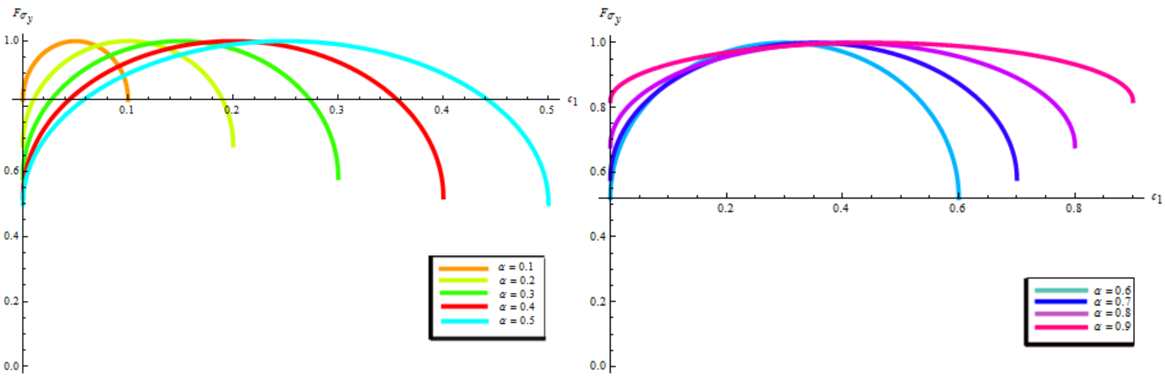

Figure 3. The quantum Fisher information $F(\rho^Q, \sigma_y)$ along the y-direction versus c_1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

The figure 3 gives the quantum Fisher information $F(\rho^Q, \sigma_y)$ along the y-direction as function of the parameter c_1 and c_2 for different values of $\alpha = c_1 + c_2$. For $\alpha \leq \frac{1}{2}$ or $\alpha \geq \frac{1}{2}$, the quantum Fisher information $F(\rho^Q, \sigma_y)$ is maximal for the states satisfying $c_1 = c_2 = \frac{\alpha}{2}$ (34) and minimal for the states which correspond $(c_1 = 0,\ c_2 = \alpha)$(37) or $(c_1 = \alpha,\ c_2 = 0)$(38). Indeed, the states ρ^Q(34) can gives a good precision.

4.3 Analytical expression for direction $F(\rho^Q, \sigma_z)$

In such case, the unitary operator is defined as $U_\theta(\sigma_z^A) = e^{i\sigma_z^A \theta}$ $(\sigma_z^A = \sigma_z \otimes \mathbb{I})$. Especially it is easy to show that for $\alpha = 0$ and for any value of β, the quantum Fisher information (32) becomes

$$F(\rho^Q, \sigma_z) = 4 - 4\frac{(c_1 - c_2)^2}{c_1 + c_2} \tag{40}$$

We can see from figure 4 that the quantum Fisher information $F(\rho^Q, \sigma_x)$ along the z-direction, behaves like the quantum Fisher information $F(\rho^Q, \sigma_y)$ along the y-direction depicted in figure 2 and exhibits the fixed maximum values which move to the left-hand when the parameter α increases. Indeed, the quantum Fisher information is maximum for $c_1 = c_2 = \frac{\alpha}{2}$. This indicates that by suitably choosing the input state, one could get the maximum QFI, which gives the minimum uncertainty of the unknown parameter from Eq.(13). Another remark in this case, that for $c_1 = c_2$ the density matrix ρ^Q(13) is a Bell-diagonal state.

- **Analysis and results** :

Having investigated and discussed the precision in the bipartite discordant states ρ^Q(13) for unitary parametrization process with different kinds of Hamiltonians; σ_x, σ_y and σ_z. It can be inferred that,

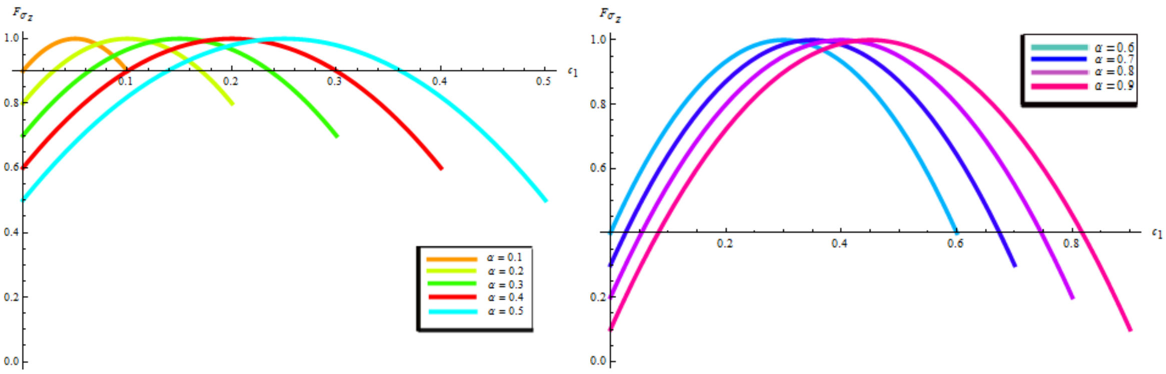

Figure 4. The quantum Fisher information $F(\rho^Q, \sigma_z)$ along the z-direction versus c_1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

these type of states are permitted to guarantee a nonzero phases precision for all the chosen generating Hamiltonians (see figures 2, 3, 4 and 5). Consequently, our analysis shows that the quantum correlation discord-like plays an important roles in quantum metrology. More specially, these quantum correlations measured by QFI are enable to enhance the phase parameter precision.

