Local quantum uncertainty and pairwise quantum Discord in Z_2-symmetric quantum spin lattices

L. Jebli[a 1], **B. Benzimoune**[b 2] and **M. Daoud**[c,d 3]

[a] *Department of Physics, Faculty of Sciences, University Ibnou Zohr, Agadir , Morocco*

[b] *LPHE-Modeling and Simulation, Faculty of Sciences, University Mohamed V in Rabat, Morocco*

[c] *Department of Physics, Faculty of Sciences-Ain Chock, University Hassan II Casablanca, Morocco*

[b] *Abdus Salam International Centre for Theoretical Physics, Trieste, Italy*

Abstract

A special emphasis is devoted to the concept of local quantum uncertainty as indicator of quantum correlations. We study quantum discord for a class of two-qubit states parameterized by two parameters. Quantum discord based on local quantum uncertainty, von Neumann entropy and trace distance (Schatten 1-norm) are explicitly derived and compared. The behavior of the local quantum uncertainty quantifier under decoherence effects is investigated .

Parler aussi ds abstarct Frosen LQU

1. Lire le papier en detail, relever les fautes de frappe et d'anglais
2. Verifier les calculs
3. Verifier si toutes les references sont citees, Est ce que elles apparaissent dans l'ordre dans le texte, est ce que il y a des references qu'il faut ajouter et me dire a quel niveau.
4. Changer les notations sur les figures: C'est indique en rouge au bas de chaque figure dans le texte

[1] email: larbi.jebli@gmail.com
[2] email: b.benzimoun@gmail.com
[3] email: m_daoud@hotmail.com

1 Introduction

Characterizing quantum correlations in multipartite quantum systems is one of the most challenging topics in quantum information theory. Various measures to quantify the degree of quantumness in a multipartite quantum system were introduced in the literature. The most familiar ones the concurrence, the entanglement of formation, the quantum discord and its various geometric versions [1, 2, 3, 4, 5]. The interest in quantum correlations other than entanglement lies in the existence of nonclassical correlations even in separable states [6, 7]. In fact, entanglement does not account for all nonclassical aspects of correlations, especially in mixed states. This yielded many works dedicated to introduce quantum correlation quantifiers beyond entanglement. As the total correlation is the sum of two contributions: a classical part and quantum part, different concepts were considered to develop the best way to distinguish between classical and quantum correlations. In this context, the entropy based quantum discord [6, 7] is probably the quantifier which has been intensively investigated in the literature for different purposes and from several perspectives (see for instance [5]). However, it must be noticed that the analytical evaluation of quantum discord which is in general very challenging. Only partial results were obtained for few two-qubit systems. To overcome such technical difficulties and to find reliable and computable quantifiers, geometric variants of quantum discord were introduced by considering different geometrical measures. Indeed, The 2-norm (Hilbert-Schmidt norm) version of the quantum discord was introduced in [8]. This quantum correlation indicator is easily computable [9, 10, 11, 12]. However, despite its computability for any bipartite quantum system, the Hilbert-Schmidt based quantum discord can increase under local operations on the unmeasured qubit. This drawback of quantum correlation quantifier based on Hilbert-Schmidt norm comes from the non-contractibility of the 2-norm (Schatten 2-norm) [13]. Now, it is well known that the only norm among the Schatten p-norm which is contractible is the Bures norm (trace norm with $p = 1$) and which constitutes a suitable tool to quantify geometrically the quantum discord (see for instance [14, 15]).

Quantifying quantum correlations in multipartite quantum systems continues to draw special attention in quantum information science. Hence, another reliable geometric quantifier of discord-like correlations was recently introduced by employing the so-called local quantum uncertainty. This quantifier uses the notion skew information introduced in [16] to determine the uncertainty in the measurement of an observable. The local quantum uncertainty is given by the minimum of the skew information over all possible local observales. This measure offers an appropriate tool to evaluate the analytical expressions of quantum correlations encompassed in any qubit-qudit bipartite system [17]. The local quantum uncertainty is related to the quantum Fisher information [18, 19, 20] which is a key ingredient in quantum metrology protocols [21]. Also, it quantifies the speed of the local (unitary) evolution of a bipartite quantum system [17].

In this paper the analytical derivation of quantum discord is essentially approached in the context of local quantum uncertainty formalism. We consider a particular family of rank-2 X states which includes various types of two-qubit states of interest in different models of collective spin systems like for instance Dicke model [22] and Lipkin-Meshkov-Glick model [23] where the quantum discord was investigated in relation with their critical properties and quantum phase transitions (see for instance [24, 25, 26, 27]). Remarkably, it has been shown that the quantum discord provides a suitable indicator to understand the role of quantum correlations in characterizing quantum phase transitions [28](see also [29]). We note also that the set of two-qubit under consideration are of special relevance in investigating quantum correlations in bipartite states extracted from multi-qubit Dicke states and their superpositions(e.g., generalized GHZ states, even and odd spin coherent states) [30]. Thus, beside the explicit derivation of local quantum uncertainty, we also the amount of quantum correlations in such states when measured by von Neumann entropy or trace distance. Another facet of this work concerns the dynamics of the local quantum uncertainty under decoherence effects induced by the unavoidable interaction of a quantum system with environment. Four typical quantum decoherence channels are considered. The explicit expressions of local quantum uncertainty are derived for each case. We will show that in some cases the local quantum uncertainty is unaffected by the decoherence channel effects.

The paper is structured as follows. In section 2, we give the explicit expressions for local quantum uncertainty, the von Neumann entropy based quantum discord and the trace norm quantum discord for a class of two-qubit states which are, as we mentioned already, relevant in investigating bipartite quantum correlations in various collective spin models. In section 3, under four quantum decoherence channels (bit flip, phase flip, bit-phase flip and generalized amplitude damping), we give the analytic expressions of local quantum uncertainty. In particular, we show the freezing character of local quantum uncertainty in some special cases. Concluding remarks close this paper.

2 General presentation

The manuscript discusses EPR steering in the context of optomechanical systems. The proposed setup consists of two optomechanical cavities, which are driven by two-mode-squeezed (TMS) light on the red mechanical sideband, and whose output is monitored by separate homodyne detectors. The TMS drive generates entanglement between the cavity modes, which is swapped to the mechanical systems by the optomechanical (beam-splitter) interaction. Evaluating the covariance matrix of the bipartite mechanical system, the authors show that the generated state fulfills the Reid EPR steering criterion.

The manuscript is divided into an introduction (section 1), a theoretical description of the optomechanical systems (section 2), and a discussion of steering (both general and in the context of the proposed setup) (section 3) and a conclusion (section 4). The presented analysis is theoretically sound. As a whole, the manuscript is, however, mediocre in terms of presentation (language, clarity) and content.

From the outset the authors fail to explain what the concrete goal of their work is. In the introduction they discuss general aspects of EPR steering and its possible application in quantum information processing. At the end of the section they mention their motivation to study optomechanical systems (high-precision measurements, quantum information processing), but they fail to connect this motivation to the concept of EPR steering. This leaves me wondering: Why do they deem an optomechanical implementation of EPR steering interesting? Is it the fundamental or the applied aspects? If it is the latter, which applications do they have in mind?

