Unified view of correlations using the square norm distance

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The distance between a quantum state and its closest state not having a certain property has been used to quantify the amount of correlations corresponding to that property. This approach allows a unified view of the various kinds of correlations present in a quantum system. In particular, using relative entropy as a distance measure, total correlations can be meaningfully separated in a quantum and a classical part thanks to an additive relation involving only distances between states. Here, we investigate a unified view of correlations using as distance measure the square norm, already used to define the so-called geometric quantum discord. We thus consider geometric quantifiers also for total and classical correlations finding, for a quite general class of bipartite states, their explicit expressions. We analyze the relationship among geometric total, quantum and classical correlations and we find that they do not satisfy anymore a closed additivity relation.

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I. INTRODUCTION

Quantum systems have properties characterized by various kinds of correlations some of which distinguish them from classical systems [1, 2]. These properties may be an essential resource for quantum computation and quantum information [3]. This makes important to distinguish among those correlations that are peculiar of quantum systems with respect to the ones present also in classical systems. A nonlocal property of quantum systems, entanglement, allows to achieve exponential speed-up in pure-state computation if it grows with the size of the system [4]. On the other hand, in the case of mixed-state computation, in certain computational tasks quantum speed-up can be achieved using separable (unentangled) states, like in the so-called deterministic quantum computation with one qubit (DQC1) protocol [5]. This speed-up has been linked [6] to the presence of quantum discord [7, 8], considered as a quantifier of the quantum part of correlations present in a bipartite system and defined as the difference between two quantum analogues of the classical mutual information [9, 10].

In the general case of a multipartite system, the various kinds of correlations present in a quantum state have been linked to the distance between the system state itself and its closest states without the desired property, allowing to look at them in a unified view [11, 12]. Relative entropy, although not symmetrical under the exchange of the entries, has been used as distance measure between states. In this approach, the decomposition of the total correlations, T, in a classical, C, and a quantum part, D, appears meaningful because T equals the sum of D and C up to a quantity L, which in turn is a relative entropy-based (REB) distance between two of the relevant closest states. The quantity L results to be, in the bipartite case, equal to the difference between REB quantum discord and its original definition [7, 8]. Using relative entropy as a distance measure, correlation quantifiers therefore satisfy a closed additivity relation among them [12].

The properties of quantum discord have been widely investigated in the last years [13]. It has been shown that it is present in almost all quantum states [14] and the relation between discord and entanglement has been discussed [15–17]. In contrast with entanglement, discord can be generated using local noise [18] and it is not monogamous [19]. Differently from what happens for entanglement, quantum discord does not present sudden death [20] during its evolution but can still present revivals even in absence of system-environment back-action [21]. Generalizations of discord to the multipartite case have been also reported following different approaches [22].

On the other hand, both the original and the REB discord require involved minimization procedures even if only von Neumann (orthogonal) measurements are used. Using more general measurements (POVM) the minimization problem gets increasingly demanding [23], with the consequence that there are only few general results. Discord analytical expressions have been obtained only for certain classes of two-qubit states, such as Bell-diagonal [24], rank-2 [23, 25] and X [26] states and, for the case of continuous variable, for Gaussian states [27]. To overcome this drawback, geometric quantum discord $D_{\rm g}$ has been introduced based on the square norm (Hilbert-Schmidt) distance between the system state and its closest classical state and it has been used to evaluate quantum correlations present in an arbitrary two-qubit state

[28]. Quantitative comparisons between REB and geometric discord have been reported [29–31] and their dynamics have been also compared, revealing qualitative differences in their time behaviors [32]. Geometric measures of total correlations, $T_{\rm g}$, and classical correlations, $C_{\rm g}$, have been also defined using the square norm distance with their explicit expressions given only for Bell-diagonal states [32].

The aim of this paper is to discuss the role and use of the square norm distance to quantify in a unified view various kinds of correlations in a two-qubit state. To this purpose we will consider a quite general class of bipartite states for which we will find explicit expressions for geometric quantifiers of total, quantum and classical correlations. In analogy to what has been done with REB correlation quantifiers, we will investigate the possibility to have closed additive relations among correlation quantifiers based on the square norm.