In other hand, using the general description of quantum interferometric power, which given explicitly as

$$\mathcal{P}(\rho_{AB}; H_A) = \frac{1}{4} \min_{H_A} F(\rho_{AB}). \tag{41}$$

We remark that $F(\rho^Q, \sigma_y) \geq F(\rho^Q, \sigma_x)$. In this case, we rewrite (41) as

$$\mathcal{P} = \frac{1}{4} \min\{F(\rho^Q, \sigma_x), F(\rho^Q, \sigma_z)\}, \tag{42}$$

It is simple to verify that the difference $F(\rho^Q, \sigma_x) - F(\rho^Q, \sigma_z)$ is positive when the parameters c_1 and c_2 satisfy the following condition

$$2(c_1 + c_2)^2 - (\sqrt{c_1} + \sqrt{c_2})^2 \geq 0. \tag{43}$$

Otherwise, we have $F(\rho^Q, \sigma_x) \leq F(\rho^Q, \sigma_z)$.

Thus, for a fixed value of $\alpha \leq \frac{1}{2}$, the quantity $F(\rho^Q, \sigma_x) - F(\rho^Q, \sigma_z)$ is non positive and the quantum interferometric power (42) writes

$$\mathcal{P} = \frac{1}{4} F(\rho^Q, \sigma_x), \tag{44}$$

which rewrites explicitly as

$$\mathcal{P} = 1 - 2[1 - (c_1 + c_2)][\sqrt{c_2} + \sqrt{c_1}]^2. \tag{45}$$

For $\alpha \geq \frac{1}{2}$, the condition(43) is satisfied for

$$0 \leq c_1 \leq \alpha_- \quad , \quad \alpha_+ \leq c_1 \leq \alpha \tag{46}$$

where

$$\alpha_\pm = \frac{1}{2}\alpha \pm \sqrt{\alpha^3 - \alpha^4} \tag{47}$$

In this case, the quantum Interferometric power is given by

$$\mathcal{P} = \frac{1}{4} F(\rho^Q, \sigma_z), \tag{48}$$

Conversely, for $c_1\epsilon[\alpha_-,\alpha_+]$ the diffrence $F(\rho^Q,\sigma_x)-F(\rho^Q,\sigma_z)$ is negative and the quantum Interferometric power reads

$$\mathcal{P}=\frac{1}{4}F(\rho^Q,\sigma_x), \tag{49}$$

The behavior of quantum interferometric power versus c_1 is given in the figure 5 for different values of $\alpha=c_1+c_2$ and for two different cases: $\alpha\leq\frac{1}{2}$ and $\alpha\geq\frac{1}{2}$

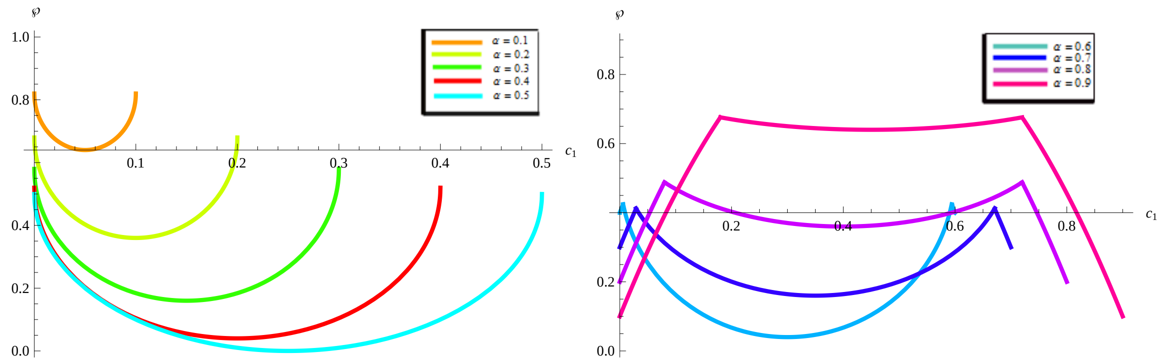

Figure 5. The variation of quantum interferometric power $\mathcal{P}$ based on QFI as function of the parameter c_1 for $\alpha\leq\frac{1}{2}(\alpha\geq\frac{1}{2})$respectively.

