Local quantum uncertainty for a class of two-qubit X states and quantum correlations dynamics under decoherence

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Abstract

A special emphasis is devoted to the concept of local quantum uncertainty as indicator of quantum correlations. We study quantum discord for a class of two-qubit states parameterized by two parameters. Quantum discord based on local quantum uncertainty, von Neumann entropy and trace distance (Schatten 1-norm) are explicitly derived and compared. The behavior of the local quantum uncertainty quantifier under decoherence effects is investigated .

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1 Introduction

Characterizing quantum correlations in multipartite quantum systems is one of the most challenging topics in quantum information theory. Various measures to quantify the degree of quantumness in a multipartite quantum system were introduced in the literature. The most familiar ones the concurrence, the entanglement of formation, the quantum discord and its various geometric versions [1, 2, 3, 4, 5]. The interest in quantum correlations other than entanglement lies in the existence of nonclassical correlations even in separable states [6, 7]. In fact, entanglement does not account for all nonclassical aspects of correlations, especially in mixed states. This yielded many works dedicated to introduce quantum correlation quantifiers beyond entanglement. As the total correlation is the sum of two contributions: a classical part and quantum part, different concepts were considered to develop the best way to distinguish between classical and quantum correlations. In this context, the entropy based quantum discord [6, 7] is probably the quantifier which has been intensively investigated in the literature for different purposes and from several perspectives (see for instance [5]). However, it must be noticed that the analytical evaluation of quantum discord which is in general very challenging. Only partial results were obtained for few two-qubit systems. To overcome such technical difficulties and to find reliable and computable quantifiers, geometric variants of quantum discord were introduced by considering different geometrical measures. Indeed, The 2-norm (Hilbert-Schmidt norm) version of the quantum discord was introduced in [8]. This quantum correlation indicator is easily computable [9, 10, 11, 12]. However, despite its computability for any bipartite quantum system, the Hilbert-Schmidt based quantum discord can increase under local operations on the unmeasured qubit. This drawback of quantum correlation quantifier based on Hilbert-Schmidt norm comes from the non-contractibility of the 2-norm (Schatten 2-norm) [13]. Now, it is well known that the only norm among the Schatten p-norm which is contractible is the Bures norm (trace norm with $p = 1$) and which constitutes a suitable tool to quantify geometrically the quantum discord (see for instance [14, 15]).

Quantifying quantum correlations in multipartite quantum systems continues to draw special attention in quantum information science. Hence, another reliable geometric quantifier of discord-like correlations was recently introduced by employing the so-called local quantum uncertainty. This quantifier uses the notion skew information introduced in [16] to determine the uncertainty in the measurement of an observable. The local quantum uncertainty is given by the minimum of the skew information over all possible local observales. This measure offers an appropriate tool to evaluate the analytical expressions of quantum correlations encompassed in any qubit-qudit bipartite system [17]. The local quantum uncertainty is related to the quantum Fisher information [18, 19, 20] which is a key ingredient in quantum metrology protocols [21]. Also, it quantifies the speed of the local (unitary) evolution of a bipartite quantum system [17].

In this paper the analytical derivation of quantum discord is essentially approached in the context of local quantum uncertainty formalism. We consider a particular family of rank-2 X states which includes various types of two-qubit states of interest in different models of collective spin systems like for instance Dicke model [22] and Lipkin-Meshkov-Glick model [23] where the quantum discord was investigated in relation with their critical properties and quantum phase transitions (see for instance [24, 25, 26, 27]). Remarkably, it has been shown that the quantum discord provides a suitable indicator to understand the role of quantum correlations in characterizing quantum phase transitions [28](see also [29]). We note also that the set of two-qubit under consideration are of special relevance in investigating quantum correlations in bipartite states extracted from multi-qubit Dicke states and their superpositions(e.g., generalized GHZ states, even and odd spin coherent states) [30]. Thus, beside the explicit derivation of local quantum uncertainty, we also the amount of quantum correlations in such states when measured by von Neumann entropy or trace distance. Another facet of this work concerns the dynamics of the local quantum uncertainty under decoherence effects induced by the unavoidable interaction of a quantum system with environment. Four typical quantum decoherence channels are considered. The explicit expressions of local quantum uncertainty are derived for each case. We will show that in some cases the local quantum uncertainty is unaffected by the decoherence channel effects.

The paper is structured as follows. In section 2, we give the explicit expressions for local quantum uncertainty, the von Neumann entropy based quantum discord and the trace norm quantum discord for a class of two-qubit states which are, as we mentioned already, relevant in investigating bipartite quantum correlations in various collective spin models. In section 3, under four quantum decoherence channels (bit flip, phase flip, bit-phase flip and generalized amplitude damping), we give the analytic expressions of local quantum uncertainty. In particular, we show the freezing character of local quantum uncertainty in some special cases. Concluding remarks close this paper.

2 Local quantum uncertainty, entropic quantum discord and geometric quantum for rank two X states

The two-qubit density matrices which display non zero entries only along the main- and anti-diagonals are usually called X-states. They generalize several two-qubit states as for instance Bell-diagonal states (see [31]), Werner states [32], isotropic states [33]. Their particular relevance was first identified in investigating the phenomenon of sudden death of entanglement [34]and since then extended to many other context in connection of quantum information theory. A generic X-state is parameterized by seven real parameters and the corresponding symmetry is fully characterized by the $su(2)\times su(2)\times u(1)$ subalgebra of the full $su(4)$ algebra describing an arbitrary two-qubit system [35]. This symmetry reduction from $su(4)$ to $su(2) \times su(2) \times u(1)$ renders easy many analytical calculations of concurrence, entanglement of formation, quantum discord and leads to interesting results in studying their properties and especially their evolution under dissipative processes were reported in the literature (see for instance [36, 37]).

In this work we consider the set of two-qubit density matrices which have the following form

$$\rho = \begin{pmatrix} c_1 & 0 & 0 & \sqrt{c_1c_2} \\ 0 & \frac{1}{2}(1-c_1-c_2) & \frac{1}{2}(1-c_1-c_2) & 0 \\ 0 & \frac{1}{2}(1-c_1-c_2) & \frac{1}{2}(1-c_1-c_2) & 0 \\ \sqrt{c_1c_2} & 0 & 0 & c_2 \end{pmatrix} \tag{1}$$

in the computational basis $\mathcal{B} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The parameters c_1 and c_2 satisfy the conditions $0 \leq c_1 \leq 1$, $0 \leq c_2 \leq 1$ and $0 \leq c_1 + c_2 \leq 1$. We assume that all entries of the matrix ρ are positives. In fact, the local unitary transformation, acting on the qubit 1 and the qubit 2 forming the system,

$$|0\rangle_k \to \exp\left(\frac{i}{2}(\theta_1 + (-)^k\theta_2)\right)|0\rangle_k \quad k = 1, 2$$

eliminates the phase factors of the off diagonal elements and the rank of the density matrix ρ remains unchanged. In the Fano-Bloch representation, the density ρ writes

$$\rho = \frac{1}{4}\sum_{\alpha,\beta} R_{\alpha\beta}\sigma_\alpha \otimes \sigma_\beta$$

where the correlation matrix $R_{\alpha\beta}$ are given by $R_{\alpha\beta} = \text{Tr}(\sqrt{\rho}\ \sigma_\alpha \otimes \sigma_\beta)$ with $\alpha, \beta = 0, 1, 2, 3$. The non-vanishing correlation matrix elements are given by

$$R_{30} = R_{03} = c_1 - c_2 \quad R_{33} = 2(c_1 + c_2) - 1, \quad R_{11} = 1 - (\sqrt{c_1} - \sqrt{c_2})^2 \quad R_{22} = 1 - (\sqrt{c_1} + \sqrt{c_2})^2. \tag{2}$$

