k-Grassmannian coherent states for $su(n)$ algebras

XXXXX[1,2,3,4] **and YYYYYY**[1,2,3]

Abstract

The aim of this article is to construct *à la* Barut–Girardello coherent states for $su(n)$ algebra. This construction uses the generalized Grassmann variables.

1 Introduction

It is well established that the formalism of coherent states is widely used in several areas of quantum physics (see [2, 33, 35, 42, 51, 59]). There exist three definitions of coherent states for a given Lie algebra (see for instance [62]): (i) The Klauder-Perelomov approach which defines the coherent states by the action of an unitary displacement operator on a reference state [51, 50] , (ii) The Barut–Girardello approach where the coherent states are defined as the eigenstates of the lowering generators [10] and (iii) The uncertainty approach in which the coherent states are obtained by minimizing the Robertson-Schrödinger uncertainty relations for Hermitian generators of a group [56, 57] (see also [5, 6]). In general for a given algebra, the three approaches yields to different sets of coherent states expect the Weyl-Heisenberg describing the harmonic oscillator.

It must noticed that contrarily to Klauder-Perelomov approach , the Barut–Girardello prescription is limited to Lie algebras with infinite-dimensional representation space. For instance, this approach does not apply for $su(2)$ algebra for which the coherent states can be defined using the first and the third approach.

In this paper, we shall be concerned with the construction of Barut–Girardello coherent states for $su(n)$ algebras in terms of generalized Grassmann variables.

2 Qudits and generalized Weyl-Heisenberg algebra

Dealing with bosonic and fermionic many particles states is simplified by considering the algebraic structures of the corresponding raising and lowering operators. For bosons the creation and annihilation operators satisfy the commutations relations

$$[b_i^-, b_j^+] = \delta_{ij}\mathbb{I}, \qquad [b_i^-, b_j^-] = [b_i^+, b_j^+] = 0. \tag{1}$$

where the unit operator $\mathbb{I}$ commute with the creation and annihilation operators b_i^+ and b_i^i. On the hand, fermions are specified by the following anti-commutation relations

$$\{f_i^-, f_j^+\} = \delta_{ij}\mathbb{I}, \qquad \{f_i^+, f_j^+\} = \{f_i^-, f_j^-\} = 0. \tag{2}$$

The Fock spaces for bosons and fermions give the realizations of the associated commutation and anti-commutation relations and subsequently the symmetric and antisymmetric waves functions. The properties of Fock states follow from the commutation and anti-commutation relations which imposes only be one particle in each state for fermions (two dimension) and multiple particles for bosons (infinite dimension). Following Wu and Vidal there is a crucial difference between fermions and qubits. In fact, a qubit is a vector in a two dimensional Hilbert space like fermions and the Hilbert space of a multi-qubit system has a tensor product structure like bosons. In this respect, the raising and lowering operators commutation rules for qubits are neither specified by relations of bosonic type (1) nor of fermionic type (2).

2.1 Qubit algebra from generalized Weyl-Heisenberg algebra

The qubits appear like objects which exhibits both bosonic and fermionic properties so that they cannot described by Fermi-like or Bose-like operators. An alterative way for the algebraic description of qubits and qudits, is possible by resorting the formalism of generalized Weyl-Heisenberg algebras. We by $|0\rangle$ the ground state and $|1\rangle$ the excited state of a two-level system and we define the lowering, raising and number operators by

$$a^- = |0\rangle\langle 1|, \qquad a^+ = |1\rangle\langle 0|, \qquad N = |1\rangle\langle 1|. \tag{3}$$

They satisfy the following commutation relations

$$[a^-, a^+] = \mathbb{I} - 2N, \qquad [N, a^+] = -a^+, \qquad [N, a^-] = +a^-. \tag{4}$$

where $\mathbb{I}$ is the unit operator. In this scheme, the qubit is described by a modified bosonic algebra and the creation and the annihilation operators satisfy the nilpotency condition: $(a^+)^2 = (a^-)^2 = 0$ like Fermi operators. This algebra turns out to be a particular case of the generalized Weyl-Heisenberg algebra $\mathcal{A}_\kappa$ introduced in [?, ?] ($\kappa \in \mathbb{R}$). The structure relations of the algebra $\mathcal{A}_\kappa$ are defined by

$$[a^-, a^+] = \mathbb{I} + 2\kappa N, \qquad [N, a^+] = -a^+, \qquad [N, a^-] = +a^- \tag{5}$$

where $\kappa \in \mathbb{R}$. For $\kappa < 0$, the Hilbert space representations is finite dimensional. The algebra $\mathcal{A}_\kappa$ reduces to the algebra (4) for $\kappa = -1$. It must be stressed that the commutation relations (4) coincide with ones defining the algebra introduced in [?] to introduce an alternative algebraic description of qubits instead of the parafermionic formulation considered in [?].

To describe a l-qubit system, we consider l copies of the algebra $\mathcal{A}_{-1}$ generated by the raising and lowering operators a_i^+ and a_i^-, the number operators N_i and the unit operator $\mathbb{I}$ such that they satisfy the relations

$$[a_i^-, a_j^+] = (\mathbb{I} - 2N_i)\ \delta_{ij}, \qquad [N_i, a_j^+] = -\delta_{ij} a_j^+, \qquad [N_i, a_j^-] = +\delta_{ij} a_j^- \qquad [a_i^-, a_j^-] = [a_i^+, a_j^+] = 0. \tag{6}$$

where $i = 1, 2, \cdots, l$. Let denote by $\mathcal{H}_i = \{|k_i\rangle, \quad k_i = 0, 1\}$ the Hilbert space for the qudit i. In view of the relation $[a_i^-, a_j^+] = 0$ for $i \neq j$, the n-qubit Hilbert space has the following tensor product structure

$$\mathcal{H}(l) = \bigotimes_{i=1}^{l} \mathcal{H}_i = \{|n_1, n_2, \cdots, n_l\rangle, \quad k_i = 0, 1\}.$$

like bosons and $\{|n_1, n_2, \cdots, n_l\rangle, n_i = 0, 1 \quad \text{for} \quad i = 1, 2, \cdots, l\}$ is its orthonormal basis.