In the figure 5 ($\alpha\leq\frac{1}{2}$), the quantum correlations discord-like (Quantum Interferometric power) measured by QFI is maximal for two important situations. These situations are gives by the states 37(38), respectively. Thus, the maximum metrological precision is achievable at the states with $\alpha=0.1$ ($\mathcal{P}_{\max}=0.83$), and it is determined by the inverse of $\mathcal{P}$ ($[4\nu\mathcal{P}]^{-1}$). This implies the saturation of inequality of Cramer-Rao bound (1).
The behavior of quantum Interferometric Power $\mathcal{P}$ changes for states with $\alpha\geq\frac{1}{2}$. In fact, the QIP changes suddenly when $c_1=\alpha_-$ and $c_1=\alpha_+$ (α_- and α_+ are given by the expressions (47)). This sudden change of QIP occurs when the states $\rho^Q(13)$ have a maximum amount of quantum correlation. Also, the behavior of quantum interferomletric Power presents three distinct phases: $0\leq c_1\leq\alpha_-$, $\alpha_-\leq c_1\leq\alpha_+$ and $\alpha_+\leq c_1\leq\alpha$. The minimal value of the Quantum Interferometric Power $\mathcal{P}$ is obtained in the intermediate phase ($\alpha_-\leq c_1\leq\alpha_+$) for the states given by (34).
Consequently, in the states with ($c_1=0$, $c_2=\alpha$)(37) or ($c_1=\alpha$, $c_2=0$)(38), and for the cases that α has to be very small, the QIP can give a good precision.
Using the prescription provided in works [49] and [50] to measure the amount of entanglement in bipartite quantum states. For the states describing in (13) and in the case ($\alpha\geq\frac{1}{2}$), the concurrence is given by

$$C=2\max\{0,A_1,A_2\}, \tag{50}$$

where

$$\begin{aligned} A_1 &= \sqrt{c_1c_2}-\tfrac{1}{2}(1-c_1-c_2)\\ A_2 &= \tfrac{1}{2}(1-c_1-c_2)-\sqrt{c_1c_2} \end{aligned} \tag{51}$$

It follows that the concurrence is given by

$$C=(\sqrt{c_1}+\sqrt{c_2})^2-1 \tag{52}$$

for

$$0\leq c_1\leq c_-,\quad c_+\leq c_1\leq\alpha \tag{53}$$

with $c_\pm = \frac{\alpha\pm\sqrt{2\alpha-1}}{2}$, and the system is entangled. However, for $c_- \leq c_1 \leq c_+$, the concurrence is zero and the entanglement disappears, i.e. the system is separable. It is important to stress that the quantum interferometric power $\mathcal{P}$ is nonzero except in the particular case $c_- = c_+ = 0.25$. This implies that even when entanglement disappears in a finite interval of c_1, the metrological precision does not vanish.

5 Delimiting quantum metrology

We pose in this part, on the untimate connection of QFI with quantum metrology and quantum correlation present in a general bipartite system [51]. In quantum parameter estimation, a quantum state ρ, undergoes a unitary transformation (in general a shift in phase) so that the evolve state becomes $\rho_\theta = e^{-i\theta H}\rho e^{i\theta H}$, where H is the Hamiltonian assumed to have non-degenerate spectrum. The parameter θ is encoded in the state ρ_θ and the task is to estimate the unobservable parameter θ. Interestingly, the lower bound on the error (or variance, $\Delta\theta$), in estimating θ, is independent of the choice of the measurements (POVMs) performed after the unitary evolution and solely determined by the dependence of the output state on the parameter θ. For a single shot experiment ($\nu = 1$), it is given by the celebrated quantum Cramr-Rao bound [52] as $\Delta\theta \geq \frac{1}{[\mathrm{F}(\rho,H)]}$.
Our goal is to quantify quantum correlation in terms of QFI and for that we define the minimum of QFI, $Q_A(\rho)$, over all local Hamiltonians H_A on A-party, as

$$\mathcal{Q}_A(\rho) \quad = \quad \min{}_{\{H_A\}}\mathrm{F}(\rho, H_A). \tag{54}$$

In the presence of the quantum correlation, we have non-zero $Q_A(\rho)$ and QFI is lower bounded as $Q_A(\rho) \leq \mathrm{F}(\rho, H_A)$. Hence, the minimum of QFI $Q_A(\rho)$, rewrites also as [51]

$$\begin{aligned} \mathcal{Q}_A(\rho) &= \min{}_{\{\overrightarrow{r}\}}\mathrm{F}(\rho, H_A), \\ &= \quad 1 - \lambda_w^{max}, \end{aligned} \tag{55}$$

where λ_w^{max} is the smallest eigenvalue of the real symmetric matrix $[W]_{ij}$, which given by

$$[w]_{ij} = \sum_{m\neq n} \frac{2\lambda_m\lambda_n}{\lambda_m+\lambda_n}\langle m|\sigma_i\otimes\mathbb{I}|n\rangle\langle n|\sigma_j\otimes\mathbb{I}|m\rangle \tag{56}$$

