Graph states from phase states for multi-qudit entangled systems

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Abstract

The description of qudits in a formalism based on a generalized variant of Weyl-Heisenberg algebras is discussed. The unitary phase operators for a multi-qudit system and the corresponding phase states (the eigenstates of the phase operator) are constructed. We discuss the dynamics of multi-qudit phase states governed by a specific Hamiltonian involving one and two-body interaction which offers a remarkable connection between phase states and generalized graph states which are of paramount importance in quantum information. Another important facet of this work concerns the construction of mutually unbiased bases from phase states. Finally, entanglement aspects of some special class of phase states are examined.

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1 Introduction

The proper quantization of the phase angle of an oscillator or single-mode electromagnetic field was considered first by Dirac in 1927 [1]. However, as pointed out later by Susskind and Glogower [2], the major difficulty in formulating in a consistent way a unitary phase operator for a quantum oscillator is the infinite character of the spectrum of the number operator. To address this issue, Pegg and Barnett suggested the formalism of truncated harmonic oscillator in which a unitary phase operator can be adequately defined [3]. In this formalism, the creation, annihilation and number operators are defined in a finite (but arbitrarily large) space. This space is spanned by finite number states and its dimension is allowed to tend to infinity only after physical quantities, such as expectation values, have been calculated in the truncated Hilbert space. The only disadvantage of the Pegg-Barnett approach is the fact that the operators do not form a closed algebra. Instead of the truncated harmonic oscillator, several attempts studied other proposals to realize the Pegg-Barnett phase operator for algebras possessing finite dimensional representations. One may quote for instance, the phase operator formulations proposed for $su(2)$ and $su(1,1)$ algebras [4], q-deformed oscillator wit q root of unity [5], non linear Weyl-Heisenberg algebras [6] and recently for $su(3)$ [7, 8] and $su(2,1)$ algebras [7]. On the other hand the definition of the phase states constitutes an important component in the theory of the phase operators. The phase states set is defined as the eigenstate basis of the phase operator. For the truncated harmonic oscillator, the properties of the eigenstates of Pegg-Barnett phase operator coincide with those normally associated with a phase [3]. Phase states associated with the phase operators of the algebras $su(2)$, $su(1,1)$, $su(3)$ and $su(2,1)$ and their properties were investigated in [4, 7].

In this paper, we shall study the construction of graph states from phase states. This will establish a relation between the formalisms of graph states and phase states which may at first sight seem unrelated. Graph states were extensively investigated and widely used in the context of quantum information theory. The interest on such states is especially due to their special and very informative entanglement structures [9, 10]. In addition, certain kinds of graph states (as for instance cluster states) can be used as resource states for measurement-based quantum computing [11, 12, 13, 14]. They have also found numerous applications in several areas of quantum information processing such as quantum error correction [15, 16] and secure quantum communication [17]. Graph states have been used in entanglement purification [18] and in examining the properties of fractional braiding statistics of anyons [19]. Furthermore, various types of multi-partite states like Greenberger-Horne-Zeilinger (GHZ) states and cluster states play an essential role in fundamental tests of quantum non-locality [20, 21]. This explains the great deal of effort devoted to the theoretical analysis of their properties [22, 23] and their experimental generation and manipulation [24, 25, 26, 27]. Now it is commonly accepted that the graph state formalism is viewed as promising tool providing a classification (although not exhaustive) of multi-qubit systems entanglement [28, 29].

As we shall be essentially concerned with the relation between quantum phase states and generalized graph states, we first discuss the realization of a multi-qudit system. In particular, we give the algebraic realization of a two level via a modified Weyl-Heisenberg algebra. In such realization qubits satisfy the exclusion Pauli principle like fermions and behaves collectively like bosons. The generalization to d-level systems is also investigated in a scheme where a qudit can be realized as the composition of $(d-1)$ qubits. For each qudit, we construct, the associated phase operator and the corresponding phase states. The phase states of the system are defined as the common eigenstates of the phase operator of each qudit. As by-product, we give the phase states for the whole system. In particular, by assuming that the dynamics of the whole system is governed by a quadratic Hamiltonian, we show that the graph states introduced in [30] can be generated from the phase states of a qudit ensemble. This correspondence provides us with the appropriate tool to examine the entanglement properties of phases states. In fact, the phase states are locally equivalent to generalized graph states. A second facet of this work concerns the relation between the phase states evolving under a quadratic Hamiltonian and the concept of mutually unbiased bases (MUBs). Due to their prominent role in several areas of quantum information, such bases were recently investigated from an angular momentum approach [31, 32]. They maximize the information extraction per measurement and minimize the effects of statistical errors [33, 34]. Besides their special use in the context of state tomography, mutually unbiased states have been shown of special relevance to enhance the performances of quantum key distribution protocols [35, 36] and in the so-called mean king problem [37, 38]. Moreover, they are useful for entanglement detection [39] and have interesting connections with symmetric informationally complete positive-operator-valued measures [40] and complex t-designs [41, 42].

This paper is organized as follows. In Section 2 we discuss the algebraic description of a d-level quantum system (qudit) using a particular variant of generalized Weyl-Heisenberg algebra. In section 3, we define the unitary phase operator and we determine the phase states for a collection of n qudits. The properties of phase operators and phase states are examined and in particular we discuss the discrete Fourier transformation between the phase state basis and the computational basis. In section 4, we assume that the Hamiltonian governing the dynamics of a collection of identical qudits is the sum of one and two body interaction. The two-body term plays an important role in establishing the connection between phase states and generalized graph states. Section 4 deals with the derivation of mutually unbiased bases from phase states is developed in section 5. The basic entanglement proprieties of phase states in relation with graph states are examined in section 6. Concluding remarks close this paper.

2 Qudits and generalized Weyl-Heisenberg algebra

Dealing with bosonic and fermionic many particles states is simplified by considering the algebraic structures of the corresponding raising and lowering operators. For bosons the creation and annihilation operators satisfy the commutations relations

$$[b_i^-, b_j^+] = \delta_{ij}\mathbb{I}, \qquad [b_i^-, b_j^-] = [b_i^+, b_j^+] = 0. \tag{1}$$

where the unit operator $\mathbb{I}$ commute with the creation and annihilation operators b_i^+ and b_i^i. On the hand, fermions are specified by the following anti-commutation relations

$$\{f_i^-, f_j^+\} = \delta_{ij}\mathbb{I}, \qquad \{f_i^+, f_j^+\} = \{f_i^-, f_j^-\} = 0. \tag{2}$$

The Fock spaces for bosons and fermions give the realizations of the associated commutation and anti-commutation relations and subsequently the symmetric and antisymmetric waves functions. The properties of Fock states follow from the commutation and anti-commutation relations which imposes only be one particle in each state for fermions (two dimension) and multiple particles for bosons (infinite dimension). Following Wu and Vidal there is a crucial difference between fermions and qubits. In fact, a qubit is a vector in a two dimensional Hilbert space like fermions and the Hilbert space of a multi-qubit system has a tensor product structure like bosons. In this respect, the raising and lowering operators commutation rules for qubits are neither specified by relations of bosonic type (1) nor of fermionic type (2).