This lack of a clear goal makes it difficult to judge the manuscript's quality and if the author's actually achieved their goal. To be precise, the manuscript presents a protocol to generate a mechanical resource state for EPR steering; it does not present a complete protocol to demonstrate steering using optomechanical systems. Although the authors talk about homodyne detection at first (figure 1), measurements are completely neglected in the rest of the manuscript. This may sound like a trivial objection, but the mechanical quantum state can only be inferred using measurements of the output light. This has two effects: It introduces additional noise (which I think should not be a problem), but it also makes it harder to argue that the measured steering is actually mechanical steering and not only due to the TMS input light (which in the current setup would also be detected by the homodyne detection). In this respect I find the presented analysis severely incomplete (it might be easy to rescue, however).

I have several miscellaneous comments and suggestions:

- In the introduction, the authors explain the concept of steering by saying that LMCC can be used to steer a quantum state nonlocally. This is not very helpful.

- Section 2 should be shortened considerably. The applied approximations have been discussed extensively in the literature and there is no reason to reproduce them here. Additionally, I see no reason to apply the rotating wave approximation, at least not to get numerical results.

- In section 3, the optomechanical parameters are taken from reference [41], which was published

in 2007. This publication is not at all "recent" in the field of optomechanics. Additionally, the thermal occupation numbers assumed to produce the plots are completely unrealistic for a 1MHz mechanical resonator. The conclusion that demonstration of EPR steering using optomechanical systems is feasible cannot be drawn based on these parameters.

- Instead of using C and $n_t h$ as independent parameters, it would be interesting to look at the dependence with respect to $C/n_t h$, which is the "thermal cooperativity".

- The authors claim that figure 2 and 3 show that the steerability is bounded by entanglement. As shown in [9] this must indeed be the case, but only if entanglement is measured in terms of Renyi-2-entropy, not logarithmic negativity (logneg)! The claim is therefore invalid!

- The observation that the logneg cannot detect EPR steerability is trivial given that the logneg is defined completely symmetric with respect of exchange of parties (as is evident from eq (35)). This should not be more than a sidenote and not a selling point of the manuscript in the abstract.

- The plots in figures 2 and 3 should be scaled such that the axis labels align correctly for better comparability.

- The labeling of the modes a^{in} does not match the description used in the text.

In conclusion, I find the manuscript lacking in several respects and cannot recommend publication in its current form.

3 Local quantum uncertainty, entropic quantum discord and geometric quantum for rank two X states

The two-qubit density matrices which display non zero entries only along the main- and anti-diagonals are usually called X-states. They generalize several two-qubit states as for instance Bell-diagonal states (see [31]), Werner states [32], isotropic states [33]. Their particular relevance was first identified in investigating the phenomenon of sudden death of entanglement [34]and since then extended to many other context in connection of quantum information theory. A generic X-state is parameterized by seven real parameters and the corresponding symmetry is fully characterized by the $su(2)\times su(2)\times u(1)$ subalgebra of the full $su(4)$ algebra describing an arbitrary two-qubit system [35]. This symmetry reduction from $su(4)$ to $su(2)\times su(2)\times u(1)$ renders easy many analytical calculations of concurrence, entanglement of formation, quantum discord and leads to interesting results in studying their properties and especially their evolution under dissipative processes were reported in the literature (see for instance [36, 37]).

3.1 Local quantum uncertainty: Definition

The concept of local quantum uncertainty is now considered as a promising quantifier of quantum correlation. This is essentially due to its easiness of computability and its reliability. It quantifies the minimal quantum uncertainty in a quantum state due to a measurement of a local observable [17]. For a bipartite quantum state ρ_{12}, the local quantum uncertainty is defined as

$$\mathcal{U}(\rho_{12}) \equiv \min_{K_1} \mathcal{I}(\rho, K_1 \otimes \mathbb{I}_2), \tag{1}$$

where K_1 is some local observable on the subsystem 1, $\mathbb{I}_2$ is the identity operator and

$$\mathcal{I}(\rho_{AB}, K) = -\frac{1}{2}\mathrm{Tr}([\sqrt{\rho_{AB}}, K]^2) \tag{2}$$

is the skew information [16, 18]. The skew information represents the non-commutativity between the state and the observable K_1. The analytical evaluation the local quantum uncertainty requirers a minimization procedure over the set of all observales acting on the part 1. A closed form for qubit-qudit systems was derived in [17]. In particular, for qubits ($\frac{1}{2}$-spin particles), the expression of the local quantum uncertainty is given by [17]

$$\mathcal{U}(\rho) = 1 - \lambda_{\max}\{W\}, \tag{3}$$

where $\lambda_{\max}$ denotes the maximum eigenvalue of the 3×3 matrix W whose matrix elements are defined by

$$\omega_{ij} \equiv \mathrm{Tr}\{\sqrt{\rho}(\sigma_i \otimes \mathbb{I}_2)\sqrt{\rho}(\sigma_j \otimes \mathbb{I}_2)\}, \tag{4}$$

with $i, j = 1, 2, 3$. The local quantum uncertainty provides an appropriate quantifier of the minimum amount of uncertainty in a bipartite quantum state. For pure bipartite states, it reduces to linear

entropy of the reduced densities of the subsystems. Also, it vanishes for classically correlated states. Another interesting property of local quantum uncertainty is its invariance under local unitary operations. This quantum correlations indicator enjoys all required properties of being a reliable quantifier [17]. Hence, in what follows, we shall employ the local quantum uncertainty to study the pairwise quantum correlation in a family of two-qubit states.

3.2 Two qubit X states

In this section, we consider a two-qubit system which is described by an X-shaped mixed state. The X states are encountered in various quantum systems in condensed matter physics. In fact, X states provide the general form of reduced density operators of arbitrary quantum spin chains with parity symmetry. Two-qubit X states include various types of quantum states usually used in investigating entanglement and quantum correlations. They have non-zero entries only along the diagonal and anti-diagonal and therefore they are parameterized by seven real parameters. The corresponding symmetry is fully characterized by the $su(2) \times su(2) \times u(1)$ subalgebra of the full $su(4)$ algebra describing an arbitrary two-qubit system. Many analytical calculations of concurrence, entanglement of formation, quantum discord can be carried out easily for X states leading to interesting results in studying their properties and especially their evolution under dissipative processes [36]. The density matrix for a two-qubit X state writes as

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{5}$$

in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The entries are subjected to constraints $\sum_{i=1}^{4} \rho_{ii} = 1$, $\rho_{11}\rho_{44} \geq |\rho_{14}|^2$ and $\rho_{22}\rho_{33} \geq |\rho_{23}|^2$.