The density matrix (1) is invariant under parity symmetry and exchange transformation (ρ commutes with $\sigma_3 \otimes \sigma_3$ and the permutation operator which exchanges the qubit state $|i, j\rangle$ to $|j, i\rangle$ leaves ρ unchanged). These symmetries simplify considerably the complexity of the analytical evaluations of bipartite correlations. Indeed, from a practical viewpoint, our interest on this type of X states (1) relies upon their simple analytical manipulation in contrast with an arbitrary two-qubit state for which one is forced to resort heavy numerical approaches.

2.1 Local quantum uncertainty: Definition

The concept of local quantum uncertainty is now considered as a promising quantifier of quantum correlation. This is essentially due to its easiness of computability and its reliability. It quantifies the minimal quantum uncertainty in a quantum state due to a measurement of a local observable [17]. For a bipartite quantum state ρ_{12}, the local quantum uncertainty is defined as

$$\mathcal{U}(\rho_{12}) \equiv \min_{K_1} \mathcal{I}(\rho, K_1 \otimes \mathbb{I}_2), \tag{3}$$

where K_1 is some local observable on the subsystem 1, $\mathbb{I}_2$ is the identity operator and

$$\mathcal{I}(\rho_{AB}, K) = -\frac{1}{2}\text{Tr}([\sqrt{\rho_{AB}}, K]^2) \tag{4}$$

is the skew information [16, 18]. The skew information represents the non-commutativity between the state and the observable K_1. The analytical evaluation the local quantum uncertainty requirers a minimization procedure over the set of all observales acting on the part 1. A closed form for qubit-qudit systems was derived in [17]. In particular, for qubits ($\frac{1}{2}$-spin particles), the expression of the local quantum uncertainty is given by [17]

$$\mathcal{U}(\rho) = 1 - \lambda_{\max}\{W\}, \tag{5}$$

where $\lambda_{\max}$ denotes the maximum eigenvalue of the 3×3 matrix W whose matrix elements are defined by

$$\omega_{ij} \equiv \mathrm{Tr}\{\sqrt{\rho}(\sigma_i \otimes \mathbb{I}_2)\sqrt{\rho}(\sigma_j \otimes \mathbb{I}_2)\}, \tag{6}$$

with $i, j = 1, 2, 3$. The local quantum uncertainty provides an appropriate quantifier of the minimum amount of uncertainty in a bipartite quantum state. For pure bipartite states, it reduces to linear entropy of the reduced densities of the subsystems. Also, it vanishes for classically correlated states. Another interesting property of local quantum uncertainty is its invariance under local unitary operations. This quantum correlations indicator enjoys all required properties of being a reliable quantifier [17]. Hence, in what follows, we shall employ the local quantum uncertainty to study the pairwise quantum correlation in a family of two-qubit states. We shall compare the discord-like local quantum uncertainty with the geometric quantum discord based on the trace distance [13, 14, 15] and the entropy based quantum discord originally defined in [?, 5]. To get the explicit form of the matrix elements (6), one needs the squared matrix $\sqrt{\rho}$ that is given

$$\sqrt{\rho} = \begin{pmatrix} \frac{c_1}{\sqrt{c_1+c_2}} & 0 & 0 & \frac{\sqrt{c_1c_2}}{\sqrt{c_1+c_2}} \\ 0 & \frac{1}{2}\sqrt{1-c_1-c_2} & \frac{1}{2}\sqrt{1-c_1-c_2} & 0 \\ 0 & \frac{1}{2}\sqrt{1-c_1-c_2} & \frac{1}{2}\sqrt{1-c_1-c_2} & 0 \\ \frac{\sqrt{c_1c_2}}{\sqrt{c_1+c_2}} & 0 & 0 & \frac{c_2}{\sqrt{c_1+c_2}} \end{pmatrix} \tag{7}$$

in the computational basis. In Fano-Bloch representation, it rewrites as

$$\sqrt{\rho} = \frac{1}{4}\sum_{\alpha,\beta} \mathcal{R}_{\alpha\beta}\sigma_\alpha \otimes \sigma_\beta$$

with $\mathcal{R}_{\alpha\beta} = \mathrm{Tr}(\sqrt{\rho}\ \sigma_\alpha \otimes \sigma_\beta)$. The non vanishing matrix correlation elements $\mathcal{R}_{\alpha\beta}$ are explicitly given by

$$\mathcal{R}_{00} = \sqrt{c_1+c_2} - \sqrt{1-c_1-c_2}, \qquad \mathcal{R}_{03} = \mathcal{R}_{30} = c_1 - c_2$$

$$\mathcal{R}_{11} = \sqrt{1-c_1-c_2} + 2\frac{\sqrt{c_1c_2}}{\sqrt{c_1+c_2}}, \quad \mathcal{R}_{22} = \sqrt{1-c_1-c_2} - 2\frac{\sqrt{c_1c_2}}{\sqrt{c_1+c_2}}, \quad \mathcal{R}_{33} = \sqrt{c_1+c_2} - \sqrt{1-c_1-c_2}.$$

Reporting the matrix (7) in the equation (6), it is simple to check that only the off-diagonal elements of the matrix W are zero and the diagonal ones are given by and the

$$\omega_{11} = \sqrt{\frac{1-(c_1+c_2)}{c_1+c_2}}(\sqrt{c_1}+\sqrt{c_2})^2, \quad \omega_{22} = \sqrt{\frac{1-(c_1+c_2)}{c_1+c_2}}(\sqrt{c_1}-\sqrt{c_2})^2, \quad \omega_{33} = \frac{(c_1-c_2)^2}{c_1+c_2}. \tag{8}$$