2.1 Qubit algebra from generalized Weyl-Heisenberg algebra

The qubits appear like objects which exhibits both bosonic and fermionic properties so that they cannot described by Fermi-like or Bose-like operators. An alterative way for the algebraic description of qubits and qudits, is possible by resorting the formalism of generalized Weyl-Heisenberg algebras. We by $|0\rangle$ the ground state and $|1\rangle$ the excited state of a two-level system and we define the lowering, raising and number operators by

$$a^- = |0\rangle\langle 1|, \qquad a^+ = |1\rangle\langle 0|, \qquad N = |1\rangle\langle 1|. \tag{3}$$

They satisfy the following commutation relations

$$[a^-, a^+] = \mathbb{I} - 2N, \qquad [N, a^+] = -a^+, \qquad [N, a^-] = +a^-. \tag{4}$$

where $\mathbb{I}$ is the unit operator. In this scheme, the qubit is described by a modified bosonic algebra and the creation and the annihilation operators satisfy the nilpotency condition: $(a^+)^2 = (a^-)^2 = 0$ like Fermi operators. This algebra turns out to be a particular case of the generalized Weyl-Heisenberg algebra $\mathcal{A}_\kappa$ introduced in [6, 7] ($\kappa \in \mathbb{R}$). The structure relations of the algebra $\mathcal{A}_\kappa$ are defined by

$$[a^-, a^+] = \mathbb{I} + 2\kappa N, \qquad [N, a^+] = -a^+, \qquad [N, a^-] = +a^- \tag{5}$$

where $\kappa \in \mathbb{R}$. For $\kappa < 0$, the Hilbert space representations is finite dimensional. The algebra $\mathcal{A}_\kappa$ reduces to the algebra (4) for $\kappa = -1$. It must be stressed that the commutation relations (4) coincide with ones defining the algebra introduced in [43] to introduce an alternative algebraic description of qubits instead of the parafermionic formulation considered in [44].

To describe a l-qubit system, we consider l copies of the algebra $\mathcal{A}_{-1}$ generated by the raising and lowering operators a_i^+ and a_i^-, the number operators N_i and the unit operator $\mathbb{I}$ such that they satisfy the relations

$$[a_i^-, a_j^+] = (\mathbb{I} - 2N_i)\,\delta_{ij}, \qquad [N_i, a_j^+] = -\delta_{ij} a_j^+, \qquad [N_i, a_j^-] = +\delta_{ij} a_j^- \qquad [a_i^-, a_j^-] = [a_i^+, a_j^+] = 0. \tag{6}$$

where $i = 1, 2, \cdots, l$. Let denote by $\mathcal{H}_i = \{|k_i\rangle, \quad k_i = 0, 1\}$ the Hilbert space for the qudit i. In view of the relation $[a_i^-, a_j^+] = 0$ for $i \neq j$, the n-qubit Hilbert space has the following tensor product structure

$$\mathcal{H}(l) = \bigotimes_{i=1}^{l} \mathcal{H}_i = \{|n_1, n_2, \cdots, n_l\rangle, \quad k_i = 0, 1\}.$$

like bosons and $\{|n_1, n_2, \cdots, n_l\rangle, n_i = 0, 1 \quad \text{for} \quad i = 1, 2, \cdots, l\}$ is its orthonormal basis.

2.2 Qudit algebra and Dicke states

For $(d-1)$-qubits, the Hilbert space is given by

$$\mathcal{H}(d-1) = \bigotimes_{i=1}^{d-1} \mathcal{H}_i = \{|n_1, n_2, \cdots, n_{d-1}\rangle, \quad k_i = 0, 1\}.$$

The corresponding creation and annihilation operators $a_i^{\pm}$ $(i = 1, 2, \cdots, d-1)$ satisfy the structure relations (6). We define the collective lowering and raising operators in the Hilbert space $\mathcal{H}(d-1)$ as follows

$$A^- = \sum_{i=1}^{d-1} a_i^- \qquad A^+ = \sum_{i=1}^{d-1} a_i^+ \tag{7}$$

in terms of the creation and annihilation operators a_i^+ and a_i^-. Here and in the following the index i refers to the system the operator is acting on, e.g.

$$a_i^{\pm} \equiv \mathbb{I} \otimes \cdots \mathbb{I} \otimes a_i^{\pm} \otimes \mathbb{I} \otimes \cdots \mathbb{I}.$$

It is simple to see that the state $|0, 0, \cdots, 0\rangle \equiv |d-1, 0\rangle$ satisfies $A^-|d-1, 0\rangle = 0$. Furthermore, using the commutation relations (6), one gets the following nilpotency conditions

$$(A^-)^d = 0 \qquad (A^+)^d = 0$$

which extends the Pauli exclusion principle for ordinary qubits (i.e., $d = 2$). The actions of the operators A^- and A^+ on the Hilbert space $\mathcal{H}(d-1)$ can be determined from the standard actions of the fermionic operators a_i^- and a_i^+. Using a recursive procedure, one verifies that repeated applications of the raising operator A^+ on the vacuum $|0, 0, \cdots, 0\rangle$ gives

$$(A^+)^k|d-1, 0\rangle = \sqrt{\frac{k!(d-k)!}{(d-1-k)!}}\ |d-1, k\rangle \tag{8}$$

where the vectors $|d-1, k\rangle$ are the symmetric Dicke states with k excitations $(k = 0, 1, 2, \cdots, d-1)$. They are defined by

$$|d-1, k\rangle = \sqrt{\frac{k!(d-1-k)!}{(d-1)!}} \sum_{\sigma \varepsilon S_{d-1}} |\underbrace{0, 0, \cdots, 0}_{d-k-1}, \underbrace{1, 1, \cdots, 1, 1}_{k}\rangle \tag{9}$$

where S_{d-1} is the permutation group of $(d-1)$ objets. The Dicke states generate an orthonormal basis of the symmetric Hilbert subspace $\mathcal{H}_s \subset \mathcal{H}$ with $\dim \mathcal{H}_s = d$. To write the explicit actions of the ladder operators $A^{\pm}$, we introduce the structure function defined by $F(k) = k(d-k)$. The equation (8) rewrites as

$$(A^+)^k|d-1, 0\rangle = \sqrt{F(k)!}\ |d-1, k\rangle \tag{10}$$

where $F(k)! = F(k)F(k-1)\cdots F(1)$ and $F(0) = 1$. After some algebra, it is simple to verify that

$$A^+|d-1, k\rangle = \sqrt{F(k+1)}\ |d-1, k+1\rangle, \qquad A^-|d-1, k\rangle = \sqrt{F(k)}\ |d-1, k-1\rangle \tag{11}$$

and the action of the creation and annihilation operators on the vectors $|d-1, 0\rangle$ and $|d-1, d-1\rangle$ gives

$$A^-|d-1, 0\rangle = 0 \qquad A^+|d-1, d-1\rangle = 0. \tag{12}$$

The number operator A is defined as

$$A|d-1,k\rangle = k\ |d-1,k\rangle. \tag{13}$$

The qudit operators A^+, A^- and A satisfy the commutation rules

$$[A^+, A^-] = (d-1)\mathbb{I} - 2A, \qquad [A^+, A] = A^+, \qquad [A^-, A] = -A^- \tag{14}$$