In the other hand, the upper bound of QFI can also be derived and it is the maximal $\mathrm{F}(\rho, H_A)$ over all possible H_A and can be calculated analytically as [51]

$$\begin{aligned} \mathcal{P}_A(\rho) &= \max{}_{\{\overrightarrow{r}\}}\mathrm{F}(\rho, H_A), \\ &= \quad 1 - \lambda_w^{min}, \end{aligned} \tag{57}$$

where λ_w^{min} is the smallest eigenvalue of the real symmetric matrix $[W]_{ij}$(56).
The $\mathcal{P}_A(\rho)$ possesses all the good properties as $\mathcal{Q}_A(\rho)$. Thus, the bounds on the QFI becomes $\mathcal{P}_A(\rho) \geq \mathrm{F}(\rho, H_A) \geq \mathcal{Q}_A(\rho)$. In particular, in the presence of quantum correlation ($\mathcal{Q}_A(\rho) \neq 0$), the error on the estimated parameter, in a single shot experiment, is given by [51]

$$\Delta\theta_{\max} \geq \Delta\theta \geq \Delta\theta_{\min} \tag{58}$$

where $\Delta\theta_{\max}$ and $\Delta\theta_{\min}$ are respectively given by

$$\Delta\theta_{\max} = \frac{1}{\mathcal{Q}_{\mathcal{A}}(\rho)}; \qquad \Delta\theta_{\min} = \frac{1}{\mathcal{P}_{A}(\rho)}. \tag{59}$$

Remarkably the quantum bipartite states have intrinsic precision in metrology with local unitaries that is inverse to the quantum correlation present in the system, while it is absent for the CQ states (see Figure 5). This intrinsic precision is also tested experimentally [40].
As above, to decide about delimiting quantum metrology based on the Fisher information, we introduce the following quantity

$$\Delta\theta = \Delta\theta_{\max} - \Delta\theta_{\min} \tag{60}$$

To compute this quantity in a two-qubit state defined by (13), and to investigate the relation given in (58). we first compare the eigenvalues of the matrix $[W]_{ij}$, which are given by

$$\begin{array}{rcl} \lambda_w^1 & = & 2[1-(c_1+c_2)][\sqrt{c_2}+\sqrt{c_1}]^2, \\ \lambda_w^2 & = & 2[1-(c_1+c_2)][\sqrt{c_2}-\sqrt{c_1}]^2, \\ \lambda_w^3 & = & \frac{(c_1-c_2)^2}{c_1+c_2}, \end{array} \tag{61}$$

Remark that λ_w^1 is always greater than λ_w^2. At this point, we evaluate the expression of $\Delta\theta_{\max}$ over the maximum between the eigenvalues λ_w^1 and λ_w^3. Also, the expression $\Delta\theta_{\min}$ can be derived over the minimum of the two eigenvalues λ_w^2 and λ_w^3.
In order to give the explicit expression of $\Delta\theta_{\max}$, it is simple to verify that the difference $\lambda_w^1 - \lambda_w^3$ is positive when the parameters c_1 and c_2 satisfy the following condition

$$(\sqrt{c_1}+\sqrt{c_2})^2 - 2(c_1+c_2)^2 \geq 0 \tag{62}$$

Otherwise, we have $\lambda_w^1 \leq \lambda_w^3$. The above condition is satisfied in the situation with $\alpha \leq \frac{1}{2}$ and the expression of $\Delta\theta_{\max}$ (60) writes

$$\Delta\theta_{\max} = \{1 - 2(1-\alpha)(\sqrt{c_1}+\sqrt{\alpha - c_1})^2\}^{-1} \tag{63}$$

For $\alpha \geq \frac{1}{2}$, the condition (62) is satisfied also for

$$\alpha_- \leq c_1 \leq \alpha_+ \tag{64}$$

where

$$\alpha_\pm = \frac{1}{2}\alpha \pm \sqrt{\alpha^3 - \alpha^4}, \tag{65}$$

and the expression of $\Delta\theta_{\max}$ takes the form (63).
Conversely, for $0 \leq c_1 \leq \alpha_-$ and $\alpha_+ \leq c_1 \leq \alpha$ the difference $\lambda_w^1 - \lambda_w^3$ is negative. In this case, one gets

$$\Delta\theta_{\max} = \{1 - \frac{1}{\alpha}\,(2c_1 - \alpha)^2\}^{-1} \tag{66}$$

Now, we determin the explicit form of the quantity $\Delta\theta_{\min}$, We consider separately the situation $\lambda_w^{min} = \lambda_w^2$ and $\lambda_w^{min} = \lambda_w^3$. We first treat the situation where $\lambda_w^2 \geq \lambda_w^3$. In this case we find the following condition

$$(\sqrt{c_1}-\sqrt{c_2})^2 - 2(c_1+c_2)^2 \geq 0. \tag{67}$$

It is interesting to note that this condition is satisfied when

$$0 \leq c_1 \leq \alpha_-; \quad \alpha_+ \leq c_1 \leq \alpha \tag{68}$$

and for $\alpha \leq \frac{1}{2}$. Hence, the explicit form of $\Delta\theta_{\min}$ coincide indeed with the expression described in (66). Along the same line of reasoning, one verifies that $\lambda_w^2 \leq \lambda_w^3$ for $\alpha_- \leq c_1 \leq \alpha_+$ (c_+ and c_- are given in (65)) and $\Delta\theta_{\min}$ is necessarily of the form

$$\Delta\theta_{\min} = (1 - 2(1-\alpha)(\sqrt{c_1} - \sqrt{\alpha - c_1})^2\}^{-1}. \tag{69}$$