2.2 Qudit algebra and Dicke states

For $(d-1)$-qubits, the Hilbert space is given by

$$\mathcal{H}(d-1) = \bigotimes_{i=1}^{d-1} \mathcal{H}_i = \{|n_1, n_2, \cdots, n_{d-1}\rangle, \quad k_i = 0, 1\}.$$

The corresponding creation and annihilation operators $a_i^{\pm}$ $(i = 1, 2, \cdots, d-1)$ satisfy the structure relations (6). We define the collective lowering and raising operators in the Hilbert space $\mathcal{H}(d-1)$ as follows

$$A^- = \sum_{i=1}^{d-1} a_i^- \qquad A^+ = \sum_{i=1}^{d-1} a_i^+ \tag{7}$$

in terms of the creation and annihilation operators a_i^+ and a_i^-. Here and in the following the index i refers to the system the operator is acting on, e.g.

$$a_i^{\pm} \equiv \mathbb{I} \otimes \cdots \mathbb{I} \otimes a_i^{\pm} \otimes \mathbb{I} \otimes \cdots \mathbb{I}.$$

It is simple to see that the state $|0, 0, \cdots, 0\rangle \equiv |d-1, 0\rangle$ satisfies $A^-|d-1, 0\rangle = 0$. Furthermore, using the commutation relations (6), one gets the following nilpotency conditions

$$(A^-)^d = 0 \qquad (A^+)^d = 0$$

which extends the Pauli exclusion principle for ordinary qubits (i.e., $d = 2$). The actions of the operators A^- and A^+ on the Hilbert space $\mathcal{H}(d-1)$ can be determined from the standard actions of the fermionic operators a_i^- and a_i^+. Using a recursive procedure, one verifies that repeated applications of the raising operator A^+ on the vacuum $|0, 0, \cdots, 0\rangle$ gives

$$(A^+)^k|d-1, 0\rangle = \sqrt{\frac{k!(d-k)!}{(d-1-k)!}}\ |d-1, k\rangle \tag{8}$$

where the vectors $|d-1, k\rangle$ are the symmetric Dicke states with k excitations $(k = 0, 1, 2, \cdots, d-1)$. They are defined by

$$|d-1, k\rangle = \sqrt{\frac{k!(d-1-k)!}{(d-1)!}} \sum_{\sigma \varepsilon S_{d-1}} |\underbrace{0, 0, \cdots, 0}_{d-k-1}, \underbrace{1, 1, \cdots, 1, 1}_{k}\rangle \tag{9}$$

where S_{d-1} is the permutation group of $(d-1)$ objets. The Dicke states generate an orthonormal basis of the symmetric Hilbert subspace $\mathcal{H}_s \subset \mathcal{H}$ with $\dim\mathcal{H}_s = d$. To write the explicit actions of the ladder operators $A^{\pm}$, we introduce the structure function defined by $F(k) = k(d-k)$. The equation (8) rewrites as

$$(A^+)^k|d-1, 0\rangle = \sqrt{F(k)!}\ |d-1, k\rangle \tag{10}$$

where $F(k)! = F(k)F(k-1)\cdots F(1)$ and $F(0) = 1$. After some algebra, it is simple to verify that

$$A^+|d-1, k\rangle = \sqrt{F(k+1)}\ |d-1, k+1\rangle, \qquad A^-|d-1, k\rangle = \sqrt{F(k)}\ |d-1, k-1\rangle \tag{11}$$

and the action of the creation and annihilation operators on the vectors $|d-1, 0\rangle$ and $|d-1, d-1\rangle$ gives

$$A^-|d-1, 0\rangle = 0 \qquad A^+|d-1, d-1\rangle = 0. \tag{12}$$

The number operator A is defined as

$$A|d-1, k\rangle = k\ |d-1, k\rangle. \tag{13}$$

The qudit operators A^+, A^- and A satisfy the commutation rules

$$[A^+, A^-] = (d-1)\mathbb{I} - 2A, \qquad [A^+, A] = A^+, \qquad [A^-, A] = -A^- \tag{14}$$

Using the commutation relation $\left[a_i^+, a_j^-\right] = 0$ for $i \neq j$, it is simple to verify that

$$[A^+, A^-] = \sum_{i,j} \left[a_i^+, a_j^-\right] = \sum_i \left[a_i^+, a_i^-\right]$$

and the operator A can be expressed as

$$A = \sum_{i=1}^{d-1} N_i$$

where N_i is the single qubit number operator ($N_i|0\rangle_i = 0$ and $N_i|1\rangle_i = 1|1\rangle_i$). It is remarkable that the creation and annihilation operator A^+ and A^- close the following trilinear relation commutation

$$[A^-, [A^+, A^-]] = 2A^-, \qquad [A^+, [A^+, A^-]] = -2A^+$$

characterizing a parafermion. Note also that the definition (7) is similar to Green decomposition in the construction of parafermions from ordinary fermions. Therefore, the operators A^+, A^- and A satisfying the relations (14) provide a simple algebraic description of d-level quantum systems (qudit). We notice also that by re-scaling the generators of the algebra (14)

$$A^\pm \longrightarrow \frac{A^\pm}{\sqrt{d-1}},$$

one recovers the algebra $\mathcal{A}_\kappa$ with $\kappa = 1/(1-d)$. This shows clearly that the generalized Weyl-Heisenberg provides the appropriate tools to describe qudit systems. In particular, this realization expresses the Hilbert states of a qudit system in terms of Dicke states of $(d-1)$ qubits. In this way, the global properties of the qubit ensemble are encoded in the qudit system. To close this section that the algebraic description of qudit systems provides us with the necessary ingredients to define the phase operator for a qudit system and subsequently the phase states for a collection of identical qudits. This constitutes the main issue of the next section.

2.3 Genealized Grassmann varaibles for multilevel systems

We consider the algebra generated by the identity 1 and $(k-1)$ commuting Grassmann variables η_i $(i = 1, 2, \cdots k-1)$ obeying the usual nilpotency conditions. Namely

$$\eta_i^2 = 0, \qquad [\eta_i, \eta_j] = 0 \tag{15}$$

We note that the Grassamnn variables are commuting. We denote by $\bar{\eta}_i$ the complex conjugate of the element η_i. The is spanned by 2^k linearly independent elements of the form $\eta_{i_1}\eta_{i_2}\cdots\eta_{i_n}$ with $i_1 < i_2 < \cdots < i_n$ for $n = 0, 1, \cdots, k-1$. For $n = 0$, the corresponding element is the identity. The η-derivative $\partial_i = \frac{\partial}{\partial \eta_i}$ satisfies

$$\partial_i \eta_j = \delta_{ij}, \quad \partial_i 1 = 0, \quad \partial_i \partial_j = \partial_j \partial_i \tag{16}$$