The eigenvalues of the density matrix ρ are given by

$$\lambda_1 = \frac{1}{2}t_1 + \frac{1}{2}\sqrt{t_1^2 - 4d_1}, \quad \lambda_2 = \frac{1}{2}t_2 + \frac{1}{2}\sqrt{t_2^2 - 4d_2}, \quad \lambda_3 = \frac{1}{2}t_2 - \frac{1}{2}\sqrt{t_2^2 - 4d_2}, \quad \lambda_4 = \frac{1}{2}t_1 - \frac{1}{2}\sqrt{t_1^2 - 4d_1}$$

with $t_1 = \rho_{11} + \rho_{44}$, $t_2 = \rho_{22} + \rho_{33}$, $d_1 = \rho_{11}\rho_{44} - \rho_{14}\rho_{41}$, and $d_2 = \rho_{22}\rho_{33} - \rho_{32}\rho_{23}$. The Fano-Bloch decomposition of the state ρ writes as

$$\rho = \frac{1}{4}\sum_{\alpha,\beta} R_{\alpha\beta}\sigma_\alpha \otimes \sigma_\beta$$

where the correlation matrix $R_{\alpha\beta}$ are given by $R_{\alpha\beta} = \mathrm{Tr}(\sqrt{\rho}\ \sigma_\alpha \otimes \sigma_\beta)$. They write

$$R_{03} = 1 - 2\rho_{22} - 2\rho_{44}, \quad R_{30} = 1 - 2\rho_{33} - 2\rho_{44}, \quad R_{11} = 2\ \mathrm{Re}(\rho_{32} + \rho_{41}), \quad R_{22} = 2\ \mathrm{Re}(\rho_{32} - \rho_{41})$$

$$R_{12} = -2i\ \mathrm{Im}(\rho_{41} - \rho_{32}) \quad R_{21} = -2i\ \mathrm{Im}(\rho_{41} + \rho_{32}), \quad R_{00} = \rho_{11} + \rho_{22} + \rho_{33} + \rho_{44} = 1, \quad R_{33} = 1 - 2\rho_{22} - 2\rho_{33}.$$

It is simple to check that the square root of the density matrix ρ writes

$$\sqrt{\rho} = \begin{pmatrix} \frac{\rho_{11}+\sqrt{d_1}}{\sqrt{t_1+2\sqrt{d_1}}} & 0 & 0 & \frac{\rho_{14}}{\sqrt{t_1+2\sqrt{d_1}}} \\ 0 & \frac{\rho_{22}+\sqrt{d_2}}{\sqrt{t_2+2\sqrt{d_2}}} & \frac{\rho_{23}}{\sqrt{t_2+2\sqrt{d_2}}} & 0 \\ 0 & \frac{\rho_{32}}{\sqrt{t_2+2\sqrt{d_2}}} & \frac{\rho_{33}+\sqrt{d_2}}{\sqrt{t_2+2\sqrt{d_2}}} & 0 \\ \frac{\rho_{41}}{\sqrt{t_1+2\sqrt{d_1}}} & 0 & 0 & \frac{\rho_{44}+\sqrt{d_1}}{\sqrt{t_1+2\sqrt{d_1}}} \end{pmatrix}. \tag{6}$$

The eigenvalues $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$, $\sqrt{\lambda_3}$ and $\sqrt{\lambda_4}$ of the matrix $\sqrt{\rho}$ can be rewritten as

$$\sqrt{\lambda_1} = \frac{1}{2}\sqrt{t_1+2\sqrt{d_1}} + \frac{1}{2}\sqrt{t_1-2\sqrt{d_1}}, \qquad \sqrt{\lambda_2} = \frac{1}{2}\sqrt{t_2+2\sqrt{d_2}} + \frac{1}{2}\sqrt{t_2-2\sqrt{d_2}}$$

$$\sqrt{\lambda_3} = \frac{1}{2}\sqrt{t_2+2\sqrt{d_2}} - \frac{1}{2}\sqrt{t_2-2\sqrt{d_2}}, \qquad \sqrt{\lambda_4} = \frac{1}{2}\sqrt{t_1+2\sqrt{d_1}} - \frac{1}{2}\sqrt{t_1-2\sqrt{d_1}}$$

In Fano-Bloch representation, the matrix $\sqrt{\rho}$ writes as

$$\sqrt{\rho} = \frac{1}{4}\sum_{\alpha,\beta} \mathcal{R}_{\alpha\beta}\sigma_\alpha \otimes \sigma_\beta$$

with $\mathcal{R}_{\alpha\beta} = \mathrm{Tr}(\sqrt{\rho}\ \sigma_\alpha \otimes \sigma_\beta)$. The non vanishing matrix correlation elements $\mathcal{R}_{\alpha\beta}$ are explicitly given by

$$\mathcal{R}_{00} = \sqrt{t_1+2\sqrt{d_1}} + \sqrt{t_2+2\sqrt{d_2}} \qquad \mathcal{R}_{03} = \frac{1}{2}\frac{R_{30}+R_{03}}{\sqrt{t_1+2\sqrt{d_1}}} - \frac{1}{2}\frac{R_{30}-R_{03}}{\sqrt{t_2+2\sqrt{d_2}}}$$

$$\mathcal{R}_{30} = \frac{1}{2}\frac{R_{30}+R_{03}}{\sqrt{t_1+2\sqrt{d_1}}} + \frac{1}{2}\frac{R_{30}-R_{03}}{\sqrt{t_2+2\sqrt{d_2}}} \qquad \mathcal{R}_{11} = \frac{1}{2}\frac{R_{11}+R_{22}}{\sqrt{t_2+2\sqrt{d_2}}} + \frac{1}{2}\frac{R_{11}-R_{22}}{\sqrt{t_1+2\sqrt{d_1}}}$$

$$\mathcal{R}_{12} = \frac{1}{2}\frac{R_{12}+R_{21}}{\sqrt{t_1+2\sqrt{d_1}}} + \frac{1}{2}\frac{R_{12}-R_{21}}{\sqrt{t_2+2\sqrt{d_2}}} \qquad \mathcal{R}_{21} = \frac{1}{2}\frac{R_{12}+R_{21}}{\sqrt{t_1+2\sqrt{d_1}}} - \frac{1}{2}\frac{R_{12}-R_{21}}{\sqrt{t_2+2\sqrt{d_2}}}$$

$$\mathcal{R}_{22} = \frac{1}{2}\frac{R_{11}+R_{22}}{\sqrt{t_2+2\sqrt{d_2}}} - \frac{1}{2}\frac{R_{11}-R_{22}}{\sqrt{t_1+2\sqrt{d_1}}} \qquad \mathcal{R}_{33} = \sqrt{t_1+2\sqrt{d_1}} - \sqrt{t_2+2\sqrt{d_2}}$$