Using the commutation relation $\left[a_i^+, a_j^-\right] = 0$ for $i \neq j$, it is simple to verify that

$$\left[A^+, A^-\right] = \sum_{i,j} \left[a_i^+, a_j^-\right] = \sum_i \left[a_i^+, a_i^-\right]$$

and the operator A can be expressed as

$$A = \sum_{i=1}^{d-1} N_i$$

where N_i is the single qubit number operator ($N_i|0\rangle_i = 0$ and $N_i|1\rangle_i = 1|1\rangle_i$). It is remarkable that the creation and annihilation operator A^+ and A^- close the following trilinear relation commutation

$$[A^-, [A^+, A^-]] = 2A^-, \qquad [A^+, [A^+, A^-]] = -2A^+$$

characterizing a parafermion. Note also that the definition (7) is similar to Green decomposition in the construction of parafermions from ordinary fermions. Therefore, the operators A^+, A^- and A satisfying the relations (14) provide a simple algebraic description of d-level quantum systems (qudit). We notice also that by re-scaling the generators of the algebra (14)

$$A^{\pm} \longrightarrow \frac{A^{\pm}}{\sqrt{d-1}},$$

one recovers the algebra $\mathcal{A}_\kappa$ with $\kappa = 1/(1-d)$. This shows clearly that the generalized Weyl-Heisenberg provides the appropriate tools to describe qudit systems. In particular, this realization expresses the Hilbert states of a qudit system in terms of Dicke states of $(d-1)$ qubits. In this way, the global properties of the qubit ensemble are encoded in the qudit system. To close this section that the algebraic description of qudit systems provides us with the necessary ingredients to define the phase operator for a qudit system and subsequently the phase states for a collection of identical qudits. This constitutes the main issue of the next section.

3 Phase operators and phase states

In the following we define phase operators (or more precisely, a unitary exponential operators) for a multipartite system comprising n qudits. This definition originates from the polar decomposition of the raising and lower operators A_i^+ and A_i^- associated with each qudit i ($i = 1, 2, \cdots, n$). We shall employ the algebraic description and in particular the Hilbert representation of the generalized Weyl-Heisenbeg algebra discussed in the previous section. Furthermore, we construct the phase states, i.e., the eigenstates of the phase operators. The properties of the phase operators and their eigenstates are discussed.

3.1 Phase operators

An ensemble of n qudits can be described by the n commuting algebras $\{A_i^+, A_i^-; \quad i = 1, 2, \cdots, n\}$ satisfying the structure relations (similar to the relations (14))

$$[A_i^+, A_j^-] = \Big((d-1)\mathbb{I} - 2A_i\Big)\ \delta_{ij}, \quad [A_i^+, A_j^+] = [A_i^-, A_j^-] = 0,$$

Here also the commutation relations $[A_i^+, A_j^-] = 0$ for $i \neq j$ infers to the Hilbert space a tensor product structure

$$\mathcal{H}_s^{\otimes n} = \{|d-1, k_1\rangle \otimes |d-1, k_2\rangle \cdots |d-1, k_n\rangle \equiv |k_1, k_2, \cdots, k_n\rangle\}$$

where the Dicke states $|d-1, k_i\rangle$ are given by (9) with $i = 1, 2, \cdots, n$. Using the polar decomposition of the qudit raising and lowering operators

$$A_i^- = E_i \ \sqrt{A_i^+ A_i^-} \qquad\qquad A_i^+ = \sqrt{A_i^+ A_i^-}\ E_i^\dagger, \tag{15}$$

one verifies that the operators E_i act on the Hilbert space $\mathcal{H}_s^{\otimes n}$ as follows

$$E_i|k_1, k_2, \cdots, k_i, \cdots, k_n\rangle = |k_1, k_2, \cdots, k_i - 1, \cdots, k_n\rangle, \qquad E_i|k_1, k_2, \cdots, 0, \cdots, k_n\rangle = |k_1, k_2, \cdots, d-1, \cdots, k_n\rangle \tag{16}$$

and equivalently the actions of the conjugate of the phase operators give

$$E_i^\dagger|k_1, k_2, \cdots, k_i, \cdots, k_n\rangle = |k_1, k_2, \cdots, k_i + 1, \cdots, k_n\rangle, \qquad E_i^\dagger|k_1, k_2, \cdots, d-1, \cdots, k_n\rangle = |k_1, k_2, \cdots, 0, \cdots, k_n\rangle \tag{17}$$

From this definition, the phase operators E_i are indeed unitary

$$E_i^\dagger E_i = E_i E_i^\dagger = \mathbb{I}.$$

By using the actions (16) and (13), one obtains the commutation relations between the phase operators E_i and the number operators A_i

$$[E_i, A_j] = \delta_{ij}\ E_i.$$

Furthermore, they satisfy the following relations

$$[E_i, E_j] = 0, \qquad [E_i, E_j^\dagger] = 0$$

and the periodicity relation

$$(E_i)^d = \mathbb{I}, \qquad (E_i^\dagger)^d = \mathbb{I}. \tag{18}$$

3.2 Phase states

Now, we consider the explicit determination of the phase states for a multi-qudit system. First, note that the phase operators E_i are pairwise commuting and can be diagonalized in the same basis. The common eigen-basis is given by the product states

$$|z_1, z_2, \cdots, z_i, \cdots, z_n\rangle = |z_1\rangle \otimes |z_2\rangle \cdots \otimes |z_i\rangle \cdots \otimes |z_n\rangle$$

where the vectors $|z_i\rangle$ $(i = 1, 2, \cdots, n)$ are the solutions of the following eigenvalue equations

$$E_i|z_i\rangle = z_i|z_i\rangle \tag{19}$$

The complex variable z_i is the eigenvalue associated with the qudit i. To solve this equation, we expand the state $|z_i\rangle$ as follows

$$|z_i\rangle = \sum_{k_i=0}^{d-1} C_{k_i} z_i^{k_i} |k_i\rangle$$

and reporting it in the equation (19) one gets

$$C_{k_i+1} = C_{k_i} \quad \text{for} \quad k_i = 0, 1, \cdots, d-2, \quad C_0 = z_i^d C_{d-1} \tag{20}$$

From this recurrence relation one shows

$$C_0 = C_1 = \cdots = C_{d-1} \quad \text{and} \quad C_0 = z_i^d C_{d-1}. \tag{21}$$

It follows the complex number z_i is subjected to the nilpotency condition $z_i^d = 1$ which is consistent with the periodicity property of the phase operators E_i (18). Thus, each variable z_i takes discrete values on the unit circle

$$z_i = \omega^{m_i}, \quad m_i = 0, 1, \cdots, d-1 \qquad \omega = e^{i\frac{2\pi}{d}}. \tag{22}$$

By imposing the normalization condition of the common eigenstates $|z_i\rangle$ of the phase operators E_i, one has $C_0 = \frac{1}{\sqrt{d}}$. The eigenvector $|z_i\rangle$, which is now labeled by the integers $m_i \in \mathbb{Z}/d\mathbb{Z}$, writes

$$|m_i\rangle = \frac{1}{\sqrt{d}} \sum_{k_i=0}^{d-1} \omega^{m_i k_i} |k_i\rangle$$