We define the generalized Grassmann variables as follows

$$\eta = \sum_{i=1}^{k-1} \eta_i \qquad \bar{\eta} = \sum_{i=1}^{k-1} \bar{\eta}_i \tag{17}$$

in terms of the nilpotent variables η_i and $\bar{\eta}_i$. We define the following symmetric η-polynomials

$$e_n(\vec{\eta}) = \sum_{i_1 < i_2 < \cdots < i_n} \eta_{i_1}\eta_{i_2}\cdots\eta_{i_n}, \quad \text{for} \quad n = 1, 2, \cdots, k-1 \quad \text{and} \quad e_0(\vec{\eta}) = 1. \tag{18}$$

where $\vec{\eta} = (\eta_1, \eta_2, \cdots, \eta_n)$. Explicitly, we have

$$e_1(\vec{\eta}) = \eta$$

$$e_2(\vec{\eta}) = \sum_{i<j} \eta_i \eta_j$$

$$e_3(\vec{\eta}) = \sum_{i<j<l} \eta_i \eta_j \eta_l$$

$$\cdots$$

$$e_n(\vec{\eta}) = \eta_1 \eta_2 \cdots \eta_k$$

we note that

$$\eta^n = n! e_n(\vec{\eta}) \quad \text{for} \quad n = 1, 2, \cdots, k-1$$

and

$$\eta^k = 0$$

The η-derivative is defined by

$$\frac{\partial}{\partial \eta} = \sum_{i=1}^{k-1} \frac{\partial}{\partial \eta_i} \qquad \frac{\partial}{\partial \bar{\eta}} = \sum_{i=1}^{k-1} \frac{\partial}{\partial \bar{\eta}_i}, \tag{19}$$

Similarly, one shows that

$$\partial_\eta^n = n! g_n \quad \text{for} \quad n = 1, 2, \cdots, k-1 \quad \text{and} \quad \partial_\eta^k = 0$$

where

$$g_n = \sum_{i_1 < i_2 < \cdots < i_n} \partial_{\eta_{i_1}} \partial_{\eta_{i_2}} \cdots \partial_{\eta_{i_n}}, \quad \text{for} \quad n = 1, 2, \cdots, k-1 \quad \text{and} \quad g_0 = 1. \tag{20}$$

We define also the functions

$$D_n(\vec{\eta}) = \sqrt{\frac{n!(k-1-n)!}{(k-1)!}} e_n(\vec{\eta}) \tag{21}$$

It is interesting to notice that

$$\eta^n = \sqrt{\frac{n!(k-1)!}{(k-1-n)!}} D_n(\vec{\eta}) = n! e_n(\vec{\eta}) \tag{22}$$

Using the equation (22), one shows

$$\eta D_n(\vec{\eta}) = \sqrt{(n+1)(k-n-1)}\ D_{n+1}(\vec{\eta}) \tag{23}$$

Using the definition of the η-derivative,

$$\frac{\partial}{\partial\eta} = \sum_{i=1}^{k-1} \frac{\partial}{\partial\eta_i} \qquad \frac{\partial}{\partial\bar{\eta}} = \sum_{i=1}^{k-1} \frac{\partial}{\partial\bar{\eta}_i}, \tag{24}$$

one shows that

$$\frac{\partial e_n(\vec{\eta})}{\partial\eta} = (k-n)\ e_{n-1}(\vec{\eta}) \ \text{ for } \ n = 1, 2, \cdots, k-1 \ \text{ and } \ \frac{\partial e_0(\vec{\eta})}{\partial\eta} = 0 \tag{25}$$

from which one gets

$$\frac{\partial D_n(\vec{\eta})}{\partial\eta} = \sqrt{n(k-n)}\ D_{n-1}(\vec{\eta}) \tag{26}$$

2.4 Generalized Grassmann Variables

Generalized Grassmann variables η and $\bar{\eta}$ of order k satisfy

$$\eta^k = \bar{\eta}^k = 0 \tag{27}$$

The sets $\{I, \eta, \cdots, \eta^{k-1}\}$ and $\{I, \bar{\eta}, \cdots, \bar{\eta}^{k-1}\}$ span isomorphic Grassmann algebras. The derivatives are formally defined by

$$\partial_\eta \eta^n = n(k-n)\ \eta^{n-1} \qquad \partial_{\bar{\eta}} \bar{\eta}^n = n(k-n)\ \bar{\eta}^{n-1} \tag{28}$$

for $n = 0, 1, \cdots, k-1$. Hence, for functions f and g such that

$$f(\eta) = \sum_{n=0}^{k-1} a_n \eta^n \qquad g(\bar{\eta}) = \sum_{n=0}^{k-1} b_n \bar{\eta}^n \tag{29}$$

where the a_n and b_n coefficients in the expansions are complex numbers, we easily show that

$$(\partial_\eta)^k f(\eta) = (\partial_{\bar{\eta}})^k g(\bar{\eta}) = 0. \tag{30}$$

As a consequence, we assume that the conditions

$$(\partial_\eta)^k = (\partial_{\bar{\eta}})^k = 0 \tag{31}$$

hold in addition to (27).

The η-integral can be defined by using the Berezin integral of Grassmann variables given by

$$\int \eta_i d\eta_j = \delta_{ij} \qquad \int d\eta_i = 0 \tag{32}$$

This gives

$$\int e_n(\vec{\eta}) d\eta = 0 \quad (n = 0, 1, \cdots, k-2) \quad \text{and} \int e_{k-1}(\vec{\eta}) d\eta = 1$$

where $d\eta = d\eta_1 d\eta_2 \cdots d\eta_{k-1}$. As result, one gets the following η-integral

$$\int \eta^n\ d\eta = 0 \quad (n = 0, 1, \cdots, k-2) \quad \text{and} \int \eta^{k-1}\ d\eta = (k-1)!$$

Clearly, the usual Berezin integration for ordinary Grassmann variables is recovered in the $k = 2$ particular case.

Similarly, for the conjugate generalized Grassamnn variables we have the following integration rules

$$\int \bar{\eta}^n d\bar{\eta} = 0 \quad (n = 0, 1, \cdots, k-2), \quad \int \bar{\eta}^{k-1} d\bar{\eta} = (k-1)! \tag{33}$$

2.5 The spin coherent states à la Barut-Girardello

The various irreducible representation classes of the group $SU(2)$ are characterized by a label j with $2j \in \mathbf{N}$. The standard irreducible matrix representation associated with j is spanned by the irreducible tensorial set

$$B_{2j+1} = \{|j, m\rangle : m = j, j-1, \ldots, -j\}, \tag{34}$$

where the vector $|j, m\rangle$ is a common eigenvector of the Casimir operator j^2 and of the Cartan operator j_z of the Lie algebra $su(2)$ of $SU(2)$. More precisely, we have the relations

$$j^2|j, m\rangle = j(j+1)|j, m\rangle, \quad j_z|j, m\rangle = m|j, m\rangle, \tag{35}$$

which are familiar in angular momentum theory. (We use lower case letters for operators and capital letters for matrices so that j^2 in (35) stands for the square of a generalized angular momentum.) The raising and lowering operators are given by

$$j_+ = \sum_{m=-j}^{j} \sqrt{(j+m+1)(j-m)}|j, m+1\rangle\langle j, m|, \quad j_- = \sum_{m=-j}^{j} \sqrt{(j+m)(j-m+1)}|j, m-1\rangle\langle j, m| \tag{36}$$