We note that the eigenvalue ω_{11} is always larger than ω_{11} so that that $\omega_{\max} = \max(\omega_{11}, \omega_{33})$. To determine the states for which $\omega_{11} \leq \omega_{33}$ or $\omega_{33} \leq \omega_{11}$, one study the sign of the quantity $\omega_{11} - \omega_{33}$. In this respect, one verifies the following identity

$$\text{sign } (\omega_{11} - \omega_{33}) = \text{sign } \left(\sqrt{(c_1 + c_2)(1 - (c_1 + c_2))} - (\sqrt{c_1} - \sqrt{c_2})^2 \right). \tag{9}$$

The set of states of type (1) can be partitioned as

$$\{\rho \equiv \rho_{c_1,c_2}, \quad 0 \leq c_1 + c_2 \leq 1\} = \bigcup_{\alpha \in [0,1]} \{\rho_\alpha \equiv \rho_{c_1,\alpha - c_1}, \quad 0 \leq c_1 \leq \alpha\}$$

with $c_1 + c_2 = \alpha$. Therefore, for a fixed value of α, we have

$$\text{sign } \left(\sqrt{(c_1 + c_2)(1 - (c_1 + c_2))} - (\sqrt{c_1} - \sqrt{c_2})^2 \right) = \text{sign } \left(2\sqrt{c_1}\ \sqrt{\alpha - c_1} + \sqrt{\alpha}\ (\sqrt{1 - \alpha} - \sqrt{\alpha}) \right) \tag{10}$$

The function sign $(\omega_{11} - \omega_{33})$ is positive for $\alpha \leq \frac{1}{2}$. This gives $\omega_{\max} = \omega_{11}$. Conversely, for $\alpha \geq \frac{1}{2}$, one verifies that the function (10) is positive for $\alpha_- \leq c_1 \leq \alpha_+$ and negative for $0 \leq c_1 \leq \alpha_-$ or $\alpha_+ \leq c_1 \leq \alpha$ where the quantities α_- and α_+ are given by

$$\alpha_\pm = \frac{\alpha}{2} \pm \frac{1}{2} \sqrt{\alpha\ \sqrt{1 - \alpha}\ \left(2\sqrt{\alpha} - \sqrt{1 - \alpha}\right)}. \tag{11}$$

Accordingly, the maximum eigenvalue of the matrix W (6) for the states ρ (1) with $\alpha \geq \frac{1}{2}$ writes as

$$\omega_{\max} = \begin{cases} \omega_{33} & \text{for} \quad 0 \leq c_1 \leq \alpha_- \\ \omega_{11} & \text{for} \quad \alpha_- \leq c_1 \leq \alpha_+ \\ \omega_{33} & \text{for} \quad \alpha_+ \leq c_1 \leq \alpha \end{cases} \tag{12}$$

To write the analytical expression of the local quantum uncertainty measure given by $\mathcal{U}(\rho) = 1 - \omega_{\max}$, the situations where the parameter α is greater or less than $\frac{1}{2}$ are treated separately.
For $\alpha \leq \frac{1}{2}$, we have

$$\mathcal{U}(\rho) = 1 - \omega_{11} = 1 - \sqrt{\frac{1 - \alpha}{\alpha}} (\sqrt{c_1} + \sqrt{\alpha - c_1})^2 \quad \text{with} \quad 0 \leq c_1 \leq \alpha. \tag{13}$$

For $\alpha \geq \frac{1}{2}$, on gets

$$\mathcal{U}(\rho) = 1 - \omega_{11} = 1 - \sqrt{\frac{1 - \alpha}{\alpha}} (\sqrt{c_1} + \sqrt{\alpha - c_1})^2 \quad \text{for} \quad \alpha_- \leq c_1 \leq \alpha_+, \tag{14}$$

and

$$\mathcal{U}(\rho) = 1 - \omega_{33} = 1 - \frac{(2c_1 - \alpha)^2}{\alpha} \quad \text{for} \quad 0 \leq c_1 \leq \alpha_- \quad \text{and} \quad \alpha_+ \leq c_1 \leq \alpha. \tag{15}$$

The quantum discord quantified by local quantum uncertainty in the states ρ (1) is depicted in the figure 1 for different values of α. We notice that the local quantum uncertainty is non zero except for $c_1 = c_2 = \frac{1}{4}$ with $\alpha = \frac{1}{2}$. The discord-like local quantum uncertainty goes beyond entanglement. Indeed, for the states ρ (1), the Wootters concurrence writes [38]

$$\mathcal{C}_{12}(\rho) = |(\sqrt{c_1} + \sqrt{c_2})^2 - 1| \tag{16}$$

with $\alpha = c_1 + c_2$. It is simply verified that for $\alpha < \frac{1}{2}$, the states are entangled. For $\alpha \geq \frac{1}{2}$, the states ρ with $(c_1, c_2) = \frac{1}{2}\big(\alpha + \sqrt{2\alpha - 1}, \alpha - \sqrt{2\alpha - 1}\big)$ and $(c_1, c_2) = \frac{1}{2}\big(\alpha - \sqrt{2\alpha - 1}, \alpha + \sqrt{2\alpha - 1}\big)$ are separable. For this special set of separable states, the local quantum uncertainty is non zero as it can verified from the equations (14) and (15). The figure 1 shows that for $\alpha \leq \frac{1}{2}$ the local quantum uncertainty is minimal in the states with $c_1 = c_2 = \frac{\alpha}{2}$ which are explicitly given by

$$\rho\big(c_1 = \frac{\alpha}{2}, c_2 = \frac{\alpha}{2}\big) = (1 - \alpha)|\psi_1\rangle\langle\psi_1| + \alpha|\psi_2\rangle\langle\psi_2| \tag{17}$$

with

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad , \quad |\psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{18}$$

On the other hand, the maximal amount of quantum correlations in the states with $\alpha \leq \frac{1}{2}$ is reached when $(c_1 = 0, c_2 = \alpha)$ or $(c_1 = \alpha, c_2 = 0)$. The maximally discord states in this case are given by

$$\rho(c_1 = 0, c_2 = \alpha) = \alpha|11\rangle\langle 11| + (1-\alpha)|\psi_1\rangle\langle\psi_1| \text{ or } \rho(c_1 = \alpha, c_2 = 0) = \alpha|00\rangle\langle 00| + (1-\alpha)|\psi_1\rangle\langle\psi_1|. \tag{19}$$

It is also important to stress that states with values of α approaching zero contain more quantum correlations. This situation is completely different for $\alpha \geq \frac{1}{2}$. In fact, as depicted in the figure 1, the local quantum uncertainty increases as the parameter α increases. Also, comparing the particular values $\alpha = 0.6$ and $\alpha = 0.9$, the figure 1 reveals that the minimal value of quantum correlations is obtained for $\alpha = 0.6$ when $(c_1 = \frac{\alpha}{2}, c_2 = \frac{\alpha}{2})$ and for $\alpha = 0.9$ when $(c_1 = 0, c_2 = \alpha)$ or $(c_1 = \alpha, c_2 = 0)$. Similarly, the maximal amount of quantum correlations is no longer obtained for states with $(c_1 = 0, c_2 = \alpha)$ or $(c_1 = \alpha, c_2 = 0)$ as for values of $\alpha \leq \frac{1}{2}$ but the states encompassing the maximal values of local quantum uncertainty are those with $(c_1 = \alpha_+, c_2 = \alpha_-)$ or $(c_1 = \alpha_-, c_2 = \alpha_+)$ where the quantities α_+ and α_- are given by (11). It is remarkable that for these two special states, the local quantum uncertainty presents a double sudden change indicating a discontinuity in the derivative of the local quantum uncertainty with respect the parameter c_1 reflecting the jump from ω_{11} to ω_{33}. This intriguing sudden change has no analogue for von Neumann based quantum discord as we shall discuss in what follows.