Hence, the eigenstates $|z_1, z_2, \cdots, z_n\rangle$ take the following form

$$|m_1, m_2, \cdots, m_n\rangle = \frac{1}{\sqrt{d^n}} \sum_{k_1, k_2, \cdots, k_n} \omega^{m_1 k_1} \omega^{m_2 k_2} \cdots \omega^{m_n k_n} |k_1, k_2, \cdots, k_n\rangle. \tag{23}$$

The phase states (23) are now labeled by the set of parameters $(m_1, m_2, \cdots, m_n)$ and satisfy:

$$E_i|m_1, m_2, \cdots, m_n\rangle = \omega_i^{m_i} |m_1, m_2, \cdots, m_n\rangle. \tag{24}$$

It is interesting to notice that the states $|m_1, m_2, \cdots, m_n\rangle$ correspond to an ordinary discrete Fourier transform of the basis $\{|k_1, k_2, \cdots, k_i, \cdots, k_n\rangle : k_i = 0, 1, \ldots, d-1\}$ of the Hilbert space $\mathcal{H}_s^{\otimes n}$. The phase states $|m_1, m_2, \cdots, m_n\rangle$ have the following remarkable properties:

(i) The equiprobability relation

$$|\langle k_1, k_2, \cdots, k_n|m_1, m_2, \cdots, m_n\rangle| = \frac{1}{\sqrt{d^n}} \qquad m_i \in \mathbb{Z}/d\mathbb{Z}. \tag{25}$$

(ii) The ortho-normalization relation

$$\langle m_1, m_2, \cdots, m_n|m'_1, m'_2, \cdots, m'_n, \rangle = \delta_{m_1,m'_1} \delta_{m_2,m'_2} \cdots \delta_{m_n,m'_n} \qquad m_i, m'_i \in \mathbb{Z}/d\mathbb{Z}. \tag{26}$$

It follows that the set of phase states $\{|m_1, m_2, \cdots, m_n\rangle; m_i \in \mathbb{Z}/d\mathbb{Z}\}$ form a basis of the Hilbert space which are obtained from the computational basis $\{|k_1, k_2, \cdots, k_n\rangle; k_i \in \mathbb{Z}/d\mathbb{Z}\}$. As consequence of the property **(i)**, they are mutually unbiased. We recall that two orthonormal bases are mutually unbiased if all mutual scalar products, between any element of the first basis with an arbitrary element of the second basis, are equals. The generation of other mutually unbiased bases can be realized by Hamiltonian evolutions involving non linear terms. This issue is discussed in Section 5.

4 Phase states evolution and generalized graph states

Graph states are quantum states of a quantum system comprising multi-partite subsystems. For a system of n constituents, each graph state is associated with a graph fully characterized by a symmetric $n \times n$ matrix. The off-diagonal matrix elements encode the pairwise interaction between the different components of the quantum system. Graph states provide a superb resource in several areas of quantum information science. Indeed, graph states offer a graphical representation to tackle the the intriguing questions related to aspects of multi-particle entanglement in terms of the adjacency matrix [22]. Graph states play a prominent role in quantum-gate-based quantum computation and quantum error correction [45] and in investigating quantum secret sharing [46]. In this section we address the interesting question concerning the derivation of generalized graph states from the phase states of a multi-qudit system. The generation of generalized graph is approached from the dynamical evolution of the phase states. This evolution is governed by a specific Hamiltonian involving two-particle interaction which ensures the generation of entangled phase states.

4.1 A generalized Hamiltonian of a collection of n qudits

Since the creation and annihilation operators A_i^+ and A_i^- $(i = 1, 2, \cdots, n)$ represent the dynamical variables associated with a n-qudit system, the Hamiltonian H governing the corresponding dynamics is naturally expressed in terms of A_i^+ and A_i^-. The evolution of the phase states (23) can be captured by the relation

$$e^{-itH}|m_1, m_2, \cdots, m_n\rangle \equiv |m_1, m_2, \cdots, m_n, t\rangle. \tag{27}$$

In general, the generation of entangled states in multipartite systems can be treated by Hamiltonian evolution involving nonlinear terms. Let us consider the time evolution is dictated by a quadratic Hamiltonian which is the sum of two terms

$$H = \sum_{i=1}^{n} a_{ii} H_i + H_{\rm int} \tag{28}$$

where

$$H_i = A_i^+ A_i^-, \qquad H_{\rm int} = \sum_{i<j} a_{ij} A_i \otimes A_j. \tag{29}$$

where the coefficients a_{ii} and a_{ij} are reals and $a_{ij} = a_{ji}$. The single qudit Hamiltonian H_i is a natural generalization of the ordinary harmonic oscillator. The second term in (28) includes the qubit-qubit interaction and the parameters a_{ij} denote the coupling strength between the qudits i and j.

This special form of the Hamiltonian H (28) is interesting for two reasons. First, the quadratic spectrum of H is the key tool which will used later to construct mutually unbiased bases from the states $|m_1, m_2, \cdots, m_n, t\rangle$ [6]. Second, the pairwise interaction term establish a link between two qudits. This allows us as we shall discuss hereafter the derivation of the generalized graph states from the evolved phase states. The qudit-qudit Hamiltonian $H_{\rm int}$ acts an entangling operator on the phase states and generates entangled phase states. The Hamiltonian H is diagonal in the computational basis:

$$H|k_1, k_2, \cdots, k_n\rangle = E_{k_1,k_2,\cdots,k_n}|k_1, k_2, \cdots, k_n\rangle \tag{30}$$

where the energy eigenvalues are given by

$$E_{k_1,k_2,\cdots,k_n} = \left[\sum_{i}^{n} a_{ii}k_i(d-k_i) + \sum_{i<j} a_{ij}k_ik_j\right].$$

The evolved phase states write explicitly as

$$|m_1, m_2, \cdots, m_n, t\rangle = \frac{1}{\sqrt{d^n}} \sum_{k_1,k_2,\cdots,k_n} \omega^{m_1k_1}\omega^{m_2k_2}\cdots\omega^{m_nk_n}e^{-itE_{k_1,k_2,\cdots,k_n}}|k_1, k_2, \cdots, k_n\rangle \tag{31}$$

The overlap between two evolved phase states $|m'_1, m'_2, \cdots, m'_n, t'\rangle$ and $|m_1, m_2, \cdots, m_n, t\rangle$ reads

$$\langle m_1, m_2, \cdots, m_n, t|m'_1, m'_2, \cdots, m'_n, t'\rangle = \frac{1}{d^n} \sum_{k_1,k_2,\cdots,k_n} e^{\sum_{j=1}^{n}(\frac{2\pi i}{d}(m_j-m'_j)k_j)}e^{-i(t'-t)E_{k_1,k_2,\cdots,k_n}}. \tag{32}$$