They satisfy the structure relations

$$[j_z, j_+] = +j_+, \quad [j_z, j_-] = -j_-, \quad [j_+, j_-] = 2j_z. \tag{37}$$

In what follows, we make the identifications

$$|j, m\rangle \longleftrightarrow |n\rangle \qquad j+m \longleftrightarrow n,$$

so that the Hilbert space B_{2j+1} is given by

$$B_{2j+1} = \{|n\rangle := 0, 1, \ldots, 2j\}.$$

In order to construct the $su(2)$ coherent states à la Barut-Girardello, we consider the following eigenvalue equation

$$j_-|\eta\rangle = \eta|\eta\rangle \qquad |\eta\rangle = \sum_{n=0}^{2j} C_n \eta^n |n\rangle. \tag{38}$$

Using the action of the lowering operator j_-, one gets the following recurrence relations

$$C_{n+1} \sqrt{(n+1)(2j-n)} = C_n, \quad \text{for} \quad n = 0, 1, \cdots, 2j-1 \tag{39}$$

with the condition

$$C_{2j}\ \eta^{2j+1} = 0. \tag{40}$$

From the equations (39) and (41), it is clear that the eigenvalue equation is solvable if the variable η is a Grassmann variable of order $(2j+1)$:

$$\eta^{2j+1} = 0. \tag{41}$$

In this case the coefficients in the expansion of the states $|\eta\rangle$ writes

$$C_n = \sqrt{\frac{(2j-n)!}{n!(2j)!}}\; C_0 \tag{42}$$

where C_0 can be fixed from the normalization condition of the states $|\eta\rangle$. As result one obtains

$$|\eta\rangle_{BG} = \mathcal{N}_{BG} \sum_{n=0}^{2j} \sqrt{\frac{(2j-n)!}{n!(2j)!}}\; \eta^n |n\rangle. \tag{43}$$

where the normalization factor is given by

$$|\mathcal{N}_{BG}|^{-2} = \sum_{n=0}^{2j} \frac{(2j-n)!}{n!(2j)!} \bar{\eta}^n \eta^n$$

In view of $\eta^{2j+1} = 0$, the $|\eta, \varphi\rangle$ states can be called $(2j+1)$-fermionic coherent states [18]. They satisfy

$$j^- |\eta, \varphi\rangle = \eta |\eta, \varphi\rangle \tag{44}$$

and are thus coherent states in the Barut–Girardello sense. In addition, they constitute an over-complete set with

$$\int |\eta, \varphi\rangle d\mu(\eta, \bar{\eta}) \langle \eta, \varphi| = \sum_{n=0}^{2j} |n\rangle\langle n| \tag{45}$$

for the $d\mu$ measure satisfying the following integral formula

$$\frac{1}{F(n)!} \int \eta^n \; d\mu(\eta, \bar{\eta}) \; |\mathcal{N}_{BG}|^2 \; \bar{\eta}^m = \delta_{n,m}. \tag{46}$$

Setting $d\mu(\eta, \bar{\eta}) |\mathcal{N}_{BG}|^2 = \sigma(\eta, \bar{\eta}) d\eta d\bar{\eta}$, the equation (46) becomes

$$\frac{1}{F(n)!} \int \sigma(\eta, \bar{\eta}) \; \eta^n \bar{\eta}^m \; d\eta d\bar{\eta} = \delta_{n,m}. \tag{47}$$

Expanding the function $\sigma(\eta, \bar{\eta})$ as

$$\sigma(\eta, \bar{\eta}) = \sum_{n=0}^{k-1} a_n \eta^{k-1-n} \bar{\eta}^{k-1-n}, \quad k = 2j+1$$

it is simple to verify that the function $\sigma(\eta, \bar{\eta})$ verifies the integral equation (47) for $a_n = F(n)!$. It follows that the measure takes the following form

$$d\mu(\eta, \bar{\eta}) = |\mathcal{N}_{BG}|^{-2} \sum_{n=0}^{k-1} F(n)! \; \eta^{k-1-n} \; \bar{\eta}^{k-1-n} \; d\eta \; d\bar{\eta} \tag{48}$$

3 The $SU(3)$ coherent states à la Barut-Girardello

3.1 The analytical representation of the generalized algebra $\mathcal{A}_\kappa(2)$

3.1.1 The algebra and the Hilbert representation

We first recall the definition of the $\mathcal{A}_\kappa(2)$ algebra. Here, we shall consider $\kappa = -\frac{1}{k}$ with $k \in \mathbb{N}^*$. This algebra is spanned by two pairs of creation and annihilation operators a_i^-, a_i^+ with $i = 1, 2$ and two number operators N_i. They satisfy the following structures relations

$$[a_i^-, a_i^+] = kI - (N_1 + N_2 + N_i), \quad [N_i, a_j^\pm] = \pm\delta_{i,j} a_i^\pm, \quad i, j = 1, 2 \tag{49}$$

and

$$[a_i^\pm, a_j^\pm] = 0, \quad i \neq j, \tag{50}$$

complemented by the Serre like relations

$$[a_i^\pm, [a_i^\pm, a_j^\mp]] = 0, \quad i \neq j. \tag{51}$$

The identity operator is denoted by I. The Hilbertian representation of this algebra on a Hilbert-Fock space $\mathcal{F}_k$ is finite dimensional. The Fock space $\mathcal{F}_k$ is given by the orthonormal set of vectors

$$\{|n_1, n_2\rangle : n_1 \in \mathbb{N}, n_2 \in \mathbb{N}; n_1 + n_2 \leq k\}.$$

The dimension of the Fock space $\mathcal{F}_k$ is

$$d = \frac{1}{2}(k+1)(k+2), \quad k \in \mathbb{N}^*. \tag{52}$$

The basis of Fock space are defined as the eigenstates of the number operators N_1 and N_2 , i.e.,

$$N_i|n_1, n_2\rangle = n_i|n_1, n_2\rangle, \quad i = 1, 2. \tag{53}$$

The raising and lowering operators $a_1^\pm$ and $a_2^\pm$ act sa

$$a_1^+|n_1, n_2\rangle = \sqrt{(n_1+1)(k - n_1 - n_2)}|n_1 + 1, n_2\rangle, \quad a_1^-|n_1, n_2\rangle = \sqrt{n_1(k + 1 - n_1 - n_2)}|n_1 - 1, n_2\rangle \tag{54}$$

and

$$a_2^+|n_1, n_2\rangle = \sqrt{(n_2+1)(k - n_1 - n_2)}|n_1, n_2 + 1\rangle, \quad a_2^-|n_1, n_2\rangle = \sqrt{n_1(k + 1 - n_1 - n_2)}|n_1, n_2 - 1\rangle. \tag{55}$$

Using the actions (54) and (55), one verifies that the creation and annihilation operators satisfy the conditions

$$(a_1^+)^{k+1} = (a_1^-)^{k+1} = 0 \quad (a_2^+)^{k+1} = (a_2^-)^{k+1} = 0 \tag{56}$$

implemented by the hybrid nilpotency conditions

$$(a_1^+)^{k+1-l}(a_2^+)^l = 0 \quad (a_1^-)^{k+1-l}(a_2^-)^l = 0 \tag{57}$$

for $l = 1, 2, \cdots, k$.