The main point of this paper is to show that relevant qualitative differences are found when one attempts to construct a unified view of correlations using different ways to measure the distance between the relevant states. In the case of a quite general class of two-qubit states (X-states), we will be able to analytically prove that, differently from what happens with REB distance, using the square norm distance measure total correlations cannot be in general separated in a quantum and a classical part satisfying an additive relation that involves only distances between states.

The paper is organized as follows, in Section II we introduce the framework of unified view of correlations both for REB and its geometric counterpart, in Section III we present the class of states on which we will base our study, X states, in Sections IV and V we obtain the pertinent closest states. Finally, in Section VI we show under which conditions the closure of correlations is not satisfied and study how often, and to what extent, this happens for X states.

II. CORRELATIONS IN A QUANTUM STATE

A natural and powerful way to quantify a given property of a quantum state consists in exploiting the distance between the state itself and its closest state without that property. Therefore, in this approach, it is necessary to choose a suitable distance measure. In this section, we first briefly review the correlation quantifiers defined by using relative entropy as a measure of distance between states and we secondly describe the geometric correlation quantifiers based on the square norm distance measure.

A. Correlation quantifiers based on relative entropy

Given two arbitrary multipartite states ρ, σ, their relative entropy is defined as $S(\rho\|\sigma) = -\mathrm{Tr}(\rho\log\sigma) - S(\rho)$,

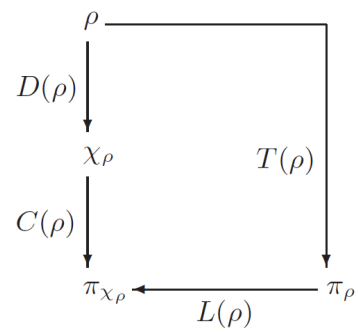

FIG. 1: Picture of the REB correlation quantifiers and the relevant states (see Ref. [12])

where $S(\rho) = -\mathrm{Tr}(\rho\log\rho)$ is the von Neumann entropy. Using relative entropy to quantify distances, the total correlations T of a state ρ are defined by the distance between ρ and the closest product state $\pi_\rho = \sigma_A \otimes \sigma_B$ (with $\sigma_{A(B)}$ density matrices for the subsystems), $T = S(\rho\|\pi_\rho)$; discord (quantum correlations) D is the distance between ρ and the closest classical state $\chi_\rho = \sum_{i,j} p_{i,j}|i\rangle\langle i| \otimes |j\rangle\langle j|$ (with $\{|i\rangle, |j\rangle\}$ independent local bases, and $p_{i,j}$ probabilities), $D = S(\rho\|\chi_\rho)$, while classical correlations are the distance between χ_ρ and its closest product state π_{χ_ρ}, $C = S(\chi_\rho\|\pi_{\chi_\rho})$ [12]. We refer to these correlation quantifiers as relative entropy-based (REB) quantifiers. In general, $T - (D + C) \neq 0$ but this difference is always equal to another quantity L defined in terms of relative entropy, that is

$$T - D - C = -L, \tag{1}$$

where $L = S(\pi_\rho\|\pi_{\chi_\rho})$. The validity of Eq. (1) is proved by the fact that all the involved REB correlation quantifiers can be written as differences between von Neumann entropies as [12]

$$\begin{aligned} T(\rho) &\equiv S(\rho\|\pi_\rho) = S(\pi_\rho) - S(\rho), \\ D(\rho) &\equiv S(\rho\|\chi_\rho) = S(\chi_\rho) - S(\rho), \\ C(\rho) &\equiv S(\chi_\rho\|\pi_{\chi_\rho}) = S(\pi_{\chi_\rho}) - S(\chi_\rho), \\ L(\rho) &\equiv S(\pi_\rho\|\pi_{\chi_\rho}) = S(\pi_{\chi_\rho}) - S(\pi_\rho). \end{aligned} \tag{2}$$

These relations allow one to draw a simple diagram, proposed in Ref. [12] and displayed in Fig. 1, where each line refers to a kind of correlations, and where the direction of each arrow is linked to the asymmetric definition of the relative entropy. For bipartite systems, the quantity L exactly quantifies the difference between the REB discord D of Eq. (2) and the original definition δ [7, 8]: $\delta = D - L$. A closed additivity relation eq. (1) thus always holds among the REB correlation quantifiers.