It is remarkable that the unitary evolution operator factorizes as

$$e^{-itH} = \prod_{i\leq j}\left(U_{ij}(t)\right)^{b_{ij}} \qquad i, j = 1, 2, \cdots, n \tag{33}$$

where the elements of the $n \times n$ matrix b are defined by

$$b_{ii} = 2a_{ii} \quad \text{and} \quad b_{ij} = a_{ij} \quad \text{for} \quad i \neq j,$$

in terms of the matrix elements a_{ij} entering in the expression H (28). The one and two body operators $U_{ii}(t)$ and $U_{ij}(t)$ with $i \neq j$ are defined by

$$U_{ii}(t) = e^{-i\frac{t}{2}A_i(d-A_i)} \quad \text{and} \quad U_{ij}(t) = e^{-itA_iA_j} \tag{34}$$

where the operators A_i $(i = 1, 2, \cdots, n)$ are the number operators. Using the factorized form of the evolution operator (33) together with the definitions (34), one verifies that the evolved phase states write as

$$|m_1, m_2, \cdots, m_n, t\rangle \equiv |\overrightarrow{m}, t\rangle = \omega^{\overrightarrow{m}.\overrightarrow{A}}\prod_{i\leq j}\left(U_{ij}(t)\right)^{b_{ij}}|+, n\rangle, \tag{35}$$

where $\overrightarrow{m}.\overrightarrow{A} = m_1A_1 + m_2A_2 + \cdots + m_nA_n$ and the state $|+, n\rangle$ is given by

$$|+, n\rangle = |+\rangle^{\otimes n} = \left[\frac{1}{\sqrt{d}}\sum_{k_i=1}^{d-1}|k_i\rangle\right]^{\otimes n} = \frac{1}{\sqrt{d^n}}\sum_{k_1,k_2,\cdots,k_n}|k_1, k_2, \cdots, k_n\rangle. \tag{36}$$

The expression (35) turns out to be of special relevance in the derivation of graph states from evolved phase states. This is discussed in what follows.

4.2 Graph states from evolved phase states

To derive generalized phase graph from the evolved phase states (31), we first recall briefly some elements of graph states formalism of relevance for our purpose. For n-qubit system, a pure graph state is a state in the Hilbert space $\{|0\rangle, |1\rangle\}^{\otimes n}$. Graph states are associated with mathematical graph $G = (V, E)$ where V is a finite set of vertices and $E \subset V \times V$ is a set of edge. Each vertex corresponds to a qubit and each edge to an entangling gate connecting some vertex pairs. Here, we shall consider undirected graphs (i.e, the edges are unordered pairs of vertices) with multiple edges as well as self-loops. We can describe an undirected graph by the elements of a $n \times n$ symmetric matrix A where the diagonal entries A_{ii} correspond to the self-loops and the off-diagonal elements A_{ij} are given by the number of the edges linking two vertices. Such a matrix is called an adjacency matrix of the graph G [46, 47, 48].

The generalized graph states are defined be means of a $n \times n$ adjacency matrix A with entries in $\mathbb{Z}/d\mathbb{Z}$ with d a prime number. Each vertex corresponds to a vector in the finite dimensional Hilbert space $\{|0\rangle, |1\rangle, \cdots, |d-1\rangle\}$. As for qubit systems, the diagonal elements A_{ii} of the matrix A give the number of self-loops around the vertex i and the off-diagonal elements A_{ij} represent the number of edges linking the vertices i and j. Given an adjacency matrix, the generalized graph states can be constructed by adopting the picture interaction via one and two qudit phase gates. We define the generalized graph state as

$$|G\rangle = \prod_{i \leq j} \left(U_{i,j} \right)^{A_{ij}} |+, n\rangle \tag{37}$$

where $|+, n\rangle$ is the n-qudit state given by (36) and the unitary operators $U_{i,i}$ and $U_{i,j}$ $(i \neq j)$ are defined by

$$U_{i,i}|k_i\rangle = \omega^{\frac{1}{2}k_i(d-k_i)}|k_i\rangle, \qquad U_{i,j}|k_i, k_j\rangle = \omega^{k_i k_j}|k_i, k_j\rangle, \tag{38}$$

The one and two body gate phase $U_{i,i}$ and $U_{i,j}$ can be obtained from the operators $U_{i,i}(t)$ and $U_{i,j}(t)$ given by (34). This will allow us to establish relationship between phases states and graph states. We consider the situation where t takes the special values

$$t = \frac{2\pi}{d}\ (d - p) \quad \text{with} \quad p \in \mathbb{Z}/d\mathbb{Z}.$$

In this case, the phase states given in (31) take the form

$$|m_1, m_2, \cdots, m_n, p\rangle = \frac{1}{\sqrt{d^n}} \sum_{k_1, k_2, \cdots, k_n} \omega^{m_1 k_1} \omega^{m_2 k_2} \cdots \omega^{m_n k_n} \omega^{p\left[\sum_{i=0}^{n} a_{ii} k_i (d-k_i) + \sum_{i<j} a_{ij} k_i k_j\right]} |k_1, k_2, \cdots, k_n\rangle \tag{39}$$

Now, we introduce the $n \times n$ matrix A whose diagonal A_{ii} and off diagonal A_{ij} matrix elements are defined by

$$A_{ii} = 2p\ a_{ii}, \quad A_{ij} = p\ a_{ij}. \tag{40}$$

The quantities a_{ii} and a_{ij} are the physical parameters entering in the expression of the Hamiltonian describing the multi-qudit system (28) and A_{ii} and A_{ij} will denote the elements of the adjacency matrix $\{A_{ij}, i, j = 1, 2, \cdots, n\}$ defining the graph states which are in correspondence with the muli-partite phase states (39). For

$A_{ii} \in \mathbb{Z}/d\mathbb{Z}$ and $A_{ij} \in \mathbb{Z}/d\mathbb{Z}$, the evolved phase states (39) are graph states of type (37) which are fully specified by the adjacency matrix A (40). The phase states (39) can be written as

$$|m_1, m_2, \cdots, m_n, p\rangle = |G_A(m_1, m_2, \cdots, m_n)\rangle \tag{41}$$

where $|G_A(m_1, m_2, \cdots, m_n)\rangle$ are the graph states associated to the adjacency matrix $A_{ij}, i, j = 1, 2, \cdots, n$ given by

$$|G_A(m_1, m_2, \cdots, m_n)\rangle = \frac{1}{\sqrt{d^n}} \sum_{k_1, k_2, \cdots, k_n} \omega^{m_1 k_1} \omega^{m_2 k_2} \cdots \omega^{m_n k_n} \omega^{\frac{1}{2}[\sum_i A_{ii} k_i (d-k_i) + 2\sum_{i<j} A_{ij} k_i k_j]} |k_1, k_2, \cdots, k_n\rangle \tag{42}$$

Using the definitions (38), it is simple to check

$$\prod_{i\leq j} U_{ij}^{A_{ij}} |+\rangle^{\otimes n} = \frac{1}{\sqrt{d^n}} \sum_{k_1, k_2, \cdots, k_n} \omega^{\frac{1}{2}[\sum_i A_{ii} k_i (k_i-1) + 2\sum_{i<j} A_{ij} k_i k_j]} |k_1, k_2, \cdots, k_n\rangle. \tag{43}$$

It follows that the phase states (39) rewrite in terms of one qudit operators U_i and two qudit controlled operators $U_{i,j}$ as

$$|G_A(m_1, m_2, \cdots, m_n)\rangle = \prod_{i=1}^{n} \omega^{m_i A_i} \prod_{i\leq j} U_{ij}^{A_{ij}} |+\rangle^{\otimes n} \tag{44}$$

where A_i is the number operator for the qudit i. The overlap between two generalized graph states write

$$\langle G_{A'}(m'_1, m'_2, \cdots, m'_n)|G_A(m_1, m_2, \cdots, m_n)\rangle = \langle +, n| \prod_{i=1}^{n} \omega^{(m_i - m'_i) A_i} \prod_{i\leq j} U_{ij}^{A_{ij} - A'_{ij}} |+, n\rangle. \tag{45}$$

Using this overlap, we shall investigate the mutual unbiasedness for n-qudit systems.