On the other hand, for $\alpha \geq \frac{1}{2}$ the difference $\lambda_w^2 - \lambda_w^3$ is non positive and in this case, the expression of $\Delta\theta_{\min}$ coincide with the form (69).
The set of eqautions (63), (66) and (69) establishes the correlation quantifiers based on quantum Fisher information for two-qubit state (13). Indeed, using these expressions, the quantity $\Delta\theta$ (59) is given by

$$\Delta\theta = \frac{\{(\sqrt{c_1} + \sqrt{\alpha - c_1})^2 - 2\alpha^2\}}{\{1 - 2(1-\alpha)(\sqrt{c_1} + \sqrt{\alpha - c_1})^2\}\{\frac{\alpha}{(\sqrt{c_1}+\sqrt{\alpha-c_1})^2} - (\sqrt{c_1} - \sqrt{\alpha - c_1})^2\}}, \tag{70}$$

and

$$\Delta\theta = \frac{8\sqrt{c_1\,(\alpha - c_1)}(1-\alpha)}{1 - [4(1-\alpha)\{\alpha - (2c_1 - \alpha)^2(1-\alpha)\}]}, \tag{71}$$

for $0 \leq c_1 \leq \alpha_- \bigcup \alpha_+ \leq c_1 \leq \alpha$ and $\alpha_- \leq c_1 \leq \alpha_+$, respectively ($\alpha \leq \frac{1}{2}$).
Conversely, in the case that $\alpha \geq \frac{1}{2}$, one verifies that

$$\Delta\theta = \frac{\{2\alpha^2 - (\sqrt{c_1} - \sqrt{\alpha - c_1})^2\}}{\{1 - 2(1-\alpha)(\sqrt{c_1} - \sqrt{\alpha - c_1})^2\}\{\frac{\alpha}{(\sqrt{c_1}-\sqrt{\alpha-c_1})^2} - (\sqrt{c_1} + \sqrt{\alpha - c_1})^2\}}, \tag{72}$$

for $0 \leq c_1 \leq \alpha_- \bigcup \alpha_+ \leq c_1 \leq \alpha$, and $\Delta\theta$ coincide with the expression (71) for $\alpha_- \leq \alpha \leq \alpha_+$.
To corroborate our analysis, we give in the figure 6 the behavior of $\Delta\theta_{\max} - \Delta\theta_{\min}$ present in the states (13) for two case ($\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$). Figure 6 with $\alpha \leq \frac{1}{2}$ ($\alpha = c_1 + c_2$) gives the numerical result of $\Delta\theta_{\max} - \Delta\theta_{\min}$ versus the parameter c_1. This result show that differnce $\Delta\theta_{\max} - \Delta\theta_{\min}$ is minimal for the states with (($c_1 = 0, c_2 = \alpha$), ($c_1 = \alpha, c_2 = 0$)) and maximal for the states with ($c_1 = c_2 = \frac{\alpha}{2}$). These two situations are addressed by ((37), (38)) and (34), respectively. Also, we note that for $\alpha = 0.5$ the precision is undefined in the states with ($c_1 = c_2 = \frac{\alpha}{2}$), this remark indicates that the quantum interferometric power $\mathcal{P}$ is vanish when $c_1 = 0.25$ (($\alpha = 0.5$) see figure 5).
The figure 6 with ($\alpha \geq \frac{1}{2}$) shows a nonvanishing precision for any value of parameter c_1 and α. On the other hand, the difference of the amount $\Delta\theta_{\max} - \Delta\theta_{\min}$ is changes when ($c_1 = \alpha_-$ and ($c_1 = \alpha_+$ (α_- and α_+ are given by the expressions (65)). This change occurs when the states ρ (13) have a maximum value of quantum Fisher information (see Figure 5 with $\alpha \geq \frac{1}{2}$). Also, The minimal value of $\Delta\theta_{\max} - \Delta\theta_{\min}$ is obtained in the phase ($0 \leq c_1 \leq \alpha_-$) for the states given by (37).

6 Entropic measures of bipartite quantum discord

6.1 Definition

Quantum discord has been calculated explicitly only for a rather limited set of two-qubit quantum states and analytical expressions for more general quantum states are not known [53]. For a state ρ_{AB}

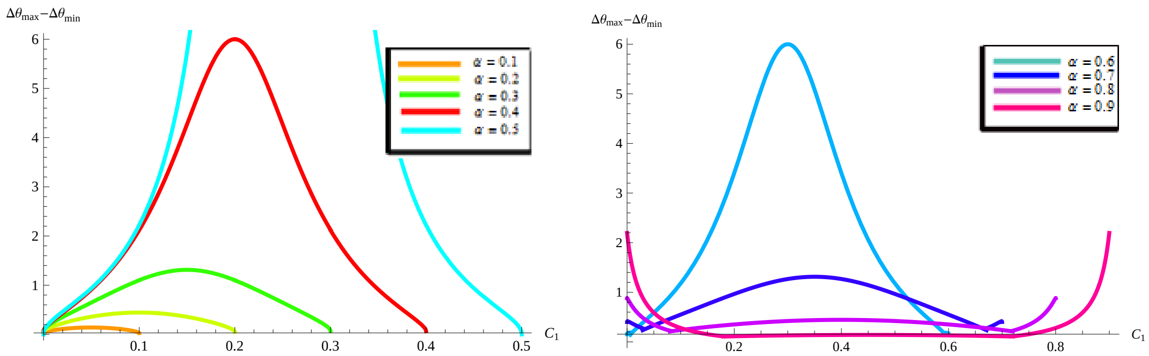

Figure 6 . $\Delta\theta_{\max} - \Delta\theta_{\min}$ versus the parameter c_1 for different values of $\alpha = c_1 + c_2$.