3.1.2 Analytical representation and generalized Grassmann like variables

We shall discuss the analytical realization of the $\mathcal{A}_k$ of in which creation operator a_i^+ acts as multiplications by the variables η_i with $i = 1, 2$. For this end, we realize the Fock space basis as

$$|n_1, n_2\rangle \longrightarrow f_{n_1,n_2}(\eta_1, \eta_2) = c_{n_1,n_2}{\eta_1}^{n_1}{\eta_2}^{n_2} \qquad a_i^+ \longrightarrow \eta_i. \tag{58}$$

Using the relations (56) and (57), the variables η_1 and η_2 satisfy

$$(\eta_1)^{k+1} = 0 \quad (\eta_2)^{k+1} = 0 \tag{59}$$

$$(\eta_1)^{k+1-l}(\eta_2)^l = 0 \quad (\eta_1)^{k+1-l}(\eta_2)^l = 0 \tag{60}$$

for $l = 1, 2, \cdots, k$. From the correspondence (61), one can write

$$|0, 0\rangle \longrightarrow 1 \tag{61}$$

where we set $C_{0,0} = 1$. Using the actions of the creation and annihilation operators (54) and (55), one verifies

$$|n_1, n_2\rangle = \sqrt{\frac{(k - n_1 - n_2)!}{k!n_1!n_2!}}{a_1^+}^{n_1}{a_2^+}^{n_2}|n_1, n_2\rangle \qquad n_1 + n_2 \leq k, \tag{62}$$

from which one writes

$$f_{n_1,n_2}(\eta_1, \eta_2) = \sqrt{\frac{(k - n_1 - n_2)!}{k!n_1!n_2!}}{\eta_1}^{n_1}{\eta_2}^{n_2} \tag{63}$$

The derivative with respect to the variables η_1 and η_2 can be easily computed. Indeed, one gets

$$\frac{\partial}{\partial\eta_1}f_{n_1,n_2}(\eta_1, \eta_2) = \sqrt{n_1(k + 1 - (n_1 + n_2))}f_{n_1-1,n_2}(\eta_1, \eta_2)\frac{\partial}{\partial\eta_2}f_{n_1,n_2}(\eta_1, \eta_2) = \sqrt{n_2(k + 1 - (n_1 + n_2))}f_{n_1,n_2-1}(\tag{64}$$

It follows that the derivatives satisfy the conditions

$$\left(\frac{\partial}{\partial\eta_1}\right)^{k+1-l}\left(\frac{\partial}{\partial\eta_2}\right)^l = 0; \quad l = 0, 1, 2, \cdots, k + 1. \tag{65}$$

3.2 The $su(3)$ algebra

To fix the notations, we first introduce the $su(3)$ algebra algebra. We denote the generators of this algebra by j_i^-, j_i^+ and h_i with $i = 1, 2$. They satisfy the following commutation relations

$$[j_i^+, j_i^-] = 2h_i, \quad [h_i, j_j^\pm] = \pm\delta_{i,j}j_i^\pm, \quad i, j = 1, 2 \tag{66}$$

and

$$[j_1^\pm, j_2^\pm] = 0, \tag{67}$$

complemented by the triple relations

$$[j_1^\pm, [j_1^\pm, j_2^\mp]] = 0, \quad [j_2^\pm, [j_2^\pm, j_1^\mp]] = 0. \tag{68}$$

Here the $su(3)$ algebra is described by two pairs of raising and lowering operators satisfying usual commutation relations (66) and (67) and triple commutation relations (68). This description is the Jacobson approach according to which the A_n Lie algebra can be defined by means of $2n$, rather than $n(n+2)$, generators satisfying commutation relations and triple commutation relations. These $2n$ Jacobson generators correspond to n pairs of creation and annihilation operators. Indeed, we note that from the triple relation give the structure relations satisfied by the operators defined

$$j_3^+ = [j_2^+, j_1^-], \quad j_3^- = [j_1^+, j_2^-]. \tag{69}$$

in terms of the j_i^-, j_i^+ $()i = 1, 2$.

The dimension $d(\lambda, \mu)$ of the irreducible representation (λ, μ) of SU_3 is given by

$$d(\lambda, \mu) = \frac{1}{2}(\lambda + 1)(\mu + 1)(\lambda + \mu + 2), \quad \lambda \in \mathbb{N}, \quad \mu \in \mathbb{N}.$$

For the irreducible representation $(0, k)$ or its adjoint $(k, 0)$ of SU_3, the dimension of representation space is given by

$$d = \frac{1}{2}(k + 1)(k + 2), \quad k \in \mathbb{N}^*. \tag{70}$$

The associated Hilbert-Fock space $\mathcal{B}_k$ is

$$\{|n_1, n_2\rangle : n_1, n_2 = 0, 1, 2, \ldots ; n_1 + n_2 = 0, 1, \ldots, k, \}$$

where the vectors generating the orthonormal basis of $\mathcal{F}_k$ are the eigenstates of the the number operators N_1 and N_2:

$$N_i|n_1, n_2\rangle = n_i|n_1, n_2\rangle, \quad i = 1, 2 \tag{71}$$

The action of the raising and lowering operators $j_1^\pm$ and $j_2^\pm$ is given by

$$j_1^+|n_1, n_2\rangle = \sqrt{F_1(n_1 + 1, n_2)}|n_1 + 1, n_2\rangle, \quad j_1^-|n_1, n_2\rangle = \sqrt{F_1(n_1, n_2)}|n_1 - 1, n_2\rangle, \tag{72}$$

and

$$j_2^+|n_1, n_2\rangle = \sqrt{F_2(n_1, n_2 + 1)}|n_1, n_2 + 1\rangle, \quad j_2^-|n_1, n_2\rangle = \sqrt{F_2(n_1, n_2)}|n_1, n_2 - 1\rangle, \tag{73}$$

where the structure functions are given by

$$F_i(n_1, n_2) = n_i[k + 1 - (n_1 + n_2)], \quad i = 1, 2. \tag{74}$$

The actions of the Cartan generators is given by

$$h_1|n_1, n_2\rangle = n_1 + \frac{1}{2}(n_2 - k)|n_1, n_2\rangle, \quad h_2|n_1, n_2\rangle = n_2 + \frac{1}{2}(n_1 - k)|n_1, n_2\rangle, \tag{75}$$