At this stage, we have the tools to evaluate the matrix elements defined by

$$\omega_{ij} = \mathrm{Tr}\Big(\sqrt{\rho}\ (\sigma_i \otimes \sigma_0)\sqrt{\rho}\ (\sigma_j \otimes \sigma_0)\Big)$$

where i and j take the values 1, 2, 3. Using the following identities

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \quad \mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij} \quad \mathrm{Tr}(\sigma_i\sigma_j\sigma_k\sigma_l) = 2(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

one shows that the matrix W is diagonal and the diagonal elements are

$$\omega_{ii} = \frac{1}{4}\Big[\sum_\beta \Big(\mathcal{R}_{0\beta}^2 - \sum_k \mathcal{R}_{k\beta}^2\Big)\Big] + \frac{1}{2}\sum_\beta \mathcal{R}_{i\beta}^2$$

where $i = 1, 2, 3$ and $\beta = 0, 1, 2, 3$. Explicitly, the elements ω_{ij} write

$$\omega_{ij} = \delta_{ij}\Big[\frac{1}{4}\sum_\beta \Big(\mathcal{R}_{0\beta}^2 - \sum_k \mathcal{R}_{k\beta}^2\Big)\Big] + \frac{1}{2}\sum_\beta \mathcal{R}_{i\beta}\mathcal{R}_{j\beta}$$

where $\beta = 0, 1, 2, 3$ and $k = 1, 2, 3$. The diagonal elements are

$$\omega_{ii} = \frac{1}{4}\Big[\sum_{\beta}\Big(\mathcal{R}_{0\beta}^2 - \sum_{k}\mathcal{R}_{k\beta}^2\Big)\Big] + \frac{1}{2}\sum_{\beta}\mathcal{R}_{i\beta}^2$$

and the off-diagonal elements are

$$\omega_{ij} = \frac{1}{2}\sum_{\beta}\mathcal{R}_{i\beta}\mathcal{R}_{j\beta} \quad i \neq j$$

Explicitly, we have

$$\omega_{11} = \frac{1}{4}\Big[4\Big(\sqrt{\lambda_1}+\sqrt{\lambda_4}\Big)\Big(\sqrt{\lambda_2}+\sqrt{\lambda_3}\Big) + \frac{(R_{11}^2 - R_{22}^2) + (R_{12}^2 - R_{21}^2) + (R_{03}^2 - R_{30}^2)}{(\sqrt{\lambda_1}+\sqrt{\lambda_4})(\sqrt{\lambda_2}+\sqrt{\lambda_3})}\Big] \tag{7}$$

$$\omega_{22} = \frac{1}{4}\Big[4\Big(\sqrt{\lambda_1}+\sqrt{\lambda_4}\Big)\Big(\sqrt{\lambda_2}+\sqrt{\lambda_3}\Big) + \frac{(R_{22}^2 - R_{11}^2) + (R_{21}^2 - R_{12}^2) + (R_{03}^2 - R_{30}^2)}{(\sqrt{\lambda_1}+\sqrt{\lambda_4})(\sqrt{\lambda_2}+\sqrt{\lambda_3})}\Big] \tag{8}$$

$$\begin{aligned}\omega_{33} \;=\; & \frac{1}{2}\Big[\Big(\sqrt{\lambda_1}+\sqrt{\lambda_4}\Big)^2 + \Big(\sqrt{\lambda_2}+\sqrt{\lambda_3}\Big)^2\Big] + \frac{1}{8}\Big[\frac{(R_{03}+R_{30})^2 - (R_{11}-R_{22})^2 - (R_{12}+R_{21})^2}{\Big(\sqrt{\lambda_1}+\sqrt{\lambda_4}\Big)^2}\Big] \\ & + \frac{1}{8}\Big[\frac{(R_{03}-R_{30})^2 - (R_{11}+R_{22})^2 - (R_{12}-R_{21})^2}{\Big(\sqrt{\lambda_2}+\sqrt{\lambda_3}\Big)^2}\Big]\end{aligned} \tag{9}$$

$$\omega_{12} = \omega_{21} = \frac{1}{2}\Big(\mathcal{R}_{11}\mathcal{R}_{21} + \mathcal{R}_{12}\mathcal{R}_{22}\Big) = \frac{1}{2}\frac{R_{11}R_{21} + R_{22}R_{12}}{(\sqrt{\lambda_1}+\sqrt{\lambda_4})(\sqrt{\lambda_2}+\sqrt{\lambda_3})} \tag{10}$$

$$\omega_{13} = \omega_{31} = 0, \quad \omega_{23} = \omega_{32} = 0 \tag{11}$$

4 Pairwise correlations for Z_2-symmetric quantum spin lattices

We will consider here an interacting pair of spins-1/2 in a spin lattice, which is governed by a Hamiltonian H that is both real and exhibits Z_2 symmetry, i.e. invariance under π-rotation around a given spin axis. By taking this spin axis as the z direction, this implies the commutation of H with the parity operator $\bigotimes_{i=1}^{N}\sigma_i^3$, where N denotes the total number of spins and σ_i^3 is the Pauli operator along the z-axis at site i. Note that a number of spin models are enclosed within these requirements as, for instance, the XXZ spin chain and the transverse field Ising model. Disregarding spontaneous symmetry breaking (see, e.g., Refs. [**?**, **?**, **?**] for a treatment of spontaneously broken ground states), the two-spin reduced density matrix at sites labelled by i and j in the basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$, with $|\uparrow\rangle$ and $|\downarrow\rangle$ denoting the eigenstates of σ^3, will be given by

$$\rho = \begin{pmatrix} a & 0 & 0 & f \\ 0 & b_1 & z & 0 \\ 0 & z & b_2 & 0 \\ f & 0 & 0 & d \end{pmatrix}. \tag{12}$$

In terms of spin correlation functions, these elements can be written as

$$\begin{aligned} a &= \frac{1}{4}\left(1 + G_z^i + G_z^j + G_{zz}^{ij}\right), \\ b_1 &= \frac{1}{4}\left(1 + G_z^i - G_z^j - G_{zz}^{ij}\right), \\ b_2 &= \frac{1}{4}\left(1 - G_z^i + G_z^j - G_{zz}^{ij}\right), \\ d &= \frac{1}{4}\left(1 - G_z^i - G_z^j + G_{zz}^{ij}\right), \\ z &= \frac{1}{4}\left(G_{xx}^{ij} + G_{yy}^{ij}\right), \\ f &= \frac{1}{4}\left(G_{xx}^{ij} - G_{yy}^{ij}\right), \end{aligned} \tag{13}$$

where $G_z^k = \langle\sigma_z^k\rangle$ $(k = i, j)$ is the magnetization density at site k and $G_{\alpha\beta}^{ij} = \langle\sigma_\alpha^i\sigma_\beta^j\rangle$ $(\alpha, \beta = x, y, z)$ denote two-point spin-spin functions at sites i and j, with the expectation value taken over the quantum state of the system. Note that, in case of translation invariance, we will have that $G_z^k = G_z^{k'}$ $(\forall\, k, k')$ and, therefore, $b_1 = b_2$. Moreover, observe also that the density operator given in Eq. (12) can be decomposed as

$$\rho = \frac{1}{4}\left[I \otimes I + \sum_{i=1}^{3}\left(c_i\sigma^i \otimes \sigma^i\right) + c_4 I \otimes \sigma^3 + c_5\sigma^3 \otimes I\right], \tag{14}$$

with

$$\begin{aligned} c_1 &= 2z + 2f, \\ c_2 &= 2z - 2f, \\ c_3 &= a + d - b_1 - b_2, \\ c_4 &= a - d - b_1 + b_2, \\ c_5 &= a - d + b_1 - b_2. \end{aligned} \tag{15}$$