of a bipartite quantum system composed of two particles or modes A and B, the quantum discord is defined by the difference between total correlation and classical correlation (see (5)), this expression can be rewritten as [?]

$$D(\rho_{AB}) = S(\rho_A) + \tilde{S}_{min} - S(\rho_{AB}). \tag{73}$$

in terms of von-Neumann entropy $S(\rho)$ after performing taken an optimization over all perfect measurement [53](seen also [?] and [?]).
Where $S(\rho_{AB})\{S(\rho_A)\}$ is the Von-Neumann entropy of the matrix density $\rho_{AB}\{\rho_A\}$, respectively. According to the density matrix (11), the expressions of $S(\rho_{AB})$ and $S(\rho_A)\}$ are explicitly given by:

$$S(\rho_{AB}) = -\lambda_+^{AB} log_2 \lambda_+^{AB} - \lambda_-^{AB} log_2 \lambda_-^{AB}. \quad , \quad S(\rho_A) = -\lambda_+^{A} log_2 \lambda_+^{A} - \lambda_-^{A} log_2 \lambda_-^{A} \tag{74}$$

Thus, the corresponding eigenvalues are defined by

$$\lambda_+^{AB} = c_1 + c_2 \quad , \quad \lambda_-^{AB} = 1 - (c_1 + c_2) \tag{75}$$

in the case of the bipartite density matrix ρ_{AB}, and

$$\lambda_+^{A} = \tfrac{1}{2}(1 + c_1 - c_2) \quad , \quad \lambda_-^{A} = \tfrac{1}{2}(1 - c_1 + c_2) \tag{76}$$

for the subsystem described by the reduced density matrix ρ_A. Here, the von-Neumann entropy defined by $S(\rho) = -xlog(x) - (1-x)log(1-x)$(x present the eigenvalues of the system corresponding) is exactly the binary entropy function $H(x)$. Therefore, the expression of quantum discord (73) for the bipartite state (11) can be simply obtained as

$$D(\rho_{AB}) = H(c_1 + c_2) - H(\frac{1}{2}(1 + c_1 - c_2)) + \tilde{S}_{min} \tag{77}$$

The final step in evaluating the quantum discord is the minimization of conditional entropy to get an explicit expression of the quantum discord in the bipartite system: the following section is an objective in this order.

6.2 Minimization of conditional entropy: Koashi-Winter relation

To minimize the conditional entropy $\tilde{S}_{min}$, we shall use the purification method and the Koashi-Winter relation [54]. This relation establishes the connection between classical correlation of a bipartite state

$\rho^Q = \rho_{AB}$ and the entanglement of formation of its complement ρ_{BC}. The density matrix described in ρ_{AB} can be written as

$$\rho_{AB} = \lambda_+^{AB}|\psi_1\rangle_{AB}\langle\psi_1| + \lambda_-^{AB}|\psi_2\rangle_{AB}\langle|\psi_2| \tag{78}$$

where the set of the eigenvalues and the corresponding eigenstates denoted by $\lambda_+^{AB}(\lambda_-^{AB}) = \lambda_+(\lambda_-)$ and $|\psi_1\rangle_{AB}(|\psi_2\rangle_{AB}) = |\psi_1\rangle(|\psi_2\rangle)$ respectively, is defined previously in (12) and (13). The purification of the state ρ_{AB} is realized by attaching a qubit C to the two-qubit system A and B. This yields

$$|\psi\rangle_{ABC} = \sqrt{\lambda_+^{AB}}|\psi_1\rangle_{AB} \otimes |0\rangle_C + \sqrt{\lambda_-^{AB}}|\psi_2\rangle_{AB} \otimes |1\rangle_C \tag{79}$$

such that the whole system ABC is described by the pure density matrix $\rho_{ABC} = |\psi\rangle\langle\psi|$ from which one has the bipartite densities $\rho_{AB} = Tr_C\rho_{ABC}$ and $\rho_{BC} = Tr_A\rho_{ABC}$. According to koachi-Winter relation [55], the minimal value of the conditional entropy coincides with the entanglement of formation of ρ_{BC}:

$$\tilde{S}_{min} = E(\rho_{BC}) = H(\frac{1}{2} + \frac{1}{2}\sqrt{1 - |C(\rho_{BC})|^2}) \tag{80}$$

where $H(x) = -xlog_2x - (1-x)log_2(1-x)$ is the binary entropy function and $C(\rho_{BC})$ is the concurrence of the density matrix ρ_{BC} given by [54]:

$$|C(\rho_{BC})|^2 = 2[1 - c_1 - c_2](\sqrt{c_1} - \sqrt{c_2})^2 \tag{81}$$

Finally, reporting (38), (39) in the definition (35), the explicit expression of quantum discord for the density ρ_{AB} is

$$D(\rho_{AB}) = H\left(c_1 + \frac{1}{2}(1-\alpha)\right) + H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - 2(1-\alpha)(\alpha - 2\sqrt{c_1(\alpha - c_1)})}\right) - H(\alpha) \tag{82}$$