The actions of the operators (j_3^+, j_3^-) defined by (????) in terms of the pairs (j_1^+, j_1^-) and (j_2^+, j_2^-) can be determined from (72)-(73). It is easy to check

$$j_3^+|n_1, n_2\rangle = \sqrt{n_1(n_2 + 1)}|n_1 - 1, n_2 + 1\rangle, \qquad j_3^-|n_1, n_2\rangle = \sqrt{(n_1 + 1)n_2}|n_1 + 1, n_2 - 1\rangle. \tag{76}$$

3.3 Partitions of the Hilbert space

3.3.1 Partition 1

The basis of the representation space $\mathcal{F}_k$ is generated by the set $\{|n_1, n_2\rangle : n_1, n_2 \text{ ranging} \mid n_1+n_2 \leq k\}$. To find the eigenstates of the operator j_1^-, it is appropriate to decompose this space as

$$\mathcal{F}_k = \bigoplus_{l=0}^{k} \mathcal{A}_{k,l} = \bigoplus_{l=0}^{k} \{|n, l\rangle : n = 0, 1, \ldots, k - l\}.$$

The dimension $d_{k,l}$ of each subspace $\mathcal{A}_{k,l}$ is $k - l + 1$. Using the equation (???), one verifies that

$$j_1^+|k - l, l\rangle = 0,$$

We decompose the ladder operators j_1^+ and j_1^- as

$$j_1^\pm = \sum_{l=0}^{k} j_1^\pm(l).$$

The actions of the components of j_1^+ and j_1^- on the subspace $\mathcal{A}_{k,l}$

$$j_1^+(l)|n, l'\rangle = \delta_{l,l'} \sqrt{F_1(n + 1, l)}|n + 1, l\rangle, \quad j_1^-(l)|n, l'\rangle = \delta_{l,l'} \sqrt{F_1(n, l)}|n - 1, l\rangle,$$

In this partition, the action of the components of j_1^+ and j_1^- on the subspace labeled by the quantum number l leave $\mathcal{A}_{k,l}$ invariant.

3.3.2 Partition 2

The second partition related to the second mode of the states $|n_1, n_2\rangle$ writes as

$$\mathcal{F}_k = \bigoplus_{l=0}^{k} \mathcal{B}_{k,l} = \bigoplus_{l=0}^{k} \{|l, n\rangle : n = 0, 1, \ldots, k - l\}.$$

The operators j_2^- and j_2^+ can be also decomposed as

$$j_2^\pm = \sum_{l=0}^{k} j_2^\pm(l).$$

The dimension of the subspace $\mathcal{B}_{k,l}$ is $k - l + 1$. The operators $j_2^\pm(l)$ leaves invariant the subspace $\mathcal{B}_{k,l}$. Indeed, the actions of the generators $j_2^\pm(l)$ write

$$j_2^+(l)|l', n\rangle = \delta_{l,l'} \sqrt{F_2(l, n + 1)}|l, n + 1\rangle, \quad j_2^-(l)|l', n\rangle = \delta_{l,l'} \sqrt{F_2(l, n)}|l, n - 1\rangle,$$

3.3.3 Partition 3

The third possible partition which is related to the invariance of the actions of the operators $j_3^\pm$ is then given by

$$\mathcal{F}_k = \bigoplus_{l=0}^{k} \mathcal{C}_{k,l} = \bigoplus_{l=0}^{k} \{|l - n, n\rangle : n = 0, 1, \ldots, l\} \tag{77}$$

Here the subspaces $\mathcal{C}_{k,l}$ are of dimension $l+1$. The appropriate decomposition of the generators $j_3^{\pm}$ in this partition is given by

$$j_3^{\pm} = \sum_{l=0}^{k} j_3^{\pm}(l),$$

where the components $j_3^{\pm}(l)$ act on the subspace $\mathcal{C}_{k,l}$ as

$$j_3^{+}(l)|l'-n,n\rangle = \delta_{l,l'}\sqrt{(l-n)(n+1)}|l-n-1,n+1\rangle, \quad j_3^{-}(l)|l'-n,n\rangle = \delta_{l,l'}\sqrt{(l-n+1)n}|l-n+1,n-1\rangle,$$

which reflects the invariance of the subspace $\mathcal{C}_{k,l}$ under the actions of the operators $j_3^{+}(l)$ and $j_3^{-}(l)$.

3.4 The eigenstates of the lowering operators $j_i^{-}(l)$ $(i=1,2,3)$

3.4.1 Eigenstates for $j_1^{-}(l)$

The eigenstates of $j_1^{-}(l)$ satisfy the eigenvalue equation

$$j_1^{-}(l)|u_l\rangle = u_l|u_l\rangle, \quad |u_l\rangle = \sum_{n=0}^{k-l} a_n u_l^n |n,l\rangle. \tag{78}$$

Using the results obtained for $su(2)$ algebra, one shows that the eigenstates of $j_1^{-}(l)$ are

$$|u_l\rangle = \mathcal{N}(u_l,\bar{u}_l)\sum_{n=0}^{k-l}\sqrt{\frac{(k-l-n)!}{n!(k-l)!}}u_l^n|n,l\rangle \tag{79}$$

where the variable u_l satisfies

$$u_l^{k-l+1} = 0$$

and the normalization factor is given by

$$|\mathcal{N}(u_l,\bar{u}_l)|^{-2} = \sum_{n=0}^{k-l}\frac{(k-l-n)!}{n!(k-l)!}\bar{u_l}^n u_l^n$$

3.4.2 Eigenstates for $j_2^{-}(l)$

The eigenstates of $j_2^{-}(l)$ satisfy the eigenvalue equation

$$j_2^{-}(l)|v_l\rangle = \theta_l|v_l\rangle, \quad |v_l\rangle = \sum_{n=0}^{k-l} b_n v_l^n |l,n\rangle. \tag{80}$$

In this case also, the eigenstates can be derived as in the $su(2)$ case. As result, one obatains

$$|v_l\rangle = \mathcal{N}(v_l,\bar{v}_l)\sum_{n=0}^{k-l}\sqrt{\frac{(k-l-n)!}{n!(k-l)!}}v_l^n|l,n\rangle. \tag{81}$$

The variable v_l is a generalized Grassmann variable satisfying the nilpotence condition

$$v_l^{k-l+1} = 0$$

and the factor $\mathcal{N}(v_l,\bar{v}_l)$ takes the form

$$|\mathcal{N}(v_l,\bar{v}_l)|^{-2} = \sum_{n=0}^{k-l}\frac{(k-l-n)!}{n!(k-l)!}\bar{v_l}^n v_l^n$$

3.4.3 Eigenstates for $j_3^-(l)$

The eigenstates of the $E_{3d}(l)$ operator are given by

$$j_3^-(l)|w_l\rangle = w_l|w_l\rangle, \quad |w_l\rangle = \sum_{n=0}^{l} c_n w_l^n |l-n, n\rangle.$$