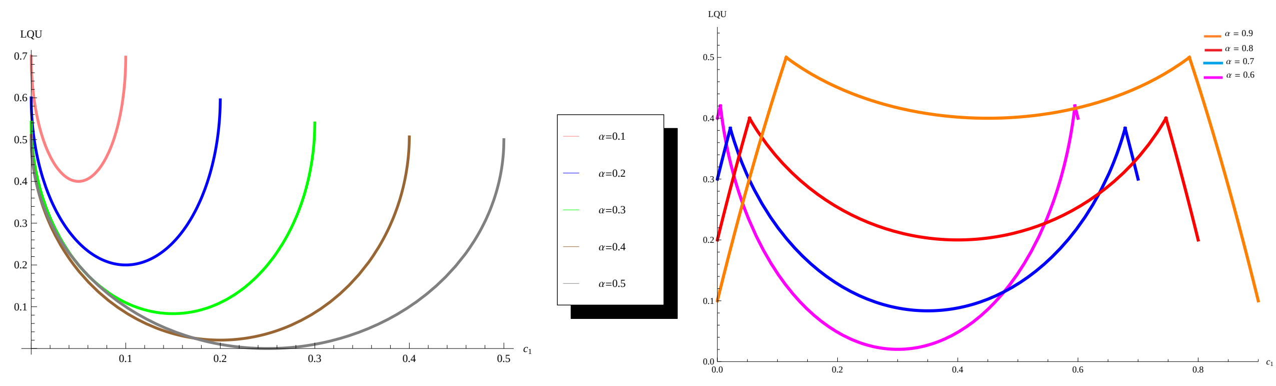

Figure 1. The Local quantum uncertainty $\mathcal{U}$ versus the parameter c_1 for different values of α.

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2.2 Entropy based quantum discord

The quantum discord in two-qubit states is defined as the difference between the mutual information and the classical correlations in a bipartite quantum system [6, 7]

$$D(\rho) = I(\rho) - C(\rho) \tag{20}$$

The total correlation is usually quantified by the mutual information I given by

$$I(\rho) = S(\rho_1) + S(\rho_2) - S(\rho), \tag{21}$$

where ρ is the state of a bipartite quantum system comprising two qubits labeled as 1 and 2, the operator $\rho_{1(2)} = \mathrm{Tr}_{1(2)}(\rho)$ is the reduced state of 1(2) and $S(\rho)$ is the von Neumann entropy of a quantum state ρ. The non vanishing eigenvalues of the density matrix ρ (1) are $\lambda_1 = c_1 + c_2$ and $\lambda_2 = 1 - (c_1 + c_2)$, so that the joint entropy writes as

$$S(\rho) = H(c_1 + c_2) \tag{22}$$

where $H(x) = -x \log_2 x - (1 - x) \log_2(1 - x)$ is the binary entropy function. The eigenvalues of the reduced density matrices ρ_1 and ρ_2 are identical. They are given by $\frac{1}{2}(1 + c_1 - c_2)$ and $\frac{1}{2}(1 - c_1 + c_2)$ so that the marginal entropy for ρ_1 and ρ_2 are given by

$$S(\rho_1) = S(\rho_2) = H\left(\frac{1 + c_1 - c_2}{2}\right). \tag{23}$$

It follows that in the states (1), the mutual information (21) writes as

$$I(\rho) = 2H\left(\frac{1 + c_1 - c_2}{2}\right) - H(c_1 + c_2). \tag{24}$$

To compute the classical correlations occurring in (20), one follows the method developed in [39] for rank-2 two-qubit states. Indeed, this method simplifies considerably the analytical derivation of the entropic quantum discord. It consists in purifying the two-qubit system by a third qubit describing the environment and making use of the KoashiWinter theorem [40]. This theorem constitutes the key ingredient in determining the quantum discord in two-qubit systems described by density matrices of rank 2. Moreover, it establishes a nice connection between the quantum discord and the entanglement of formation (for more details see the references [41, 42, 43]). In this approach, the classical correlation $C(\rho)$ is expressed in term of the entanglement of formation between the second qubit and the third qubit representing the environment. To apply this approach for the class of states of interest in this work and to employ the Koashi-Winter theorem, we first consider the purification of the states of type (1) re-expressed as follows

$$\rho = \lambda_1 |\Phi_1\rangle\langle\Phi_1| + \lambda_2 |\Phi_2\rangle\langle\Phi_2|$$

where λ_1 and λ_2 are the eigenvalues of ρ and $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are the corresponding eigenstates

$$|\Phi_1\rangle = \frac{\sqrt{c_1}}{\sqrt{c_1 + c_2}}|0,0\rangle + \frac{\sqrt{c_2}}{\sqrt{c_1 + c_2}}|1,1\rangle, \quad |\Phi_2\rangle = \frac{1}{\sqrt{2}}|0,1\rangle + \frac{1}{\sqrt{2}}|.1,0\rangle \tag{25}$$

Attaching a qubit 3 to the two-qubit system $1-2$, we write the purification of ρ as

$$|\Phi\rangle_{123} = \sqrt{\lambda_1}|\Phi_1\rangle \otimes |0\rangle_3 + \sqrt{\lambda_2}|\Phi_2\rangle \otimes |1\rangle_3 \tag{26}$$

such that the whole system 123 is described by the pure state $\rho_{123} = |\Phi\rangle_{123}\langle\Phi|$ from which one extracts the density matrix $\rho_{23} = {\rm Tr}_1\rho^{123}$ associated to the subsystem $2-3$. Suppose now that a von Neumann measurement $\{M_0, M_1\}$ is performed on the qubit 1 (here also we need positive operator valued measurement of rank one that is proportional the one dimensional projector). From the viewpoint of the whole system in the pure state $|\Phi\rangle_{123}$, the measurement gives rise to an ensemble of states for the system composed by the qubits 2 and 3. The Koashi-Winter theorem states the classical correlation is given by

$$C(\rho) = S(\rho_2) - E(\rho_{23}). \tag{27}$$

where $S(\rho_2)$ is given by (23) and the entanglement of formation $E(\rho_{23})$ is defined by