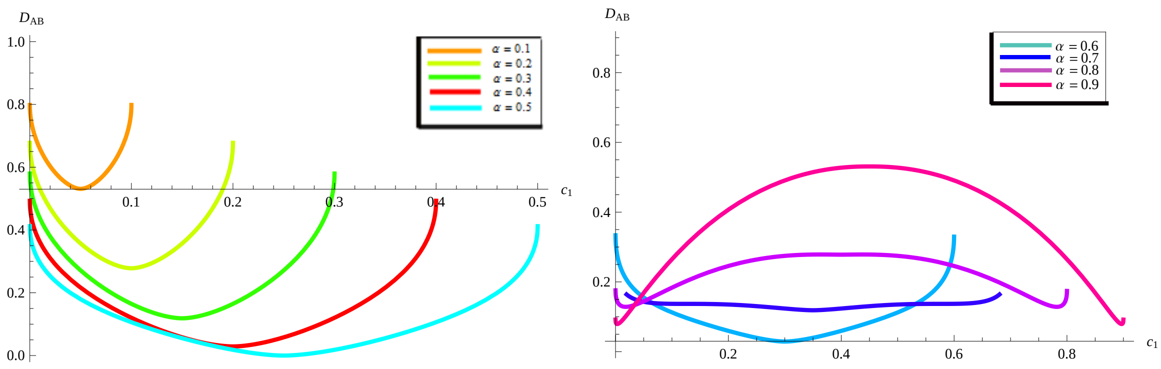

Figure 7. Quantum discord $D(\rho_{AB})$ as function of the parameter c_1 for $\alpha \leq \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$.

In figure 7, we plot the quantum discord $D(\rho_{AB})$ as function of the parameters c_1 and c_2 for different values of $\alpha = c_1 + c_2$ (α=0.1, ...,0.9). For states with $\alpha \leq \frac{1}{2}$, the quantum discord decreases as α increases, also it reaches the minimal value for $c_1 = \frac{\alpha}{2}$. These "minimally discordant" state are given by (34). In addition, the maximal value of quantum discord is obtained in the states with $(c_1 = 0, c_2 = \alpha)$ and $(c_1 = \alpha, c_2 = 0)$ which are previously given by 37(38) respectively.
In this case, we can conclude that quantum discord $D(\rho_{AB})$ for the states with $(\alpha \leq \frac{1}{2})$ has the same properties of "Quantum Interferometric Power" $\mathcal{P}$.
In the situation of states with $\alpha \geq \frac{1}{2}$. The quantum discord for the states with $\alpha = 0.6$ is minimal

at $c_1 = 0.3$. this minimam discordant state given by (34), the quantum discord is almost constant in the states with $\alpha = 0.7$, However, the situation is completely different for $\alpha = 0.8$ and $\alpha = 0.9$ the quantum discord $D(\rho_{AB})$ is evolve continuously for these states. Indeed, for the states with $\alpha = 0.9$ ($c_1 = 0.45$ for instence) the quantum discord is maximum (see condition (34)).

7 Concluding remarks

In summary, we have analyzed the role of quantum correlation of the bipartite mixed-state in quantum metrology. In fact, we investigate the role of quantum discord in phase parameter estimation: whenever a system shares discord, quantum Mechanics predicts that any local measurement has a degree of uncertainty which translates into an improved sensitivity in parameter estimation. The Local Quantum Uncertainty and the Quantum Interferometric Power are parent discord-like measures, which quantify the minimum amount of precision in interferometric phase estimation.The best suited measure of quantum correlation, which laid the basis of investigation, is the one based on quantum Fisher information originally introduced in [46]. Our results suggest that the bipartite mixed states type-discordant can be a promising resource for realizing quantum technology. In fact, for such states quantum discord is present. This adds to the evidence that quantum discord may be responsible for some quantum enhancements. An interesting question is to establish if the metrological measures of discord, which have been introduced for the bipartite quantum systems, can be extended to quantify multipartite correlations; we hope to treat this issue in a forthcoming work.

References

[1] N. Gisin, G. Ribordy, W. Tittel and H. Zbinden, Rev. Mod. Phys. **74** (2002) 145.

[2] P.W. Shor, SIAM J. Comp. **26** (1997) 1484.

[3] L.K. Grover, Phys. Rev. Lett. **79** (1997)325 .

[4] T. Baumgratz, M. Cramer and M. B. Plenio, Phys. Rev. Lett. **113** (2014) 140401.

[5] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Rev. Mod. Phys.**81** (2009) 865.

[6] D.Girolami, Journal of Physics. **626** (2015) 1.

[7] D. Girolami, T.Tufarelli and G. Adesso, Phys. Rev. Lett. **110** (2013) 240402.

[8] K. Modi, H. cable, M. Williamson and V.Vedral, Phys.Rev. X **1** (2011) 021022.