5 Mutually Unbiased Bases from phase states

The multi-partite temporally stable phase states can be used to derive mutually unbiased bases associated with a collection of n qudits whose dynamics is governed by a quadratic hamiltonian of type (28). Let us recall that two orthonormal bases $\{|a\alpha\rangle : \alpha = 0, 1, \ldots, d-1\}$ and $\{|b\beta\rangle : \beta = 0, 1, \ldots, d-1\}$ in a d-dimensional Hilbert space (with an inner product $\langle\,|\,\rangle$) are said to be mutually unbiased iff

$$|\langle a\alpha | b\beta\rangle| = \delta_{a,b}\delta_{\alpha,\beta} + \frac{1}{\sqrt{d}}(1 - \delta_{a,b}). \tag{46}$$

For fixed d, it is known that the number of mutually unbiased bases is less than or equal to $d+1$ and this number is exactly $d+1$ when d is the power of a prime number [50, 51].

5.1 Mutual unbiasedness of generalized graph states

To discuss the mutual unbiasedness of generalized graph states and the construction of MUBs using evolved phase states (39), we consider the situation where the elements A_{ii} and A_{ij}, which take discrete values in $\mathbb{Z}/d\mathbb{Z}$, of the form

$$A_{ii} = p_i \in \mathbb{Z}/d\mathbb{Z}, \qquad A_{ij} = p_{ij} \in \mathbb{Z}/d\mathbb{Z}. \tag{47}$$

To simplify our notations, we denote the graph states $|G_A(m_1, m_2, \cdots, m_n)\rangle$ by $|\overrightarrow{m}, A\rangle$ so that

$$|\overrightarrow{m}, A\rangle = \frac{1}{\sqrt{d^n}} \sum_{k_1,k_2,\cdots,k_n} \omega^{m_1k_1}\omega^{m_2k_2}\cdots\omega^{m_nk_n}\omega^{\frac{1}{2}[\sum_i p_ik_i(d-k_i)+2\sum_{i<j}p_{ij}k_ik_j]}|k_1, k_2, \cdots, k_n\rangle \tag{48}$$

The overlap between two phase states of type (48) is given by

$$\langle\overrightarrow{m}, A|\overrightarrow{m}', A'\rangle = \frac{1}{d^n} \sum_{k_1,k_2,\cdots,k_n} \omega^{\sum_{i=1}^n (m_i-m_i')k_i}\omega^{\frac{1}{2}\left[\sum_{i=1}^{i=n}(p_i-p_i')k_i(d-k_i)+2\sum_{i<j}(p_{ij}-p_{ij}')k_ik_j\right]}. \tag{49}$$

It can be can be cast in the following closed form which

$$\langle\overrightarrow{m}, A|\overrightarrow{m}', A'\rangle = \frac{1}{d^n} \sum_{k_1,k_2,\cdots,k_n} e^{\frac{i\pi}{d}(\overrightarrow{v}^t.\overrightarrow{k}+\overrightarrow{k}^t.D.\overrightarrow{k})} \tag{50}$$

where $\overrightarrow{k} = (k_1, k_2, \cdots, k_n)^t$, $\overrightarrow{v} = (v_1, v_2, \cdots, v_n)^t$ with $v_i := 2(m_i - m_i') - (p_i' - p_i)d$ (the superscript t stands for matrix transposition) and the elements of the $n \times n$ matrix D are defined by

$$D_{ii} = u_i := p_i' - p_i, \quad D_{ij} := p_{ij} - p_{ij}' \text{ for } i \neq j.$$

The expression (50) can be cast into a compact form involving the generalized quadratic Gauss sum defined by [52]

$$\mathcal{S}(u, v, w) := \sum_{k=0}^{|w|-1} e^{i\pi(uk^2+vk)/w}, \tag{51}$$

where u, v and w are integers such that u and w are mutually prime, $uw \neq 0$. Nonzero values of $S(u, v, w)$ require that $uw+v$ must be even. Indeed, by diagonalizing the matrix D, one can rewrite the quadratic portion in the exponent of the overlap (50) without cross terms. The matrix D is symmetric with entries in $\mathbb{Z}/d\mathbb{Z}$ can be diagonalized via a congruence transformation C such that [51]

$$D_{\rm d} = C^tDC = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n).$$

It follows that the scalar product (50) rewrites now as

$$\langle\overrightarrow{m}, A|\overrightarrow{m}', A'\rangle = \frac{1}{d^n} \sum_{l_1,l_2,\cdots,l_n} e^{\frac{i\pi}{d}(\overrightarrow{v}'^t.\overrightarrow{l}+\overrightarrow{l}^t.D_{\rm d}.\overrightarrow{l})} \tag{52}$$

or equivalently

$$\langle\overrightarrow{m}, A|\overrightarrow{m}', A'\rangle = \frac{1}{d^n} \sum_{l_1,l_2,\cdots,l_n} e^{i\pi(\sum_{i=1}^n v_i'l_i+\lambda_il_i^2)/d} \tag{53}$$

where the vectors $\overrightarrow{v}'$ and $\overrightarrow{l}$ are defined by

$$\overrightarrow{v}' = C^t\overrightarrow{v} \qquad \overrightarrow{l} = C^t\overrightarrow{k}.$$

Using (51), the overlap (53) can be expressed as

$$\langle\overrightarrow{m}, A|\overrightarrow{m}', A'\rangle = \frac{1}{d^n}\prod_{i=1}^n \mathcal{S}(\lambda_i, v_i', d). \tag{54}$$

It is interesting to note that for a pair of generalized graph states such that $p_{ij} = p'_{ij}$ (i.e., $D_{ij} = 0$) for $i \neq j$, the overlap (50) rewrites

$$\langle \overrightarrow{m}, A | \overrightarrow{m}', A' \rangle = \frac{1}{d^n} \prod_{i=1}^{n} \mathcal{S}(u_i, v_i, d) \tag{55}$$

with

$$u_i := p'_i - p_i \qquad v_i := -(p'_i - p_i)d + 2(m_i - m'_i) \tag{56}$$

We notice that $du_i + v_i$ is even. Clearly, when the difference matrix $D = A - A'$ is diagonal, the overlap between a pair of generalized graph states factorize and write in terms of generalized quadratic Gauss sum.