$$E(\rho_{23}) = H(\frac{1}{2} + \frac{1}{2}\sqrt{1-|C_{23}|^2}). \tag{28}$$

Using the Wootters formula [38], the concurrence C_{23} for the bipartite state ρ_{23} writes

$$|C_{23}|^2 = 2(1-c_1-c_2)(\sqrt{c_1}-\sqrt{c_2})^2. \tag{29}$$

Reporting the Koashi-Winter relation (27) in the definition (20) and using the equation (21), the quantum discord in the states ρ takes the simple form

$$D(\rho) = S(\rho_1) + E(\rho_{23}) - S(\rho), \tag{30}$$

which can be rewritten, using the expressions (22), (23) and (28), as

$$D(\rho) = H\left(\frac{1+c_1-c_2}{2}\right) + H(\frac{1}{2} + \frac{1}{2}\sqrt{1-2(1-c_1-c_2)(\sqrt{c_1}-\sqrt{c_2})^2}) - H(c_1+c_2), \tag{31}$$

in terms of the parameters c_1 and c_2. Setting $\alpha = c_1 + c_2$, the quantum discord in the states ρ_α, with a fixed value of α, is given by

$$D(\rho_\alpha) = H\left(\frac{1-\alpha}{2} + c_1\right) + H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-2(1-\alpha)(\alpha - 2\sqrt{c_1}\sqrt{\alpha-c_1})}\right) - H(\alpha). \tag{32}$$

The figure 2 shows that the quantum discord $D(\rho_\alpha)$ exhibits similar behavior as the quantum discord based on von Neumann entropy for $\alpha \leq \frac{1}{2}$. The minimal amount of quantum correlations is obtained in the states ρ_α with $(c_1 = c_2 = \frac{\alpha}{2})$ (17) and the maximally correlated states are the states with $(c_1 = 0, c_2 = \alpha)$ and $(c_1 = \alpha, c_2 = 0)$ given by (19). However, for the density matrices ρ_α with $\alpha \geq \frac{1}{2}$, we observe that the entropic quantum discord $D(\rho_\alpha)$ and discord-like local quantum uncertainty $\mathcal{U}(\rho_\alpha)$ have very different behavior. Indeed, for $\alpha = 0.6$ or $\alpha = 0.7$ the maximal amount of entropic quantum discord in the states ρ_α is reached for $(c_1 = 0, c_2 = \alpha)$ and $(c_1 = \alpha, c_2 = 0)$. This is the case with the local quantum uncertainty for which the maximally correlated states are given by $(c_1 = \alpha_+, c_2 = \alpha_-)$

or $(c_1 = \alpha_-, c_2 = \alpha_+)$ where α_+ and α_- are given by (11). Also, the states ρ_α with $\alpha = 0.8$ or $\alpha = 0.9$, the maximum of entropic quantum discord is reached in the states $(c_1 = c_2 = \frac{\alpha}{2})$ (17) but the figure 1 shows that the maximal amount of the quantum correlations $\mathcal{U}(\rho_\alpha)$ is obtained when $(c_1 = \alpha_+, c_2 = \alpha_-)$ or $(c_1 = \alpha_-, c_2 = \alpha_+)$. Furthermore, from figure 2, it can be inferred that the minimal value of the quantum discord $D(\rho_\alpha)$ for $\alpha = 0.8$ or $\alpha = 0.9$ is not obtained in the states with $(c_1 = 0, c_2 = \alpha)$ and $(c_1 = \alpha, c_2 = 0)$ as it is the case with local quantum uncertainty (see figure 1).

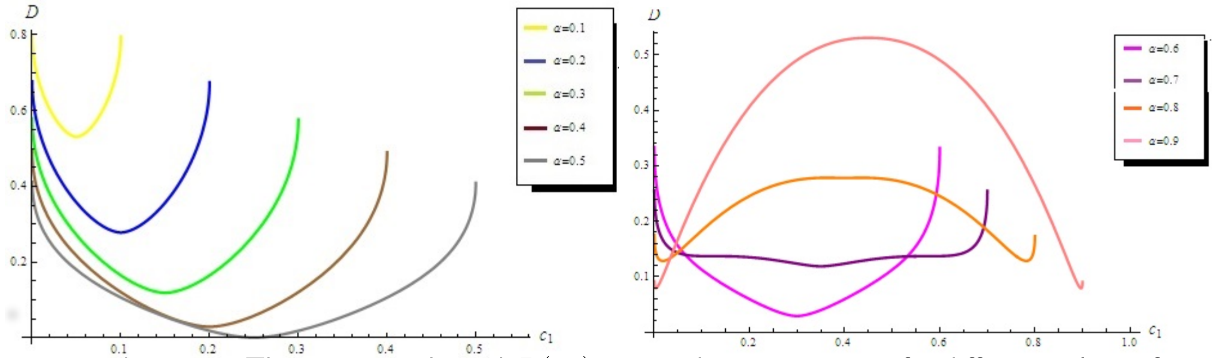

Figure 2. The quantum discord $D(\rho_\alpha)$ versus the parameter c_1 for different values of α.

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Il faut verifier que la premiere courbe est bonne. La seconde courbe est fausse: pour deux raisons: la premiere c'est que c1 doit varier entre 0 et alpha pour chaque valeur fixe de alpha. La seconde, la fiction discorde est symetrique par rapport a l'axe alpha/2

2.3 Geometric quantum discord

The optimization that involves quantum discord based on von Neumann entropy is very challenging for a generic bipartite system. Hence, despite the great deal of efforts, analytical expressions of this quantifier are available for only few two-qubit states as for instance X states of rank 2. To define reliable and computable quantifiers, several geometric approaches were considered in the literature. The first geometric variant of quantum discord was introduced by Dakić, Vedral and Brukner [8]. This geometric quantifier of quantum correlation in bipartite systems is based on the Hilbert-Schmidt distance which is defined in terms of Schatten 2-norm. Unfortunately, the Hilbert-Schmidt quantum discord is not a bona fide measure of bipartite quantum correlations [13]. This drawback is essentially due to the lack of contractivity of the HilbertSchmidt norm under trace-preserving quantum channels [44]. Indeed, it has been shown that among all Schatten p-norms only the Schatten 1-norm is contractive under trace-preserving quantum channels [14]. The trace distance quantum discord has been explicitly derived for general Bell-diagonal states [14, 45] and has been also extended for arbitrary X states [46]. By adopting Schatten 1-norm, the trace distance quantum discord for a bipartite state ρ is defined by

$$D_{\rm g}(\rho) = \frac{1}{2} \min_{\chi \in \Omega} ||\rho - \chi||_1, \tag{33}$$