[9] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88** (2001) 017901.

[10] A. Streltsov, H. Kampermann and D. Bruss, Phys. Rev. Lett. **106** (2011) 160401.

[11] K. Modi, A. Brodutch, H. Cable, T. Paterek and V. Vedral, Rev. Mod. Phys. **84** (2012) 1655.

[12] S.Pirandola, Scientific Reports . **4** (2014) 6956.

[13] V. Giovannetti, S. Lloyd, and L. Maccone, Nat. Photonics. **5** (2011) 222.

[14] T.K. Chaun et al, Phys.Rev.Lett. **109** (2012) 070501.

[15] F. G. S. L. Brando, M. Piani and P. Horodeski, arxiv:1310.8640v1 (2013).

[16] V. Giovannetti, S. Lloyd, and L. Maccone, Phys. Rev. Lett. **96** (2006) 010401 .

[17] S. Boixo and R. D. Somma, Phys. Rev. A **77** (2008) 052320.

[18] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88** (2001) 017901.

[19] I. S. Oliveira, T. J. Bonagamba, R. S. Sarthour, J. C.C. Freitas and E.R. Azevedo, NMR Quantum Information Processing. Elsevier, 2011.

[20] S. Lloyd, Science. **321** (2008) 1463.

[21] V. Giovannetti, S. Lloyd, and L.Maccone, Nat. Photonics. **5** (2011) 222.

[22] V. Giovannetti, S. Lloyd, and L.Maccone, Phys. Rev. Lett.**96** (2006) 010401.

[23] C. C. Gerry and J. Mimih, Contemp. Phys. **51**(2010)497.

[24] A. W. Chin, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. **109** (2012) 233601.

[25] W. Zhong, Z. Sun, J. Ma, X. Wang and F. Nori, Phys. Rev. A **87** (2013) 022337.

[26] D. C. Brody, J. Phys. A: Math. Theor. **44** (2011) 252002.

[27] J. Liu, X.Jing, and X. Wang, Phys. Rev. A **88** (2013) 042316.

[28] J. Joo, W. J.Munro, and Timothy P. Spiller, Phys. Rev. Lett. **107** (2011) 083601.

[29] Y. M. Zhang, X. W. Li, W. Yang, and G. R. Jin, Phys. Rev. A **88** (2013) 043832.

[30] Y. C. Liu, G. R. Jin, and L. You, Phys. Rev. A **82** (2010) 045601

[31] X. Jing, J. Liu, W. Zhong, and X. Wang, Commun.Theor. Phys. **61** (2013) 1 .

[32] P. B. Slater, J. Phys. A: Math. Gen. **29** (1996)10.

[33] A. Merberg and S. J. Miller, Course Notes for Math 162: Mathematical Statistics(2008).

[34] C. W. Helstrom, Quantum Detection and Estimation Theory. Academic Press. **126** (1976).

[35] Holevo, A. S. "Probabilistic aspects of quantum theory". North-Holland, Amsterdam 2.2.1 (1982): 2.

[36] S. Luo, Phys. Rev. Lett. **91** (2003) 180403.

[37] P. Gibilisco, D. Imparato, T. Isola, Mathematical Physics **59**(2007)

[38] S. L.Luo, Theoretical and Mathematical Physics. **143** (2005) 2.

[39] D. Girolami, T. Tufarilli and Adesso.G, phys.Rev. Lett. **110** (2013) 240402.

[40] D. Girolami et al,phys.Rev.Lett **112** (2014)21041.

[41] D. Petz, Linear Alegbra Appl. **244** (1996) 81-96.

[42] Dakie. B et al, Nat.phys. **8** (2012) 666-670.

[43] S. Luo and Q. Zhang, Phys. Rev. A **69** (2004) 032106.

[44] J. Liu, X. Jing and X. Wang, Phys.Rev. A **88** (2013) 042316.

[45] L. Pezz and A. Smerzi, Phys. Rev. Lett. **102** (2009) 100401 .

[46] Jing Liu, Xiaoxing Jing, Wei Zhong and Xiaoguang Wang, Communications in Theoretical Physics, **61** (2014)45 .

[47] Jing Liu, Xiaoxing Jing, Wei Zhong and Xiaoguang Wang, Communications in Theoretical Physics, **61** (2014)45 .

[48] M. Daoud,R. Ahl Laamara and W. Kaydi, Physics Letters A **378** (2014) 3510.

[49] W.K. Wootters, Phys. Rev. Lett. **80** (1998) 2245; W.K. Wootters, Quant. Inf. Comp. **1** (2001) 27-44.

[50] S. Hill and W.K. Wootters, Phys. Rev. Lett. **78** (1997) 5022.

[51] Manabendra N. Bera, Role of quantum correlation in metrology beyond standard quantum limit, arXiv preprint arXiv:1405.5357(2014).

[52] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72** (1994)3439.

[53] M. Daoud and R. Ahl Laamara, International Journal of Quantum Information. **10** (2012) 1250060.

[54] Jiang and J. Du, J. Phys. A: Math. Theor. **44** (2011) 415304.

[55] M. Koachi and A. Winter, Phys. Rev. A **69** (2004) 022309.