The solution of this eigenvalue equation is

$$|w_l\rangle = \mathcal{N}(w_l, \bar{w}_l) \sum_{n=0}^{l} \sqrt{\frac{(l-n)!}{n!l!}} w_l^n |l-n, n\rangle. \tag{82}$$

where w_l is a generalized Grassmann variable of order $l+1$:

$$w_l^{l+1} = 0,$$

and

$$\mathcal{N}(w_l, \bar{w}_l) = \sum_{n=0}^{l} \frac{(l-n)!}{n!l!} \bar{w_l}^n w_l^n$$

4 $SU(3)$ Barut-Girardello like coherent states

The $su(3)$ lowering operators j_1^- and j_2^- commute and admit a common set of eigenstates satisfying the following eigenvalue equations

$$j_1^-|\theta_1, \theta_2\rangle = \theta_1||\theta_1, \theta_2\rangle\rangle, \quad j_2^-|\theta_1, \theta_2\rangle = \theta_2||\theta_1, \theta_2\rangle\rangle \tag{83}$$

where the state $|\theta_1, \theta_2\rangle$ is

$$|\theta_1, \theta_2\rangle = \sum_{l=0}^{k} \sum_{n=0}^{k-l} C_{n,l} \theta_1^n \theta_2^l |n, l\rangle. \tag{84}$$

Using (72), the eigenvalue equation of the first mode gives the following recurrence relations

$$C_{n+1,l}\sqrt{F_1(n+1, l)} = C_{n,l}, \quad n = 0, 1, 2, \cdots k-l-1 \tag{85}$$

and the conditions

$$\theta_1^{k-l+1} \theta_2^l = 0 \tag{86}$$

for $l = 0, 1, \ldots, k-1$. Similarly, using Using (72), the second eigenvalue equation in (83) leads to the following recurrence relations

$$C_{n,l+1}\sqrt{F_2(n, l+1)} = C_{n,l} \quad n = 0, 1, 2, \cdots k-l-1 \tag{87}$$

together with the conditions (86) for $l = 0, 1, \ldots, k-1$. From the recurrence relations (87), one shows

$$C_{n,l}\sqrt{F_1(n, l)F_1(n-1, l) \cdots F_1(1, l)} = C_{0,l}. \tag{88}$$

On the other hand, using the set of recurrence relation (87), it is simple to verify that

$$C_{0,l}\sqrt{F_2(0,l)F_2(0,l-1)\cdots F_2(0,1)} = C_{0,0}. \tag{89}$$

It follows that the expansion coefficients $C_{n,l}$ are given by

$$C_{n,l} = \frac{C_{0,0}}{\sqrt{F_1(n,l)F_1(n-1,l)\cdots F_1(1,l)}\sqrt{F_2(0,l)F_2(0,l-1)\cdots F_2(0,1)}}, \tag{90}$$

and using the expressions of the structure function F_1 and F_2, one shows that

$$C_{n,l} = C_{0,0}\sqrt{\frac{(k-n-l)!}{k!\ n!\ l!}}. \tag{91}$$

Finally the eigenstates $|\theta_1,\theta_2\rangle$ write

$$|\theta_1,\theta_2\rangle = C_{0,0}\sum_{l=0}^{k}\sum_{n=0}^{k-l}\sqrt{\frac{(k-n-l)!}{k!\ n!\ l!}}\theta_1^n\theta_2^l|n,l\rangle. \tag{92}$$

where the coefficient $C_{0,0}$ can be fixed from the normalization condition of the states $|\theta_1,\theta_2\rangle$. Indeed, one gets

$$|C_{0,0}|^{-2} = \sum_{l=0}^{k}\sum_{n=0}^{k-l}\sqrt{\frac{(k-n-l)!}{k!\ n!\ l!}}\theta_1^n\bar{\theta}_1^n\theta_2^l\bar{\theta}_2^l$$

4.1 $SU(3)$ Vector coherent states

Following the vector coherent states formalism discussed in Ref. [**?**], we shall give a similar construction for $SU(3)$ coherent states in terms of generalized Grassmann variables.

4.1.1 Vector coherent states for j_1^-

To construct $SU(3)$ vector coherent states, we define the $(k+1)\times(k+1)$-matrix

$$\mathbf{U} = \mathrm{diag}(u_0, u_1, \ldots, u_k),$$

and the $(k+1)\times 1$-vector

$$[n,l] = \begin{pmatrix} 0 \\ \vdots \\ |n,l\rangle \\ \vdots \\ 0 \end{pmatrix},$$

where the $|n,l\rangle$ entry appears on the l-th line (with $l = 0,1,\ldots,k$). Let us define the $(k+1)\times 1$-vector

$$[u_l] = \mathcal{N}(u_l,\bar{u}_l)\sum_{n=0}^{k-l}\sqrt{\frac{(k-l-n)!}{n!(k-l)!}}u_l^n[n,l]. \tag{93}$$

or equivalently

$$[u_l] = \begin{pmatrix} 0 \\ \vdots \\ |u_l\rangle \\ \vdots \\ 0 \end{pmatrix}, \tag{94}$$

The states (100) are refereed as vector coherent states. In this matrix presentation, the operator j_1^- can be represented as

$$j_1^- = \mathrm{diag}\left(j_1^-(0), j_1^-(1), ..., j_1^-(k)\right).$$

It is simple to verify that j_1^- satisfies

$$j_1^-[u_l] = u_l[u_l].$$

from which one obtains the following matrix eigenvalue equation

$$j_1^- \begin{pmatrix} |u_0\rangle \\ |u_1\rangle \\ \vdots \\ |u_k\rangle \end{pmatrix} = U \begin{pmatrix} |u_0\rangle \\ |u_1\rangle \\ \vdots \\ |u_k\rangle \end{pmatrix} \tag{95}$$

?? ??

Other properties of vector phase states $[l, m, \varphi]$ can be deduced from those of phase states $|l, m, \varphi\rangle$. For instance, we obtain

- The temporal stability condition

  $$e^{-iHt}[l, m, \varphi] = [l, m, \varphi + t]$$

  for t real.

- The closure relation

  $$\bigoplus_{l=0}^{k} \sum_{m=0}^{k-l} [l, m, \varphi][l, m, \varphi]^\dagger = \mathbf{I_d},$$

  where $\mathbf{I_d}$ is the unit matrix of dimension $d \times d$ with d given by (70).

Similar vector phase states can be obtained for E_{2d} by permuting the n and l quantum numbers occurring in the derivation of the vector phase states for E_{1d}.