B. Correlation quantifiers based on square norm

As already noted, the REB correlation quantifiers have the drawback that their analytical expressions are known

only for certain classes of states [24–27] and require in general numerical minimizations. A more manageable quantifier, named geometric quantum discord, has been recently introduced for quantum correlations as the square norm distance between the system state ρ and its closest classical state χ_ρ [28]

$$D_{\rm g}(\rho) = \|\rho - \chi_\rho\|^2, \tag{3}$$

where $\|\cdot\|^2 = {\rm Tr}(\cdot)^2$ is the square norm distance in the Hilbert-Schmidt space and χ_ρ has the form $\chi_\rho = \sum_{i,j} p_{i,j}|i\rangle\langle i| \otimes \rho_j^B$ (with $\{|i\rangle\}$ a local basis on part A, ρ_j^B arbitrary states of part B and $p_{i,j}$ probabilities). The advantage of this definition is that $D_{\rm g}(\rho)$ can be analytically evaluated for an arbitrary two-qubit state and for some multipartite or higher dimensional systems more easily than REB quantum discord [28, 33]. It is worth noticing that the geometric definition of quantum discord of Eq. (3) is equal to the one obtained in analogy to the original definition of quantum discord and using the square norm distance measure, that is $D_{\rm g}(\rho) = \min_{\Pi^A}\|\rho - \Pi^A(\rho)\|^2$, where $\Pi^A(\rho)$ is the classical state resulting after a von Neumann measurement on part A [33].

Because geometric discord is useful to quantify quantum correlations in a system, it looks a natural extension to use square norm also to define quantifiers of total and classical correlations as [32]

$$T_{\rm g}(\rho) \equiv \|\rho - \pi_\rho\|^2, \quad C_{\rm g}(\rho) \equiv \|\chi_\rho - \pi_{\chi_\rho}\|^2, \tag{4}$$

where π_ρ and π_{χ_ρ} are, respectively, the product states closest to ρ and χ_ρ within the square norm distance measure. We refer to the quantifiers based on square norm distance as geometric correlation quantifiers. One can further define the quantity

$$L_{\rm g}(\rho) \equiv \|\pi_\rho - \pi_{\chi_\rho}\|^2, \tag{5}$$

as the analogous of the REB quantity L of Eq. (2).

In analogy of what happens for REB discord, $D_{\rm g}$ can be written as a difference of purities $D_{\rm g} = {\rm Tr}(\rho - \chi_\rho)^2 = {\rm Tr}\rho^2 - {\rm Tr}\chi_\rho^2$. In fact, analyzing the results of Ref. [28], one can show that ${\rm Tr}(\rho\chi_\rho) = {\rm Tr}(\chi_\rho^2)$. Differently, as we shall see in Sec. VI, the other geometric correlation quantifiers $T_{\rm g}$ and $C_{\rm g}$ do not hold this property.

We shall now study the relationship among geometric correlation quantifiers by firstly finding their explicit expressions for a quite general class of bipartite states. We will also investigate if in general an additivity relation analogous to that of Eq. (1) is satisfied when geometric correlation quantifiers are used.

III. TWO-QUBIT X STATES

In this section we describe the class of two-qubit states we are going to use in our analysis. To our aim it is useful to represent the states in the Bloch representation that, for an arbitrary two-qubit state, is

$$\rho = \frac{1}{4}[\mathbb{1}\otimes\mathbb{1} + \sum_i x_i\sigma_i\otimes\mathbb{1} + \sum_i y_i\mathbb{1}\otimes\sigma_i + \sum_{i,j} T_{ij}\sigma_i\otimes\sigma_j], \tag{6}$$

where $\mathbb{1}$ is the 2×2 identity matrix, $\sigma_{i,j}$ $(i,j = 1,2,3)$ are the three Pauli matrices, $x_i = {\rm Tr}[\rho(\sigma_i\otimes\mathbb{1})]$ and $y_i = {\rm Tr}[\rho(\mathbb{1}\otimes\sigma_i)]$ are components of the local Bloch vectors $\vec{x} = \{x_1, x_2, x_3\}$ and $\vec{y} = \{y_1, y_2, y_3\}$, and $T_{ij} = {\rm Tr}[\rho(\sigma_i\otimes\sigma_j)]$ are components of the correlation tensor T.