In particular, for translation invariant systems, we have that $c_4 = c_5$. In order to determine classical and quantum correlations, we first evaluate the mutual information as given by Eq. (**??**). The eigenvalues of ρ read

$$\begin{aligned}
\lambda_0 &= \frac{1}{4}\left[(1+c_3)+\sqrt{(c_4+c_5)^2+(c_1-c_2)^2}\right], \\
\lambda_1 &= \frac{1}{4}\left[(1+c_3)-\sqrt{(c_4+c_5)^2+(c_1-c_2)^2}\right], \\
\lambda_2 &= \frac{1}{4}\left[(1-c_3)+\sqrt{(c_4-c_5)^2+(c_1+c_2)^2}\right], \\
\lambda_3 &= \frac{1}{4}\left[(1-c_3)-\sqrt{(c_4-c_5)^2+(c_1+c_2)^2}\right].
\end{aligned} \tag{16}$$

Therefore, the mutual information is given by

$$I(\rho) = S(\rho^A) + S(\rho^B) + \sum_{\alpha=0}^{3} \lambda_\alpha \log \lambda_\alpha, \tag{17}$$

where

$$\begin{aligned}
S(\rho^A) &= -\left(r_1^A \log r_1^A + r_2^A \log r_2^A\right), \\
S(\rho^B) &= -\left(r_1^B \log r_1^B + r_2^B \log r_2^B\right),
\end{aligned} \tag{18}$$

with $r_1^A = (1+c_5)/2$, $r_2^A = (1-c_5)/2$, $r_1^B = (1+c_4)/2$, and $r_2^B = (1-c_4)/2$. Classical correlations can be obtained by following a procedure that is similar to those of Refs. [**?**, **?**], but applying it now for the case of the general density matrix given by Eq. (12). We first introduce a set of projectors for a local measurement on part B given by $\{B_k = V\Pi_k V^\dagger\}$, where $\{\Pi_k = |k\rangle\langle k| : k = 0, 1\}$ is the set of projectors on the computational basis ($|0\rangle \equiv |\uparrow\rangle$ and $|1\rangle \equiv |\downarrow\rangle$) and $V \in U(2)$. Note that the projectors B_k represent therefore an arbitrary local measurement on B. We parametrize V as

$$V = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\phi} \\ \sin\frac{\theta}{2}e^{i\phi} & -\cos\frac{\theta}{2} \end{pmatrix}, \tag{19}$$

where $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. Note that θ and ϕ can be interpreted as the azimuthal and polar angles, respectively, of a qubit over the Bloch sphere. By using Eq. (**??**) and the equation $\Pi_k \sigma^i \Pi_k = \delta_{i3}(-1)^k \Pi_k$, with δ_{i3} denoting the Kronecker symbol, we can show that the state of the system after measurement $\{B_k\}$ will change to one of the states

$$\rho_0 = \frac{1}{2}\left(I + \sum_{j=1}^{3} q_{0j}\sigma^j\right) \otimes \left(V\Pi_0 V^\dagger\right), \tag{20}$$

$$\rho_1 = \frac{1}{2}\left(I + \sum_{j=1}^{3} q_{1j}\sigma^j\right) \otimes \left(V\Pi_1 V^\dagger\right), \tag{21}$$

where

$$\begin{aligned} q_{k1} &= (-1)^k c_1 \left[\frac{w_1}{1+(-1)^k c_4 w_3}\right], \\ q_{k2} &= (-1)^k c_2 \left[\frac{w_2}{1+(-1)^k c_4 w_3}\right], \\ q_{k3} &= (-1)^k \left[\frac{c_3 w_3 + (-1)^k c_5}{1+(-1)^k c_4 w_3}\right], \end{aligned} \tag{22}$$

with $k = 0, 1$ and

$$\begin{aligned} w_1 &= \sin\theta\cos\phi, \\ w_2 &= \sin\theta\sin\phi, \\ w_3 &= \cos\theta. \end{aligned} \tag{23}$$

Then, by evaluating von Neumann entropy from Eqs. (20) and (21) and using that $S(V\Pi_k V^\dagger) = 0$, we obtain

$$S(\rho_k) = -\frac{(1+\theta_k)}{2}\log\frac{(1+\theta_k)}{2} - \frac{(1-\theta_k)}{2}\log\frac{(1-\theta_k)}{2}, \tag{24}$$

with

$$\theta_k = \sqrt{\sum_{j=1}^{3} q_{kj}^2}. \tag{25}$$

Therefore, the classical correlation for the spin pair at sites i and j will be given by

$$C(\rho) = \max_{\{B_k\}} \left(S(\rho^A) - \frac{(S_0+S_1)}{2} - c_4 w_3 \frac{(S_0-S_1)}{2} \right), \tag{26}$$

where $S_k = S(\rho_k)$. For some cases, the maximization in Eq. (26) can be worked out and an expression purely in terms of the spin correlation functions can be obtained (e.g., the XXZ and Ising chains below). In general, however, $C(\rho)$ has to be numerically evaluated by optimizing over the angles θ and ϕ. Once classical correlation is obtained, insertion of Eqs. (17) and (26) into Eq. (**??**) can be used to determine the quantum discord.

5 The XXZ spin chain

Let us illustrate the discussion of classical and quantum correlations between two spins by considering the XXZ spin chain, whose Hamiltonian is given by

$$H_{XXZ} = -\frac{J}{2}\sum_{i=1}^{L} \left(\sigma_i^x\sigma_{i+1}^x + \sigma_i^y\sigma_{i+1}^y + \Delta\sigma_i^z\sigma_{i+1}^z\right), \tag{27}$$

where periodic boundary conditions are assumed, ensuring therefore translation symmetry. We will set the energy scale such that $J = 1$ and will be interested in a nearest-neighbor spin pair at sites i and $i+1$. Concerning its symmetries, the XXZ chain exhibits $U(1)$ invariance, namely, $\left[H, \sum_i \sigma_z^i\right] = 0$, which provides a stronger constraint over the elements of the density matrix than the Z_2 symmetry.