where the trace distance is defined by $||\rho - \chi||_1 = \mathrm{Tr}\sqrt{(\rho-\chi)^\dagger(\rho-\chi)}$. It measures the distance between the state ρ and the classical-quantum state χ belonging to the set Ω of classical-quantum states. A generic state $\chi \in \Omega$ is of the form $\chi = \sum_k p_k \ \Pi_{k,1} \otimes \rho_{k,2}$ where $\{p_k\}$ is a probability distribution, $\Pi_{k,1}$ are the orthogonal projector associated with the qubit 1 and $\rho_{k,2}$ is density matrix associated with the second qubit. The minimization in (33) was analytically worked out in [46] for a generic X state. Therefore according to the result reported in [46], the 1-norm geometric discord in the states ρ (1) is expressed as

$$D_{\rm g}(\rho) = \frac{1}{2}\sqrt{\frac{R_{11}^2\max\{R_{33}^2, R_{22}^2+R_{03}^2\} - R_{22}^2\min\{R_{11}^2, R_{33}^2\}}{\max\{R_{33}^2, R_{22}^2+R_{03}^2\} - \min\{R_{11}^2, R_{33}^2\} + R_{11}^2 - R_{22}^2}}, \tag{34}$$

where the correlation matrix elements are given by (2). We notice that for the states under consideration quantities $R_{11}^2 - R_{33}^2 + R_{03}^2$ and $R_{22}^2 - R_{33}^2 + R_{03}^2$ are non negatives. Indeed, one has $R_{11}^2 - R_{33}^2 + R_{03}^2 = 2(1-(c_1+c_2))(\sqrt{c_1}+\sqrt{c_2})^2$ and $R_{22}^2 - R_{33}^2 + R_{03}^2 = 2(1-(c_1+c_2))(\sqrt{c_1}-\sqrt{c_2})^2$. It follows that one can further simplify the expression (34) and the quantum discord (34) rewrites

$$D_{\rm g}(\rho) = \frac{1}{2}\bigg[\Theta(|R_{33}|-|R_{11}|)\ |R_{11}| + \Theta(|R_{11}|-|R_{33}|)\ \sqrt{\frac{R_{11}^2(R_{22}^2+R_{03}^2) - R_{22}^2R_{33}^2}{R_{11}^2 - R_{33}^2 + R_{03}^2}}\bigg] \tag{35}$$

where $\Theta(.)$ is the usual Heaviside function. It turns out that one should treat the two distinct cases $|R_{33}| \leq |R_{11}|$ and $|R_{11}| \leq |R_{33}|$. In this respect, we set $c_1 + c_2 = \alpha$. It is simple to verify that for $0 \leq \alpha \leq \frac{2}{3}$, $|R_{11}|$ is always larger than $|R_{33}|$. For the two-qubit states $\rho \equiv \rho_\alpha$ with $\frac{2}{3} \leq \alpha \leq 1$, one verifies that $|R_{33}| \leq |R_{11}|$ when $c_1 \in [c_-, c_+]$ and $|R_{11}| \leq |R_{33}|$ when $c_1 \in [0, c_-] \cup [c_+, \alpha]$ where

$$c_\pm = \frac{\alpha}{2} \pm \sqrt{(1-\alpha)(2\alpha-1)}. \tag{36}$$

are the solutions of the equation $|R_{11}| = |R_{33}|$. As by product, the expression (35) becomes

$$D_{\rm g}(\rho_\alpha) = \frac{1}{2}\sqrt{\bigg(1-(\sqrt{c_1}+\sqrt{\alpha-c_1})^2\bigg)^2 + 4\sqrt{c_1(\alpha-c_1)}(\sqrt{c_1}-\sqrt{\alpha-c_1})^2} \tag{37}$$

for $0 \leq \alpha \leq \frac{2}{3}$. For the states ρ_α with $\frac{2}{3} \leq \alpha \leq 1$, one gets

$$D_{\rm g}(\rho_\alpha) = \begin{cases} \frac{1}{2}\bigg(1-(\sqrt{c_1}-\sqrt{\alpha-c_1})^2\bigg) & \text{for } \ 0 \leq c_1 \leq c_- \\ \frac{1}{2}\sqrt{\bigg(1-(\sqrt{c_1}+\sqrt{\alpha-c_1})^2\bigg)^2 + 4\sqrt{c_1(\alpha-c_1)}(\sqrt{c_1}-\sqrt{\alpha-c_1})^2} & \text{for } \ c_- \leq c_1 \leq c_+ \\ \frac{1}{2}\bigg(1-(\sqrt{c_1}-\sqrt{\alpha-c_1})^2\bigg) & \text{for } \ c_+ \leq c_1 \leq \alpha \end{cases} \tag{38}$$

In figure 3, the 1-norm geometric quantum discord shows a quasi similar behavior as the quantum correlations measured by local quantum uncertainty in figure 1. In fact, for $\alpha < \frac{1}{2}$ with the trace distance, the quantum correlations are maximal in the states ρ_α with $(c_1 = 0, c_2 = \alpha)$ and $(c_1 = \alpha, c_2 = 0)$ given by (19). Also, the amount of quantum correlations is minimal in the state with $(c_1 = c_2 = \frac{\alpha}{2})$ given by (17). This agrees with the measure of quantum correlations through the local quantum uncertainty. This similarity between the two quantifiers becomes to be slightly different starting from $\alpha = \frac{\alpha}{2}$. Indeed, the geometric discord is quasi linear with respect to the parameter c_1. It vanishes in the states ($c_1 = c_2 = \frac{1}{4}$). This state is the only separable state in the set of two-qubit states (1). It is remarkable that for $\alpha > \frac{1}{2}$, the geometric discord exhibits also a double sudden change when ($c_1 = c_+, c_2 = c_-$) and ($c_1 = c_-, c_2 = c_+$) ($c_\pm$ are given by (36)) to be compared with the points ($c_1 = \alpha_+, c_2 = \alpha_-$) and ($c_1 = \alpha_-, c_2 = \alpha_+$) given by (11) where the sudden change occurs for the local quantum uncertainty. Clearly, the states ρ_α at which the geometric discord presents a double sudden change are different from those obtained for local quantum uncertainty. This dissimilarity poses a serious challenge and especially when when one needs to employ the sudden change of quantum correlations in a multipartite system to understand the quantum phase transitions. In this sense, a future issue would to investigate which quantifier, trace distance or local quantum unerctainty, is then suitable as indicator of quantum phase transitions.