In particular, we put our attention to the class of X states, which are those states having non-zero elements only along the main diagonal and anti-diagonal of the density matrix. The general structure of an X density matrix is thus, in the standard basis $\mathcal{B} = \{|1\rangle \equiv |11\rangle, |2\rangle \equiv |10\rangle, |3\rangle \equiv |01\rangle, |4\rangle \equiv |00\rangle\}$,

$$\rho_X = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14}e^{i\gamma_{14}} \\ 0 & \rho_{22} & \rho_{23}e^{i\gamma_{23}} & 0 \\ 0 & \rho_{23}e^{-i\gamma_{23}} & \rho_{33} & 0 \\ \rho_{14}e^{-i\gamma_{14}} & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{7}$$

where ρ_{ij} $(i,j = 1,2,3,4)$ and $\gamma_{i,j}$ are all real, positive numbers. Bell states, Werner states and Bell diagonal states belong to this class of states [34]. X-structure density matrices may arise in a wide variety of physical situations and are also experimentally achievable [35]. For example, X states are encountered as eigenstates in all the systems with odd-even symmetry like in the Ising and the XY models [36]. Moreover, in many physical evolutions of open quantum systems an initial X structure is maintained in time [37]. The parameters of the Bloch representation of Eq. (6) for an X state are thus expressed in terms of the the density matrix elements of Eq. (7) as [38]

$$\begin{aligned} x_3 &= \rho_{11} + \rho_{22} - \rho_{33} - \rho_{44}\,, \\ y_3 &= \rho_{11} - \rho_{22} + \rho_{33} - \rho_{44}\,, \\ T_{11} &= 2\cos(\gamma_{14})\rho_{14} + 2\cos(\gamma_{23})\rho_{23}\,, \\ T_{12} &= -2\sin(\gamma_{14})\rho_{14} + 2\sin(\gamma_{23})\rho_{23}\,, \\ T_{21} &= -2\sin(\gamma_{14})\rho_{14} - 2\sin(\gamma_{23})\rho_{23}\,, \\ T_{22} &= -2\cos(\gamma_{14})\rho_{14} + 2\cos(\gamma_{23})\rho_{23}\,, \\ T_{33} &= \rho_{11} - \rho_{22} - \rho_{33} + \rho_{44}, \end{aligned} \tag{8}$$

with $x_1 = x_2 = y_1 = y_2 = T_{13} = T_{23} = T_{31} = T_{32} = 0$.

We are interested in the explicit expressions of the geometric correlation quantifiers for an X state. In order to obtain them, we first need to find the relevant closest states when the distance is measured by the square norm.

IV. CLOSEST PRODUCT STATE

In this section we are interested in finding the product state closest to a two-qubit X state in the square norm distance measure, with the aim to obtain the geometric quantifier of total correlations. Indicating with

$\rho_A = \frac{1}{2}[\mathbb{1} + \sum_i a_i\sigma_i]$ and $\rho_B = \frac{1}{2}[\mathbb{1} + \sum_i b_i\sigma_i]$ generic single-qubit states with Bloch vectors, respectively, $\vec{a} = \{a_1, a_2, a_3\}$ and $\vec{b} = \{b_1, b_2, b_3\}$, an arbitrary product state π is given by their tensor product as

$$\pi = \rho_A \otimes \rho_B = \frac{1}{4}\left[\mathbb{1}\otimes\mathbb{1} + \sum_i a_i\sigma_i\otimes\mathbb{1} + \sum_i b_i\mathbb{1}\otimes\sigma_j + \sum_{i,j} a_i b_j \sigma_i\otimes\sigma_j\right]. \quad (9)$$

The distance F between an arbitrary two-qubit state ρ as given in Eq. (6) and the product state π using the square norm is then

$$F = \mathrm{Tr}(\rho-\pi)^2 = \frac{1}{4}\left[\sum_i (x_i - a_i)^2 + \sum_i (y_i - b_i)^2 + \sum_{i,j}(T_{ij} - a_i b_j)^2\right]. \quad (10)$$