4.1.2 Vector coherent states for j_2^-

A similar construction of $SU(3)$ vector coherent states can be obtained for j_2^- by adopting the second partition of the representation space od $su(3)$ algebra. In this case, we define the $(k+1)\times(k+1)$-matrix

$$\mathbf{V} = \mathrm{diag}(v_0, v_1, \ldots, v_k),$$

and we define the $(k+1)\times 1$-vector

$$[v_l] = \mathcal{N}(v_l, \bar{v}_l)\sum_{n=0}^{k-l}\sqrt{\frac{(k-l-n)!}{n!(k-l)!}}v_l^n[n,l]. \tag{96}$$

or equivalently

$$[v_l] = \begin{pmatrix} 0 \\ \vdots \\ |v_l\rangle \\ \vdots \\ 0 \end{pmatrix}, \tag{97}$$

The states (100) are refereed as vector coherent states. In this matrix presentation, the lowering operator j_2^- can be represented as

$$j_2^- = \mathrm{diag}\left(j_2^-(0), j_2^-(1), ..., j_2^-(k)\right).$$

It is simple to verify that j_2^- satisfies

$$j_2^-[v_l] = v_l[v_l].$$

from which one obtains the following matrix eigenvalue equation

$$j_2^- \begin{pmatrix} |v_0\rangle \\ |v_1\rangle \\ \vdots \\ |v_k\rangle \end{pmatrix} = V \begin{pmatrix} |v_0\rangle \\ |v_1\rangle \\ \vdots \\ |v_k\rangle \end{pmatrix} \tag{98}$$

4.1.3 Vector coherent states for j_3^-

We define the $(k+1)\times(k+1)$ diagonal matrix

$$\mathbf{W} = \mathrm{diag}(w_0, w_1, \ldots, w_k),$$

and the column vector of dimension $(k+1)\times 1$

$$[[n-l, n]] = \begin{pmatrix} 0 \\ \vdots \\ |l-n, n\rangle \\ \vdots \\ 0 \end{pmatrix},$$

where the $|l-n, n\rangle$ state occurs on the l-th line (with $l = 0, 1, \ldots, k$). We introduce the vector coherent state as

$$[w_l] = \mathcal{N}(w_l, \bar{w}_l) \sum_{n=0}^{l} \sqrt{\frac{(l-n)!}{n!l!}} w_l^n [n, l]. \tag{99}$$

which can be rewritten also as

$$[w_l] = \begin{pmatrix} 0 \\ \vdots \\ |w_l\rangle \\ \vdots \\ 0 \end{pmatrix}, \tag{100}$$

The matrix representation of lowering operator j_3^- is given by

$$j_3^- = \text{diag}\left(j_3^-(0), j_3^-(1), ..., j_3^-(k)\right).$$

It is simple to verify that j_3^- satisfies

$$j_3^-[w_l] = w_l[w_l].$$

from which one obtains the following matrix eigenvalue equation

$$j_3^- \begin{pmatrix} |w_0\rangle \\ |w_1\rangle \\ \vdots \\ |w_k\rangle \end{pmatrix} = W \begin{pmatrix} |w_0\rangle \\ |w_1\rangle \\ \vdots \\ |w_k\rangle \end{pmatrix} \tag{101}$$

5 The $su(r+1)$ algebra

We introduce the Jacobson operators generating the $su(r+1)$ algebra viewed as Lie triple system. We review the construction of the corresponding Fock space.

5.1 Jacobson generators

First, let us introduce the notion of Lie triple system. Let V a vector space over a field F which is assumed to be either real or complex. The vector space V equipped vector with a trilinear mapping

$$[x, y, z] : V \otimes V \otimes V \longrightarrow V$$

is called Lie triple system if the following identities are satisfied:

$$[x, x, x] = 0,$$

$$[x, y, z] + [y, z, x] + [z, x, y] = 0,$$

$$[x, y, [u, v, w]] = [[x, y, u], v, w] + [u, [x, y, v], w] + [u, v, [x, y, w]].$$

According this definition, we will introduce the generalized A_r statistics as Lie triple system. In this respect, the algebra $\mathcal{G}$ defined by the generators a_i^+ and a_i^- $(i = 1, 2, ..., r)$ mutually commuting

$$([a_i^-, a_j^-] = [a_i^+, a_j^+] = 0)$$

and satisfying the triple relation

$$[[a_i^+, a_j^-], x_k^+] = +\delta_{jk} a_i^+ + \delta_{ij} a_k^+ \tag{102}$$

$$[[a_i^+, a_j^-], a_k^-] = -\delta_{ik} a_j^- - \delta_{ij} a_k^-, \tag{103}$$

is closed under the ternary operation

$$[x, y, z] = [[x, y], z]$$

and define a Lie triple system. The elements $a_i^\pm$ are termed Jacobson generators.

5.2 Fock representations

An Hilbertian representation is simply derived using the relations structures (102) and (103) defining $su(r + 1)$ algebra. Since, the algebra is spanned by r pairs of Jacobson generators, it is natural to assume that the Fock space $\mathcal{F}$ is given by

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^n, \tag{104}$$

where

$$\mathcal{H}^n \equiv \{|n_1, n_2, \cdots, n_r\rangle\ , n_i \in \mathbf{N}, \sum_{i=1}^{r} n_i = n > 0\}$$

and $\mathcal{H}^0 \equiv \mathbf{C}$. The action of $a_i^\pm$, on $\mathcal{F}$, are defined by

$$a_i^\pm |n_1, \cdots, n_i, \cdots, n_r\rangle \ = \sqrt{F_i(n_1, \cdots, n_i \pm 1, \cdots, n_r)} |n_1, \cdots, n_i \pm 1, \cdots, n_r\rangle \tag{105}$$

where the functions F_i are called the structure functions. Using the triple structure relations of A_r statistics, one obtains (**Donner reference**) the following expressions

$$F_i(n_1, \cdots, n_i, \cdots, n_r) = n_i(k - (n_1 + n_2 + \cdots + n_r)), \tag{106}$$

in terms of the quantum numbers $n_1, n_2, \cdots, n_r$. The dimension of the irreducible representation space $\mathcal{F}$ is determined by the condition:

$$k - (n_1 + n_2 + \cdots + n_r) > 0. \tag{107}$$

It is clear that there exists a finite number of basis states satisfying the condition $n_1 + n_2 + \cdots + n_r \leq k - 1$. The dimension is given, in this case, by $\frac{(k-1+r)!}{(k-1)!r!}$. This is exactly the dimension of the symmetric representation of $su(r)$ algebras. A quantum cannot not contain more than $(k - 1)$ particles. This condition restricting the number of particles that can be accommodated in a given state can be interpreted as a generalization of the usual exclusion Pauli. For $k = 2$, one has a collection of r independent fermions and for for large k, we have

$$[a_i^-, a_j^+] \approx k\delta_{ij} \tag{108}$$

which is equivalent to a collection of independent bosons.

6 Closing remarks

Blablabla??????????????????????

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