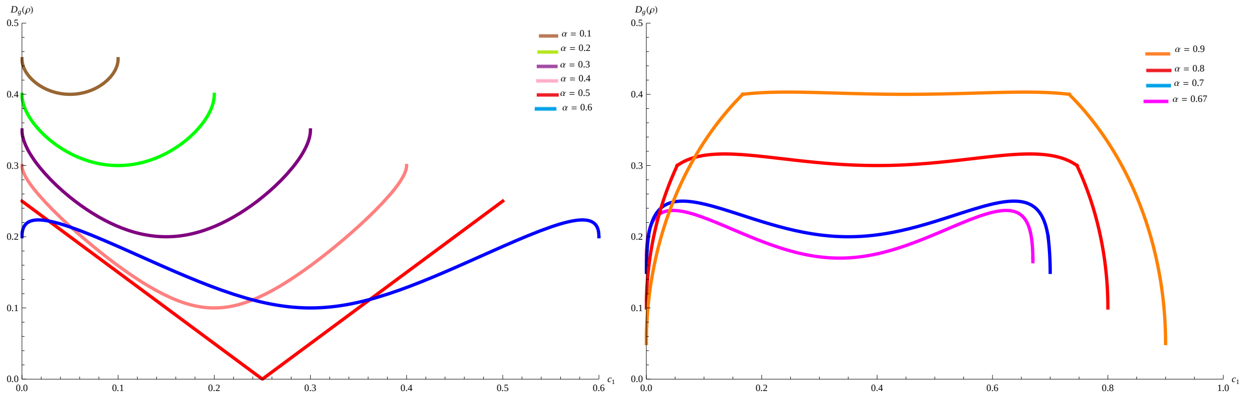

Figure 3. The geometric quantum discord $D_{\mathrm{g}}(\rho_\alpha)$ versus the parameter c_1 for $\alpha \leq \frac{2}{3}$ and $\alpha \geq \frac{2}{3}$.

3 Local quantum uncertainty under decoherence

Because of the unavoidable interaction of a bipartite quantum system with the environment, the dynamics of quantum correlations under constitutes an important issue which has received a great deal of attention [48]-[55]. Different decoherence scenarios (Markovian or non-Markovian) were investigated for different quantifiers of quantum correlations. In particular, it has been shown that entanglement suffers from sudden death [56]-[61] and the entropic quantum discord is more robust than entanglement [62]. In fact, when an two-qubit state is under the influence of a local noisy environment, the entanglement can suddenly disappear hile the quantum discord shows more resilience against the decoherence effects. Dynamics of geometric discord based on 1-Schatten norm was also studied for some two-qubit states. In particular, it has been shown that this quantum indicator exhibits in Bell diagonal states the so-called freezing phenomenon, the quantum correlations remain constant during the evolution of the system [63]. In this section, we investigate the dynamics of quantum discord quantified by local quantum uncertainty. In order to simplify our purpose, we restrict our focus to two-qubit density matrices (1) with $c_1 = c_2 = c$ which take the form of Bell-diagonal states . They are given by

$$\rho(c_1 = c_2 = c) = \frac{1}{4}\bigg(\sigma_0 \otimes \sigma_0 + \sigma_1 \otimes \sigma_1 + (1-4c)\sigma_2 \otimes \sigma_2 - \sigma_3 \otimes \sigma_3\bigg) \tag{39}$$

where $0 \leq c \leq \frac{1}{2}$. For open quantum system, the Markovian dynamics can be entirely specified by a quantum channel $\mathcal{E} : \rho \longrightarrow \mathcal{E}(\rho)$ whose action on the state can be completely characterized as follows

$$\mathcal{E}(\rho) = \sum_{ij}(E_i \otimes E_j)\rho(E_i \otimes E_j)^\dagger$$

where E_i denotes the Kraus operators describing the decohering process of a single qubit. The Kraus operators satisfy the trace-preserving condition $\sum_i (E_i)^\dagger E_i = \mathbb{I}$. For several decoherence scenarios, the action of the decoherence channel is in general parameterized by a time dependent decoherence probability p . In what follows, we will consider the dynamic behavior of local quantum uncertainty in the states (39) for certain noise channels (i.e., phase flip, bit flip, and bit-phase flip and generalized amplitude damping)

3.1 The depolarizing quantum channel

The depolarizing channel is a decohering process used to modeling three different types of errors: (i) bit flip error, (ii) phase flip error or (iii) both [1].

(i) Bit flip error: For bit flip quantum channel, the Kraus operators are

$$E_0 = \sqrt{1-p/2}\,\sigma_0 \qquad E_1 = \sqrt{p/2}\,\sigma_1 \tag{40}$$

Under the local action of the bit flip channel, the density matrix (39) writes in the Fano-Bloch representation as

$$\rho_{\rm BF} = \frac{1}{4}\big(\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} R_{ii}^{\rm BF}\sigma_i \otimes \sigma_i\big) \tag{41}$$

where

$$R_{11}^{\rm BF} = 1, \qquad R_{22}^{\rm BF} = (1-4c)(1-p)^2, \qquad R_{33}^{\rm BF} = (4c-1)(1-p)^2.$$

Obtaining the local quantum uncertainty in the states (41) requires the expressions of the square root of the density matrix and the closed form of the matrix elements ω_{ij} given by (6). Lengthy but straightforward calculation gives

$$\omega_{11}^{\rm BF} = \sqrt{1-(1-p)^4(1-4c)^2} \qquad \omega_{22}^{\rm BF} = 0 \qquad \omega_{33}^{\rm BF} = 0 \tag{42}$$

and the local quantum uncertainty is simply given by

$$\mathcal{U}(\rho_{\rm BF}) = 1 - \sqrt{1-(1-p)^4(1-4c)^2}. \tag{43}$$

(ii) Phase flip error : The phase flip channel describes the quantum noise process with loss of quantum information without loss of energy. In the operator-sum representation formalism, the Kraus operators for single qubit phase flip write

$$E_0 = \sqrt{1 - p/2}\sigma_0 \qquad E_1 = \sqrt{p/2}\sigma_3 \tag{44}$$

Under phase flip channel, the evolved quantum state writes as

$$\rho_{\rm PF} = \frac{1}{4}\big(\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} R_{ii}^{\rm PF}\sigma_i \otimes \sigma_i\big) \tag{45}$$

where the correlation elements are given by

$$R_{11}^{\rm PF} = (1-p)^2 \quad R_{22}^{\rm PF} = (1-4c)(1-p)^2 \quad R_{33}^{\rm PF} = (4c-1).$$

In this decohering scenario, the matrix elements (6) take the form

$$\omega_{11}^{\rm PF} = 2\sqrt{2}\sqrt{c(1-2c)} \qquad \omega_{22}^{\rm PF} = 2\sqrt{2}\sqrt{c(1-2c)}\sqrt{1-(1-p)^4} \qquad \omega_{33}^{\rm PF} = \sqrt{1-(1-p)^4}. \tag{46}$$

We notice that $\omega_{22}^{\rm PF} \leq \omega_{11}^{\rm PF}$ and $\omega_{22}^{\rm PF} \leq \omega_{33}^{\rm PF}$. This implies that $\omega_{\rm max}^{\rm PF}$ is given by ($\omega_{11}^{\rm PF}$ or $\omega_{33}^{\rm PF}$). For a given value of c, the condition $\omega_{11}^{\rm PF} \geq \omega_{33}^{\rm PF}$ is satisfied when the probability p is such that $0 \leq p \leq 1 - \sqrt{|4c-1|}$. It is remarkable that in this case, the local quantum uncertainty,

$$\mathcal{U}(\rho_{\rm PF}) = 1 - 2\sqrt{2}\sqrt{c(1-2c)}, \tag{47}$$

remains constant (i.e, time independent). In this interval the local quantum uncertainty exhibits a freezing behavior. This reflects that the local quantum uncertainty is robust against the phase flip errors. This frozeen behavior is followed by a sudden change at the critical point $p_c = 1 - \sqrt{|4c-1|}$. Hence for $1 - \sqrt{|4c-1|} \leq p \leq 1$, the local quantum uncertainty is given by

$$\mathcal{U}(\rho_{\rm PF}) = 1 - \sqrt{1-(1-p)^4}, \tag{48}$$

and decays monotonically to disappear complectly when $p \longrightarrow 1$.