Indeed, $U(1)$ invariance ensures that the element f of the reduced density matrix given by Eq. (12) vanishes. Moreover, the ground state has magnetization density $G_z^k = \langle \sigma_z^k \rangle = 0$ $(\forall\, k)$, which implies that

$$\begin{aligned} a &= d = \frac{1}{4}\left(1 + G_{zz}\right), \\ b_1 &= b_2 = \frac{1}{4}\left(1 - G_{zz}\right), \\ z &= \frac{1}{4}\left(G_{xx} + G_{yy}\right), \\ f &= 0. \end{aligned} \tag{28}$$

where, due to translation invariance, we write $G_{\alpha\beta} = \langle \sigma_\alpha^i \sigma_\beta^{i+1} \rangle$ $(\forall\, i)$. Due to the fact that $a = d$, we will have that $c_4 = c_5 = 0$, which considerably simplifies the computation of classical and quantum correlations. Moreover, we will have that $c_1 = c_2 = 2z$ and $c_3 = 4a - 1$. Then, the maximization procedure in Eq. (26) can be analytically worked out [**?**], yielding

$$C(\rho) = \frac{(1-c)}{2}\log\left(1-c\right) + \frac{(1+c)}{2}\log\left(1+c\right), \tag{29}$$

with $c = \max\left(|c_1|, |c_2|, |c_3|\right)$. For the mutual information $I(\rho)$ we obtain

$$I(\rho) = 2 + \sum_{i=0}^{3} \lambda_i \log \lambda_i, \tag{30}$$

where

$$\begin{aligned} \lambda_0 &= \frac{1}{4}\left(1 - c_1 - c_2 - c_3\right), \\ \lambda_1 &= \frac{1}{4}\left(1 - c_1 + c_2 + c_3\right), \\ \lambda_2 &= \frac{1}{4}\left(1 + c_1 - c_2 + c_3\right), \\ \lambda_3 &= \frac{1}{4}\left(1 + c_1 + c_2 - c_3\right). \end{aligned} \tag{31}$$

In order to compute $C(\rho)$ and $Q(\rho)$ we write c_1, c_2, and c_3 in terms of the ground state energy density. By using the Hellmann-Feynman theorem [**?**, **?**] for the XXZ Hamiltonian (27), we obtain

$$\begin{aligned} c_1 &= c_2 = \frac{1}{2}\left(G_{xx} + G_{yy}\right) = \Delta \frac{\partial \varepsilon_{xxz}}{\partial \Delta} - \varepsilon_{xxz}, \\ c_3 &= G_{zz} = -2\frac{\partial \varepsilon_{xxz}}{\partial \Delta}, \end{aligned} \tag{32}$$

where ε_{xxz} is the ground state energy density

$$\varepsilon_{xxz} = \frac{\langle \psi_0 | H_{XXZ} | \psi_0 \rangle}{L} = -\frac{1}{2}\left(G_{xx} + G_{yy} + \Delta G_{zz}\right), \tag{33}$$

with $|\psi_0\rangle$ denoting the ground state of H_{XXZ}. Eqs. (32) and (33) hold for a chain with an arbitrary number of sites, allowing the discussion of correlations either for finite or infinite chains. Indeed, ground state energy as well as its derivatives can be exactly determined by Bethe Ansatz technique [**?**], which

allows us to obtain the correlation functions c_1, c_2, and c_3. In Fig. **??**, we plot classical and quantum correlations between nearest-neighbor pairs for an infinite XXZ spin chain.

Note that, in the classical Ising limit $\Delta \to \infty$, we have a fully polarized ferromagnet. The ground state is then a doublet given by the vectors $|\uparrow\uparrow\cdots\uparrow\rangle$ and $|\downarrow\downarrow\cdots\downarrow\rangle$, yielding the mixed state

$$\rho = \frac{1}{2}|\uparrow\uparrow\cdots\uparrow\rangle\langle\uparrow\uparrow\cdots\uparrow| + \frac{1}{2}|\downarrow\downarrow\cdots\downarrow\rangle\langle\downarrow\downarrow\cdots\downarrow|. \tag{34}$$

Indeed, this is simply a classical probability mixing, with $C(\rho) = I(\rho) = 1$ and $Q(\rho) = 0$. The same applies for the antiferromagnetic Ising limit $\Delta \to -\infty$, where a doubly degenerate ground state arises. Moreover, observe that the classical (quantum) correlation is a minimum (maximum) at the infinite order QCP $\Delta = -1$. On the other hand, both correlations are discontinuous at the first-order QCP $\Delta = 1$. This is indeed in agreement with the usual behavior of entanglement both at infinite and first-order QPTs. For an infinite-order QCP, entanglement commonly display a maximum at the QCP [**?**, **?**, **?**], while for a first-order QCP, entanglement usually exhibits a jump at the QCP [**?**, **?**]. Nevertheless, we note that in the specific case of the ferromagnetic QCP $\Delta = 1$ and for pairwise entanglement measures such as concurrence [**?**] and negativity [**?**], no jump is detected, being hidden by the operation max [**?**]. It is interesting to observe the behavior of the functions $|c_1| = |c_2|$ and $|c_3|$ that govern the classical and quantum correlations. For $\Delta < -1$, we have that $|c_1| = |c_2| < |c_3|$, which means that the classical correlation is governed by $|c_3|$. For $-1 < \Delta < 1$, we have that $|c_1| = |c_2| > |c_3|$, with the crossing occurring exactly at the infinite-order QCP. Therefore, the correlations are governed by different parameters in different phases. For $\Delta \geq 1$, we obtain $|c_1| = |c_2| = 0$ and $|c_3| = 1$, which implies that $C(\rho) = 1$ and $Q(\rho) = 0$. These results are shown in Fig. **??** below.

6 The transverse field Ising model

Let us consider now the Ising chain in a transverse magnetic field, whose Hamiltonian is given by

$$H_I = -J\sum_{i=1}^{L}\left(\sigma_i^x\sigma_{i+1}^x + g\sigma_i^z\right), \tag{35}$$

with periodic boundary conditions assumed, namely, $\sigma_{L+1}^x = \sigma_1^x$. As before, we will set the energy scale such that $J = 1$ and will be interested in a nearest-neighbor spin pair at sites i and $i+1$. This Hamiltonian is Z_2-symmetric and can be exactly diagonalized by mapping it to a spinless free fermion model with single orbitals. This is implemented through the Jordan-Wigner transformation

$$\begin{aligned}\sigma_i^z &= 1 - 2c_i^\dagger c_i,\\ \sigma_i^x &= -\prod_{j<i}\left(1 - 2c_j^\dagger c_j\right)\left(c_i + c_i^\dagger\right),\end{aligned} \tag{36}$$

where $c_i^\dagger$ and c_i are the creation and annihilation fermion operators at site i, respectively. By rewriting Eq. (35) in terms of $c_i^\dagger$ and c_i we obtain

$$\begin{aligned} H_I &= -J\sum_{i=1}^{L}\left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i + c_i^\dagger c_{i+1}^\dagger + c_i c_{i+1}\right) \\ &\quad -Jg\sum_{i=1}^{L}\left(1-2c_i^\dagger c_i\right). \end{aligned} \tag{37}$$