5.2 Mutually unbiased bases

Based on the previous analysis of the mutual unbiasedness of generalized graph states, we shall consider the construction of mutually unbiased bases when d is a prime integer. We shall focus in what follows on generalized graph states such that the difference of corresponding adjacency matrices is diagonal. Using (48), this kind of generalized graph states can be written as

$$|\overrightarrow{m}, A\rangle = \prod_{i<j} (U_{ij})^{A_{ij}} |\overrightarrow{m}, \overrightarrow{p}\rangle$$

for some fixed values of the elements $A_{ij} (i \neq j)$ and the integers $p_i \equiv A_{ii}$ $(i = 1, 2, \cdots, n)$ take the values $0, 1, \cdots, d-1$. The explicit expression of the states $|\overrightarrow{m}, \overrightarrow{p}\rangle$ is given by

$$|\overrightarrow{m}, \overrightarrow{p}\rangle = \frac{1}{\sqrt{d^n}} \sum_{k_1, k_2, \cdots, k_n} \omega^{m_1 k_1} \omega^{m_2 k_2} \cdots \omega^{m_n k_n} \omega^{\frac{1}{2}[\sum_i p_i k_i (d - k_i)]} |k_1, k_2, \cdots, k_n\rangle \tag{57}$$

Clearly, in this case the overlap between the generalized graph states $|\overrightarrow{m}, A\rangle$ and $|\overrightarrow{m}', A'\rangle$ coincides with the overlap between $|\overrightarrow{m}, \overrightarrow{p}\rangle$ and $|\overrightarrow{m}', \overrightarrow{p}'\rangle$. This is given by

$$\langle \overrightarrow{m}, \overrightarrow{p} | \overrightarrow{m}', \overrightarrow{p}' \rangle = \frac{1}{d^n} \prod_{i=1}^{n} \mathcal{S}(u_i, v_i, d) \tag{58}$$

in terms of the generalized quadratic Gauss sum defined by (51) with $u_i := p'_i - p_i$ and $v_i := -(p'_i - p_i)d + 2(m_i - m'_i)$. It is clear that to construct the mutually unbiased bases it is sufficient to take the off-diagonal elements $A_{ij} = 0$. This implies that the two-body interaction term in the Hamiltonian H (28) is not relevant in our analysis. Using the results reported in [52, 53] (see also [31]), one shows that when d is a prime integer and for $\overrightarrow{p} \neq \overrightarrow{p}'$, the overlap (58) is simply given by

$$\langle \overrightarrow{m}, \overrightarrow{p} | \overrightarrow{m}', \overrightarrow{p}' \rangle = \frac{1}{\sqrt{d^n}}. \tag{59}$$

In this case we obtain the following d^n mutually unbiased bases

$$B_{\overrightarrow{p}} := \{|\overrightarrow{m}, \overrightarrow{p}\rangle : m_i = 0, 1, \ldots, d-1\} \tag{60}$$

where the components p_i $(i = 1, 2, \cdots, n)$ of the vector $\overrightarrow{p}$ take the values $p_i = 0.1, \ldots, d-1$. On the other hand, it is clear that any basis $B_{\overrightarrow{p}}$ and the computational basis

$$B_c := \{|k_1, k_2, \cdots, k_n\rangle : k_i = 0, 1, \ldots, d-1\}, \tag{61}$$

are mutually unbiased. Hence, for d a prime integer, the d^n bases $B_{\overrightarrow{p}}$ and the computational basis B_c constitute a complete set of $d^n + 1$ of mutually unbiased bases. This shows the usefullness of generalized graph states formalism in generating the mutually unbiased bases for n qudit system governed by a quadratic Hamiltonian.

6 Phase states of GHZ-type

In the expression of the quadratic Hamiltonian, the one-body term play an essential role in constructing the set of mutually unbiased bases $B_{\overrightarrow{p}}$ (60). In this section, we discuss the importance of the two-body interaction which acts as an entangling operator on the phase states (23). Hence, in this section discuss some entanglement aspects of generalized graph states obtained from the muli-qudit phase states. In particular, we shall focus on generalized graph states (48) which are locally equivalent to the maximally entangled n-qudit states of GHZ type

$$|\mathrm{GHZ}_{n,d}\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |\underbrace{k, k, \cdots, k}_{n}\rangle.$$

We recall that two n-qudit quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$ are local unitary (LU) equivalent if there exists a local unitary operator $U = \bigotimes_{i=1}^{n} U_i$ such that $U|\psi_1\rangle = |\psi_2\rangle$, where each U_i is a single-qudit unitary operation.

6.1 Bipartite qudit phase states

To examine the entanglement aspects of the phase states, we first rewrite the phase states (48), associated with the adjacency matrix A, as

$$|\overrightarrow{m}, A\rangle = \omega^{m_1 A_1} \otimes \omega^{m_2 A_2} \otimes \cdots \otimes \omega^{m_n A_n} |\psi_A, n\rangle \tag{62}$$

with A_i is the number operator of the qudit i $(i = 1, 2, \cdots, n)$ and the state $|\psi_M\rangle$ is given by

$$|\psi_A, n\rangle = \frac{1}{\sqrt{d^n}} \sum_{k_1, k_2, \cdots, k_n} \omega^{\frac{1}{2}\sum_i p_i k_i (d - k_i) + 2\sum_{i<j} p_{ij} k_i k_j} |k_1, \cdots, k_i, \cdots, k_n\rangle$$

where p_i and p_{ij} denote respectively the diagonal and off-diagonal entries of the matrix A. The phase states $|\overrightarrow{m}, A\rangle$ and $|\psi_A, n\rangle$ are local unitary equivalent. The operations ω^{A_i} $(i = 1, 2, \cdots, n)$, usually called the self-controlled phase gates, are local unitary operations which preserve the amount of the entanglement in the states $|\psi_A, n\rangle$. It follows that the basic entanglement properties of the phase states are encoded in the generalized graph states of type $|\psi_A\rangle$. We remark that when $p_{ij} = 0$ for all $i \neq j$, the pure states are unentangled $|\psi_A, n\rangle$. It is clear that the entanglement in these states arises from the cross terms $p_{ij} k_i k_j$. Thus, to examine the entanglement properties of generalized graph states, we shall focus on the states characterized by adjacency matrices with zero-elements along the main diagonal. This means that the Hamiltonian H (28) governing the system reduces to H_{int} given (29). In this case, the state $|\psi_A, n\rangle$, for a two qudit system, is given

$$|\psi_A, 2\rangle = \frac{1}{\sqrt{d}} \sum_{k_1=0}^{d-1} |k_1\rangle |\psi_{k_1}\rangle = \frac{1}{\sqrt{d}} \sum_{k_2=0}^{d-1} |\psi_{k_2}\rangle |k_2\rangle \tag{63}$$

where the states $|\psi_{k_1}\rangle$ and $|\psi_{k_2}\rangle$ are defined by

$$|\psi_{k_i}\rangle = \frac{1}{\sqrt{d}} \sum_{k_j} \omega^{p_{ij} k_i k_j} |k_j\rangle, \qquad \text{for} \quad i \neq j = 1, 2. \tag{64}$$

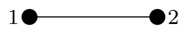

Figure 1: $n = 2$ graph with $p_{12} = 1$

The graph state $|\psi_A, 2\rangle$ is represented in Figure 1. The state $|m_1, m_2, A\rangle$ (62) and the state $|\psi_A, 2\rangle$ are local unitary equivalents. Indeed, from the expressions (62) and (63), one gets

$$|m_1, m_2, A\rangle = (\omega_1^{m_1 A_1} \otimes \omega_2^{A_2 m_2} F) \frac{1}{\sqrt{d}} \sum_{k_1} |k_1, k_1\rangle \tag{65}$$

where the unitary operator F is defined by

$$F = \frac{1}{\sqrt{d}} \sum_{k_i, k_j} \omega^{p_{ij} k_i k_j} |k_i\rangle\langle k_j|. \tag{66}$$

It follows that the phase state $|m_1, m_2, A\rangle$ is local unitary equivalent to the maximally entangled state $|\mathrm{GHZ}_{2,d}\rangle$. This local unitary equivalence writes as

$$|m_1, m_2, A\rangle = (\omega_1^{m_1 A_1} \otimes \omega_2^{A_2 m_2} F) |\mathrm{GHZ}_{2,d}\rangle \tag{67}$$

and implies that the state $\mathrm{GHZ}_{2,d}\rangle$ can be represented also by the graph in Figure 1. This result can be extended to three and more qudits.