The explicit form of the product state π_ρ closest to ρ is determined by the values of the variables a_i, b_i, as functions of the known state parameters x_i, y_i, T_{ij}, giving the absolute minimum of the distance F. Deriving F with respect to a_i and b_j we construct the system $(i, j = 1, 2, 3)$

$$a_i = \frac{x_i + \sum_j T_{ij} b_j}{1 + \sum_j b_j^2}, \quad b_j = \frac{y_j + \sum_i T_{ij} a_i}{1 + \sum_i a_i^2}. \quad (11)$$

In the case of X states, defined in Eqs. (7) and (8), the above system reduces to

$$\begin{aligned} a_1 &= \frac{T_{11}b_1 + T_{12}b_2}{1+\sum_j b_j^2}, \quad b_1 = \frac{T_{11}a_1 + T_{21}a_2}{1+\sum_j a_j^2}, \\ a_2 &= \frac{T_{21}b_1 + T_{22}b_2}{1+\sum_j b_j^2}, \quad b_2 = \frac{T_{12}a_1 + T_{22}a_2}{1+\sum_j a_j^2}, \\ a_3 &= \frac{x_3 + T_{33}b_3}{1+\sum_j b_j^2}, \quad b_3 = \frac{y_3 + T_{33}a_3}{1+\sum_i a_i^2}. \end{aligned} \quad (12)$$

It can be shown (see appendix A) that the absolute minimum of F is obtained by putting $a_1 = a_2 = b_1 = b_2 = 0$ and by taking the solutions for a_3, b_3 of the system

$$a_3 = \frac{x_3 + T_{33}b_3}{1+b_3^2}, \quad b_3 = \frac{y_3 + T_{33}a_3}{1+a_3^2}. \quad (13)$$

Indicating with $\bar{a}_3$ and $\bar{b}_3$ the solutions of Eqs. (13) and substituting in Eq. (9), the product state closest to an X state has the form

$$\pi_{\rho_X} = \frac{1}{4}[\mathbb{1}\otimes\mathbb{1} + \bar{a}_3\sigma_3\otimes\mathbb{1} + \bar{b}_3\mathbb{1}\otimes\sigma_3 + \bar{a}_3\bar{b}_3\sigma_3\otimes\sigma_3]. \quad (14)$$

An important point, as we shall see later in the paper, is that the parameters $\bar{a}_3$, $\bar{b}_3$ only depend on x_3, y_3, T_{33}, while $T_{11}, T_{12}, T_{21}, T_{22}$ do not play any role.

We observe that, in general, $\bar{a}_3 \neq x_3$ and $\bar{b}_3 \neq y_3$, that means that the product state closest to an X state in the square norm distance measure is not given by the product of its marginals $\mathrm{Tr}_B(\rho_X)\otimes\mathrm{Tr}_A(\rho_X)$, differently from what happens for any quantum state when the distance is measured by the relative entropy [12].

V. CLOSEST CLASSICAL STATE AND ITS CLOSEST PRODUCT STATE

We now face the problem of finding explicit expressions of the classical state closest to a an X state in the square norm distance measure. To this aim, we follow a reported procedure that permits to obtain the closest classical state given an arbitrary two-qubit state [28]. We point out that, although explicit expressions for geometric quantum discord for X states have been already reported in the literature [39], this is not the case for the expressions of the corresponding closest classical states. Here we also give the product state closest to the obtained closest classical state, required for calculating the geometric quantifier of classical correlations (see Eq. (4)).

The general result for geometric quantum discord of two-qubit states is $D_{\mathrm{g}}(\rho) = \frac{1}{4}[\|\vec{x}\|^2 + \|T\|^2 - k_{max}]$, where k_{max} is the largest eigenvalue of matrix $K = \vec{x}\vec{x}^T + TT^T$ (T^T is the transpose of matrix T) [28]. The eigenvalues of matrix K for an X state, in terms of the density matrix elements, are

$$\begin{aligned} k_1 &= 4(\rho_{14}+\rho_{23})^2, \quad k_2 = 4(\rho_{14}-\rho_{23})^2, \\ k_3 &= 2[(\rho_{11}-\rho_{33})^2 + (\rho_{22}-\rho_{44})^2]. \end{aligned} \quad (15)$$

We observe that k_1 is always larger than k_2, so that only two distinct cases have to be separately treated, that is $k_1 \leq k_3$ and $k_1 > k_3$.