(iii) Bit-phase flip error: The corresponding Kraus operators are given by

$$E_0 = \sqrt{1 - p/2}\ \sigma_0 \qquad E_1 = \sqrt{p/2}\ \sigma_2, \tag{49}$$

and their action on a state of type (39) leads to

$$\rho_{\rm BPF} = \frac{1}{4}\big(\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} R_{ii}^{\rm BPF}\sigma_i \otimes \sigma_i\big) \tag{50}$$

where the Fano-Bloch components writes now as

$$R_{11}^{\rm BPF} = (1-p)^2 \quad R_{22}^{\rm BPF} = (1-4c) \quad R_{33}^{\rm BPF} = (4c-1)(1-p)^2.$$

Reporting the square root of the state $\rho_{\rm BPF}$ in the expressions of the matrix elements (6), one obtains

$$\omega_{11}^{\rm BPF} = 2\sqrt{2}\sqrt{c(1-2c)} \qquad \omega_{22}^{\rm BPF} = \sqrt{1-(1-p)^4} \qquad \omega_{33}^{\rm BPF} = 2\sqrt{2}\sqrt{c(1-2c)}\sqrt{1-(1-p)^4} \quad (51)$$

The local quantum uncertainty can be derived similarly to the phase flip process by replacing $\omega_{22}^{\rm PF}$ by $\omega_{33}^{\rm BPF}$ and $\omega_{33}^{\rm PF}$ by $\omega_{22}^{\rm BPF}$. This gives

$$\mathcal{U}(\rho_{\rm BPF}) = 1 - 2\sqrt{2}\sqrt{c(1-2c)} \quad \text{for} \quad 0 \leq p \leq p_c \tag{52}$$

with $p_c = 1 - \sqrt{|4c-1|}$ and

$$\mathcal{U}(\rho_{\rm BPF}) = 1 - \sqrt{1-(1-p)^4} \quad \text{for} \quad p_c \leq p \leq 1. \tag{53}$$

It is interesting to note that, similarly to the phase flip channel, the local quantum uncertainty exhibits a freezing behavior in the interval $[0, p_c]$. Clearly, this behavior is essentially due to the phase flip errors since when only the bit flip error acts on the system the local quantum uncertainty is monotonically decreasing. Furthermore, it is remarkable that for both phase flip and Bit-phase flip, the freezing phenomenon occurs in the same interval. To investigate the width of this interval to guarantee the freezing of the quantum discord for long periods, we consider separately the situations where $0 \leq c \leq \frac{1}{4}$ and $\frac{1}{4} \leq c \leq \frac{1}{2}$. For $0 \leq c \leq \frac{1}{4}$, the critical point p_c increases as the parameter c increases. As it can verified from the equations (47) and (52), increasing the parameter c, to get a large freezing interval, is accompanied by a diminution of the amount of quantum correlations in the system. Similarly, for the states with $\frac{1}{4} \leq c \leq \frac{1}{2}$, one concludes that larger freezing intervals are also obtained for states with less quantum correlations and the price to pay is the missing of quantum correlations.

3.2 Generalized amplitude damping

Now we consider the dynamics of the states (39) under the effect of a amplitude-damping channel which describes the dissipative interaction between the system and the environment. This process may be modeled by treating the environment as a large collection of independent harmonic oscillators interacting weakly with the system. In the operator-sum representation formalism, the evolution of the system is described by the following four Kraus operators

$$E_0 = \frac{\sqrt{p}}{2}\big[(1+\sqrt{1-\gamma})\sigma_0 + (1-\sqrt{1-\gamma})\sigma_3\big], \quad E_1 = \sqrt{p\gamma}\ \sigma_+,$$
$$E_2 = \frac{\sqrt{1-p}}{2}\big[(\sqrt{1-\gamma}+1)\sigma_0 + (\sqrt{1-\gamma}-1)\sigma_3\big], \quad E_3 = \sqrt{(1-p)\gamma}\ \sigma_- \tag{54}$$

where $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$, p and γ are the decoherence probabilities [1]. To simplify the calculation of the local quantum uncertainty, we fix $p = \frac{1}{2}$. In this situation, under the generalized amplitude damping, the states (39) evolve as

$$\rho_{\rm GAD} = \frac{1}{4}\big(\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} R_{ii}^{\rm GAD}\sigma_i \otimes \sigma_i\big) \tag{55}$$

where

$$R_{11}^{\mathrm{GAD}} = (1-\gamma), \quad R_{22}^{\mathrm{GAD}} = (1-4c)(1-\gamma), \quad R_{33}^{\mathrm{GAD}} = (4c-1)(1-\gamma)^2.$$

After some lengthy but feasible algebraic manipulations of the matrix elements (6), needed to derive the quantum correlations in the state (55), one gets

$$\omega_{11}^{\mathrm{GAD}} = \sqrt{1-(1-\gamma)^2(1-4c)^2}, \quad \omega_{22}^{\mathrm{GAD}} = \sqrt{\gamma(2-\gamma)}, \quad \omega_{33}^{\mathrm{GAD}} = \sqrt{\gamma(2-\gamma)}\sqrt{1-(1-\gamma)^2(1-4c)^2}. \tag{56}$$

It is simple to verify that $\omega_{33}^{\mathrm{GAD}} \leq \omega_{22}^{\mathrm{GAD}} \leq \omega_{11}^{\mathrm{GAD}}$ and the local quantum uncertainty is given by

$$\mathcal{U}(\rho_{\mathrm{GAD}}) = 1 - \sqrt{1-(1-\gamma)^2(1-4c)^2}. \tag{57}$$

4 Concluding remarks

Quantum correlations in a composite system can be measured by employing the local quantum uncertainty.

We show that the quantum correlations quantified by local quantum uncertainty remain constant during the evolution of a class of two qubits under specific decoherence channels. This remarkable result is known in the literature as quantum correlation freezing. This result can bring a tool in understanding the inevitable decoherence due to the interaction with the environment and possibly open new ways to exploit quantum correlations from a practical point of view.

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