In order to diagonalize H_I we consider fermions in momentum space

$$\begin{aligned} c_k &= \frac{1}{\sqrt{L}}\sum_{j=1}^{L} c_j e^{-ikr_j}, \\ c_k^\dagger &= \frac{1}{\sqrt{L}}\sum_{j=1}^{L} c_j^\dagger e^{ikr_j}, \end{aligned} \tag{38}$$

where $c_k^\dagger$ and c_k are creation and annihilation fermion operators with momentum k, respectively, and r_j is the fermion position at site j. The wave vectors $\overrightarrow{k}$ satisfy the relation $ka = 2\pi q/L$, where a denotes the distance between two nearest-neighbor sites and $q = -M, -M+1, \cdots, M-1, M$, with $M = (L-1)/2$ and L taken, for simplicity, as an even number. Then, by inverting Eq. (38) and inserting the result in Eq. (37), we obtain

$$\begin{aligned} H_I &= J\sum_k \Big[2\left(g-\cos ka\right)c_k^\dagger c_k \\ &\quad + i\sin ka\left(c_{-k}^\dagger c_k^\dagger + c_{-k}c_k\right) - g\Big]. \end{aligned} \tag{39}$$

Diagonalization is then obtained by eliminating the terms $c_{-k}^\dagger c_k^\dagger$ and $c_{-k}c_k$ from the Hamiltonian given by Eq. (39), which do not conserve the particle number. This is indeed achieved through the Bogoliubov transformation in which new fermion operators γ_k and $\gamma_k^\dagger$ are introduced as linear combination of c_k and $c_k^\dagger$

$$\begin{aligned} \gamma_k &= u_k c_k - i v_k c_{-k}^\dagger, \\ \gamma_k^\dagger &= u_k c_k^\dagger + i v_k c_{-k}, \end{aligned} \tag{40}$$

where u_k and v_k are real numbers parametrized by $u_k = \sin\frac{\theta_k}{2}$ and $v_k = \cos\frac{\theta_k}{2}$. This parametrization naturally arises as a consequence of the fermionic algebra $\{\gamma_k, \gamma_{k'}^\dagger\} = \delta_{kk'}$, $\{\gamma_k^\dagger, \gamma_{k'}^\dagger\} = \{\gamma_k, \gamma_{k'}\} = 0$, with $\delta_{kk'}$ standing for the Kronecker delta symbol. Moreover, to recast the Hamiltonian in a diagonal form we define θ_k by demanding that $\tan\theta_k = \sin ka/(g-\cos ka)$. Therefore, by expressing H_I in terms of Bogoliubov fermions and by imposing the trace invariance of the Hamiltonian, Eq. (39) becomes

$$H_I = \sum_k \varepsilon_k\left(\gamma_k^\dagger\gamma_k - \frac{1}{2}\right), \tag{41}$$

with $\varepsilon_k = 2J\sqrt{1+g^2-2g\cos ka}$. Hamiltonian (41) is diagonal, with ground state given by the γ-fermion vacuum. The procedure above also applies for the evaluation of the matrix elements of the

reduced density operator given by Eq. (13), which amounts for the computation of the magnetization density G_z and the two-point functions $G_{\alpha\beta}$. This can be achieved by using that $G_{zz} = G_z^2 - G_{xx}G_{yy}$ [**?**] and by expressing the remaining correlation functions as

$$\begin{aligned} G_{xx} &= \frac{2}{L}\sum_{q=-M}^{M}\left[\cos\left(\frac{2\pi q}{L}\right)v_q^2 + \sin\left(\frac{2\pi q}{L}\right)u_q v_q\right], \\ G_{yy} &= \frac{2}{L}\sum_{q=-M}^{M}\left[\cos\left(\frac{2\pi q}{L}\right)v_q^2 - \sin\left(\frac{2\pi q}{L}\right)u_q v_q\right], \\ G_z &= \frac{1}{L}\sum_{q=-M}^{M}\left(1-2v_q^2\right), \end{aligned} \tag{42}$$

where

$$\begin{aligned} u_q v_q &= \frac{1}{2}\frac{\sin\left(\frac{2\pi q}{L}\right)}{\sqrt{1+g^2-2g\cos\left(\frac{2\pi q}{L}\right)}} \\ v_q^2 &= \frac{1}{2}\left[1-\frac{\left(g-\cos\left(\frac{2\pi q}{L}\right)\right)}{\sqrt{1+g^2-2g\cos\left(\frac{2\pi q}{L}\right)}}\right]. \end{aligned} \tag{43}$$

Hence, we exactly determine the two-spin reduced density matrix. Classical and quantum correlations can then be directly obtained from Eqs. (**??**) and (**??**). By numerically computing the classical correlation in Eq. (26) for nearest-neighbor spin pairs at sites i and $i+1$, we can show that the maximization is achieved for any g by the choice $\theta = \pi/2$ and $\phi = 0$. Then, the measurement that maximizes $J(\rho : \{B_k\})$ is given by $\{|+\rangle\langle+|, |-\rangle\langle-|\}$, with $|+\rangle$ and $|-\rangle$ denoting the up and down spins in the x direction, namely, $|\pm\rangle = (|\uparrow\rangle \pm |\uparrow\rangle)/\sqrt{2}$. This numerical observation implies that $w_1 = 1$, $w_2 = w_3 = 0$. Therefore, Eq. (26) is ruled by the spin functions $c_1 = G_{xx}^{i,i+1}$ and $c_4 = c_5 = G_z^i$, i.e.

$$C(\rho) = H_{bin}\left(p_1\right) - H_{bin}\left(p_2\right) \tag{44}$$

where H_{bin} is the binary entropy

$$H_{bin}(p) = -p\log p - (1-p)\log\left(1-p\right) \tag{45}$$

and

$$\begin{aligned} p_1 &= \frac{1}{2}\left(1+G_z^i\right), \\ p_2 &= \frac{1}{2}\left(1+\sqrt{\left(G_{xx}^{i,i+1}\right)^2 + \left(G_z^i\right)^2}\right) \end{aligned} \tag{46}$$

We plot $C(\rho)$ and $Q(\rho)$ in Fig. **??** for a chain with 1024 sites. Note that, for $g = 0$ the system is a classical Ising chain, whose ground state is a doublet given by the vectors $|++\cdots+\rangle$ and $|--\cdots-\rangle$. Therefore, the system is in the mixed state

$$\rho = \frac{1}{2}|++\cdots+\rangle\langle++\cdots+| + \frac{1}{2}|--\cdots-\rangle\langle--\cdots-|, \tag{47}$$

with $C(\rho) = I(\rho) = 1$ and $Q(\rho) = 0$. On the other hand, in the limit $g \to \infty$ the system is a paramagnet (vanishing magnetization in the x direction), with all spins in state $|\uparrow\rangle$. Therefore the system will be described by the density operator

$$\rho = |\uparrow\uparrow\cdots\uparrow\rangle\langle\uparrow\uparrow\cdots\uparrow|, \tag{48}$$

which is a pure separable state, containing neither classical nor quantum correlations.