6.2 Tripartite qudit phase states

For three qudits ($n = 3$), we consider generalized graph states associated with adjacency matrices with with zeros along the main diagonal ($p_1 = p_2 = p_3 = 0$). In this case the phase states (48) become

$$|m_1, m_2, m_3, A\rangle = \frac{1}{\sqrt{d^3}} \sum_{k_1, k_2, k_3} \omega^{m_1 k_1} \omega^{m_2 k_2} \omega^{m_3 k_3} \omega^{p_{12} k_1 k_2 + p_{23} k_2 k_3 + p_{13} k_1 k_3} |k_1, k_2, k_3\rangle. \tag{68}$$

To discuss the corresponding entanglement properties, we shall consider two types of connected the connected graphs G. The first one is determined by the following edge set $E(G) = \{(12), (23)\}$ in which the matrix element p_{13} is zero (see Figure 1). The second one corresponds to the situation where the three qudits are connected has edge set $E(G) = \{(12), (23), (13)\}$ (see Figure 2). For the first set of graphs, the state (68) becomes

$$|m_1, m_2, m_3, A\rangle == (\omega^{m_1 A_1} \otimes \omega^{A_2 m_2} \otimes \omega^{A_3 m_3}) \frac{1}{\sqrt{d}} \sum_{k_2} |\psi_{k_2}\rangle |k_2\rangle |\psi_{k_2}\rangle \tag{69}$$

where $|\psi_{k_2}\rangle$ is defined as in (64). This class of generalized graph states are is local unitary equivalents to the state $|\mathrm{GHZ}_{3,d}\rangle$. Indeed, one has

$$|m_1, m_2, m_3, A\rangle == (\omega^{A_1 m_1} \otimes \omega^{A_2 m_2} F \otimes \omega^{A_3 m_3} F) |\mathrm{GHZ}_{3,d}\rangle \tag{70}$$

where the unitary operation F is defined by (66). This shows that the generalized graph states belonging to the first type of connected graph are maximally entangled. The second set of generalized graph states are not local unitary equivalents to states of type Greenberger-Horne-Zeilinger. Indeed, the states

$$|m_1, m_2, m_3, A\rangle = \frac{1}{\sqrt{d^3}} \sum_{k_1,k_2,k_3} \omega^{m_1 k_1} \omega^{m_2 k_2} \omega^{m_3 k_3} \omega^{k_1 k_2 + k_2 k_3 + k_1 k_3} |k_1, k_2, k_3\rangle \tag{71}$$

are local unitary equivalents to the states

$$|m_1, m_2, m_3, A\rangle \sim \frac{1}{\sqrt{d^3}} \sum_{k_1,k_2,k_3} \omega^{k_1 k_2} \omega^{k_2 k_3} \omega^{k_1 k_3} |k_1, k_2, k_3\rangle,$$

but do not exist local unitary transformations such that these states can be converted in states of GHZ-type.

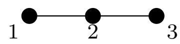

Figure 2: $n = 3$ graph with $(p_{12} = p_{23} = 1, p_{13} = 0)$.

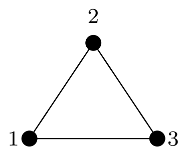

Figure 3: $n = 3$ graph with $(p_{12} = p_{23} = p_{13} = 1)$.

It is worth noticing that the property of local unitary equivalence between generalized graph states and states of GHZ-type discussed for two and three qudit systems can be extended to multi-qudit system when only one qudit is entangled with the remaining qudits forming the system. This situation is described by the the n-qudit star-shape graph G, i.e. with edge set $E(G) = \{(12), (13), \ldots, (1n)\}$. The non vanishing entries of the corresponding adjacency matrix are $p_{12} = p_{13} = \cdots = p_{1n} = 1$. In this case, it simple to verify that the generalized graph states

$$|m_1, m_2, \cdots, m_n, A\rangle = \frac{1}{\sqrt{d^n}} \sum_{k_1,k_2,\cdots,k_n} \omega^{m_1 k_1} \omega^{m_2 k_2} \cdots \omega^{m_n k_n} \omega^{k_1 k_2 + k_1 k_3 + \cdots + k_1 k_n} |k_1, k_2, \cdots, k_n\rangle. \tag{72}$$

are local unitary equivalents to the multi-qubit state $|\mathrm{GHZ}_{n,d}\rangle$. Indeed, we have

$$|m_1, m_2, \cdots, m_n, A\rangle = (\omega^{A_1 m_1} \otimes \omega^{A_2 m_2} F, \cdots, \otimes \omega^{A_n m_n} f) |\mathrm{GHZ}_{n,d}\rangle \tag{73}$$

which generalizes the results (67) and (70).

7 Concluding remarks

To close this paper, we summarize the main results. The algebraic description of qudits is formulated via a generalized variant of Weyl-Heisenberg algebra possessing finite dimensional representations. This provides the appropriate tool to define consistently the unitary phase operators for an ensemble of qudits and subsequently to determine the corresponding phase states along the lines developed in [3] (see also [4, 6]). In investigating the dynamics of multi-partite phase states governed by a quadratic Hamiltonian of type (28), we have established a subtle connection between evolved phase states and the formalism of generalized graph states which are of paramount importance in various areas of quantum information theory such as the study of multi-particle entanglement. Also, in connection with the phase operator and multi-partite phase states for a collection of qudits, the present work addresses two important issues: the generation of mutually unbiased bases by employing of the phase state formalism. The second question concerns the correspondence between the phase states and multipartite GHZ states. More prcesicely, we investigated some particular cases where the generalized graph states are locally unitary equivalents to generalized GHZ states. It is well known that this kind of maximally entangled states are suitable in various quantum protocols tasks such as quantum computation and quantum communication. In this sense, the entangled multi-qudit phase states are expected to play a significant role to encode quantum information and to protect it against the incoherence effects. It must be emphasized also that the muti-partite phase states are a kind of the co-called cluster states which can be implemented for instance in spin systems via Ising interaction between neighboring particles on a lattice. On the other hand, our study is mainly motivated by the possible practical applications of multi-particle entangled phase states in quantum key distribution with different mutually unbiased bases and the characterization of their quantum correlations by exploiting the entanglement properties of generalized graph states and their equivalence with generalized Greenberger- Horne-Zeilinger states.

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