The QPT from ferromagnetic to paramagnetic state is a second-order QPT and occurs at $g = 1$. Signatures of this QPT can be found out by looking at the derivatives of either classical or quantum correlations. Indeed, the QPT can be identified as a pronounced minimum of the first derivative of the classical correlation, which is exhibited in Fig. **??**. Note that the minimum logarithmically diverges at $g = 1$ as the thermodynamic limit is approached (see inset of Fig. **??**). In the case of quantum correlations, its first derivative shows an inflexion point around $g = 1$, as displayed in Fig. **??**. Indeed, by looking at its second derivative in Fig. **??**, the QPT is identified by a pronounced maximum, which shows quadratic logarithmic divergence at $g = 1$ as the thermodynamic limit is approached (see inset of Fig. **??**).

The behavior of the quantum discord is therefore rather different from the entanglement behavior, whose first derivative is already divergent at the QCP. Remarkably, the scaling of pairwise entanglement derivative in this case (see e.g. Refs. [**?**, **?**]) is much closer to the scaling of the classical correlation derivative (as given by Fig. **??**) than that of the quantum correlation derivative (as given by Fig. **??**). As in the case of the XXZ model, it is interesting to observe that the spin functions $c_1 = G_{xx}^{i,i+1}$ and $c_4 = c_5 = G_z^i$, which govern the correlations in the Ising chain [see Eqs. (44)-(46)], exhibit a crossing at the QCP. This is shown in Fig. **??** for a chain with 1024 sites.

7 The LMG model

The discussion of correlations above can also be applied in collective systems. As an illustration, we will consider here the LMG model [**?**], which describes a two-level Fermi system $\{|+\rangle, |-\rangle\}$, with each level having degeneracy Ω. The Hamiltonian for LMG model is given by

$$\begin{aligned} H = \lambda \sum_{m=1}^{\Omega} \frac{1}{2} \left(c_{+m}^{\dagger} c_{+m} - c_{-m}^{\dagger} c_{-m} \right) \\ -\frac{1}{2N} \sum_{m,n=1}^{\Omega} \left(c_{+m}^{\dagger} c_{-m} c_{+n}^{\dagger} c_{-n} + c_{-n}^{\dagger} c_{+n} c_{-m}^{\dagger} c_{+m} \right). \end{aligned} \tag{49}$$

The operators $c_{+m}^{\dagger}$ and $c_{-m}^{\dagger}$ create a particle in the upper and lower levels, respectively. This Hamiltonian can be taken as describing an effective model for many-body systems, with one level just below the Fermi level and and the other level just above, with the level below being filled with Ω particles [**?**]. Alternatively, the LMG model can be seen as a one-dimensional ring of spin-1/2 particles with infinite

range interaction between pairs. Indeed, the Hamiltonian can be rewritten as

$$H = \lambda S_z - \frac{1}{N}\left(S_x^2 - S_y^2\right), \tag{50}$$

where $S_z = \sum_{m=1}^{N} \frac{1}{2}(c_{+m}^{\dagger} c_{+m} - c_{-m}^{\dagger} c_{-m})$ and $S_x + iS_y = \sum_{m=1}^{N} c_{+m}^{\dagger} c_{-m}$ [**?**]. The system undergoes a second-order QPT at $\lambda = 1$. As $\Omega \to \infty$, the ground state, as given by the Hartree-Fock (HF) approach, reads

$$|HF\rangle = \prod_{m=1}^{\omega} a_{0m}^{\dagger} |-\rangle, \tag{51}$$

where we have introduced new levels labelled by 0 and 1 governed by the operators

$$\begin{aligned} a_{0m}^{\dagger} &= \cos\alpha\, c_{-m}^{\dagger} + \sin\alpha\, c_{+m}^{\dagger}, \\ a_{1m}^{\dagger} &= -\sin\alpha\, c_{-m}^{\dagger} + \cos\alpha\, c_{+m}^{\dagger}. \end{aligned} \tag{52}$$

In Eq. (52), α is a variational parameter to be adjusted in order to minimize energy, which is achieved according to the choice

$$\begin{aligned} \lambda < 1 \quad &\Rightarrow \quad \cos 2\alpha = \lambda, \\ \lambda \geq 1 \quad &\Rightarrow \quad \alpha = 0. \end{aligned} \tag{53}$$

Despite being an approximation, the HF ground state provides the exact description of the critical point (for recent discussions of the exact spectrum of the LMG model, see Refs. [**?**, **?**]). The pairwise density operator for general modes $i \equiv (+m)$ and $j \equiv (-n)$ is given by

$$\rho_{i,j} = \begin{pmatrix} \langle M_i M_j \rangle & 0 & 0 & 0 \\ 0 & \langle M_i N_j \rangle & \langle c_i^{\dagger} c_j \rangle & 0 \\ 0 & \langle c_j^{\dagger} c_i \rangle & \langle N_i M_j \rangle & 0 \\ 0 & 0 & 0 & \langle N_i N_j \rangle \end{pmatrix}, \tag{54}$$

where $M_k = 1 - N_k$ and $N_k = c_k^{\dagger} c_k$, with $k = i, j$. By evaluating the matrix elements of ρ for the HF ground state, we obtain

$$\begin{aligned} \langle M_{+m} M_{-n} \rangle &= \sin^2\alpha \cos^2\alpha \left(1 - \delta_{mn}\right), \\ \langle M_{+m} N_{-n} \rangle &= \cos^2\alpha \delta_{mn} + \cos^4\alpha \left(1 - \delta_{mn}\right) \\ \langle N_{+m} M_{-n} \rangle &= \sin^2\alpha \delta_{mn} + \sin^4\alpha \left(1 - \delta_{mn}\right) \\ \langle N_{+m} N_{-n} \rangle &= \sin^2\alpha \cos^2\alpha \left(1 - \delta_{mn}\right) \\ \langle c_{+m}^{\dagger} c_{-n} \rangle &= \sin\alpha \cos\alpha \delta_{mn} \\ \langle c_{-n}^{\dagger} c_{+m} \rangle &= \sin\alpha \cos\alpha \delta_{mn}. \end{aligned} \tag{55}$$

Note that Eq. (54) displays Z_2 symmetry and, therefore, classical and quantum correlations can be computed by using Eq. (26). Note also that, for $m \neq n$, the density matrix is diagonal and the state is completely pairwise uncorrelated. On the other hand, for $m = n$, there is an equal amount of

classical and quantum correlations between the modes. These correlations vanish for $\lambda > 1$, which is the fully polarized state. The result is plotted in Fig. **??**. We can then observe that the derivatives of both classical correlation and quantum discord exhibit a signature of the QPT (see inset of Fig. **??**). These signatures are in agreement with the caracterizations in terms of entanglement [**?**, **?**] and Fisher information [**?**].

8 Concluding remarks

Quantum correlations in a composite system can be measured by employing the local quantum uncertainty.

We show that the quantum correlations quantified by local quantum uncertainty remain constant during the evolution of a class of two qubits under specific decoherence channels. This remarkable result is known in the literature as quantum correlation freezing. This result can bring a tool in understanding the inevitable decoherence due to the interaction with the environment and possibly open new ways to exploit quantum correlations from a practical point of view.

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