The closest classical state χ_ρ is obtainable by a minimization procedure with respect to the parameters $\vec{x}, \vec{y}, T$ of the original state ρ expressed in the Bloch representation [28]. In the following we use this procedure to obtain, for these two cases, the explicit expressions of the closest classical state and of its closest product state for an X state defined by the parameters of the Bloch representation given in Eq. (8).

A. Case 1: $k_1 \leq k_3$

When the X state has density matrix elements such that the condition $k_1 \leq k_3$ is fulfilled, we find that its closest classical state in the square norm distance measure has the form

$$\chi^{(1)}_{\rho_X} = \frac{1}{4}[\mathbb{1}\otimes\mathbb{1} + x_3\sigma_3\otimes\mathbb{1} + y_3\mathbb{1}\otimes\sigma_3 + T_{33}\sigma_3\otimes\sigma_3], \quad (16)$$

where the superscript (1) refers to the case 1 ($k_1 \leq k_3$) of our analysis.

Seeing that $\chi^{(1)}_{\rho_X}$ is still an X state, we can apply the results of Sec. IV to calculate its closest product state $\pi^{(1)}_{\chi_{\rho_X}}$. The diagonal elements of ρ_X and χ_{ρ_X} are equal, $(\rho^{(1)}_X)_{ii} = (\chi_{\rho^{(1)}_X})_{ii}$, and, as said before, the solutions $\bar{a}_3$ and $\bar{b}_3$ of Eq. (13) only depend on the components x_3, y_3, T_{33} containing the diagonal density matrix elements. As a consequence, for X states lying in this case 1, the product state closest to $\chi^{(1)}_{\rho_X}$ coincides with the product state closest to ρ_X given in Eq. (14): $\pi^{(1)}_{\chi_{\rho_X}} = \pi_{\rho_X}$.

B. Case 2: $k_1 > k_3$

If the X state has density matrix elements such that $k_1 > k_3$, we obtain for the closest classical state

$$\chi^{(2)}_{\rho_X} = \frac{1}{4}\Big\{\mathbb{1}\otimes\mathbb{1} + y_3\mathbb{1}\otimes\sigma_3 + \tilde{T}_{11}\sigma_1\otimes\sigma_1 + \tilde{T}_{12}\sigma_1\otimes\sigma_2 + \tilde{T}_{21}\sigma_2\otimes\sigma_1 + \tilde{T}_{22}\sigma_2\otimes\sigma_2\Big\}, \quad (17)$$

where the superscript $^{(2)}$ refers to the case 2 ($k_1 > k_3$) of our analysis and

$$\begin{aligned}
\tilde{T}_{11} &= \frac{1}{2}\left\{T_{11}[1+\cos(\gamma_{14}+\gamma_{23})]) - T_{21}\sin(\gamma_{14}+\gamma_{23})\right\} \\
&= [\cos(\gamma_{23})+\cos(\gamma_{14})]\,(\rho_{14}+\rho_{23})\ , \\
\tilde{T}_{12} &= \frac{1}{2}\left\{T_{12}[1+\cos(\gamma_{14}+\gamma_{23})]) - T_{22}\sin(\gamma_{14}+\gamma_{23})\right\} \\
&= [\sin(\gamma_{23})-\sin(\gamma_{14})]\,(\rho_{14}+\rho_{23})\ , \\
\tilde{T}_{21} &= \frac{1}{2}\left\{T_{21}[1-\cos(\gamma_{14}+\gamma_{23})]) - T_{11}\sin(\gamma_{14}+\gamma_{23})\right\} \\
&= -\,[\sin(\gamma_{23})+\sin(\gamma_{14})]\,(\rho_{14}+\rho_{23})\ , \\
\tilde{T}_{22} &= \frac{1}{2}\left\{T_{22}[1-\cos(\gamma_{14}+\gamma_{23})]) - T_{12}\sin(\gamma_{14}+\gamma_{23})\right\} \\
&= [\cos(\gamma_{23})-\cos(\gamma_{14})]\,(\rho_{14}+\rho_{23})\,. \qquad (18)
\end{aligned}$$

Here again $\chi^{(2)}_{\rho_X}$ has an X structure, so that considerations made in Sec. IV apply when one looks for its closest product state. In particular, solving Eq. (13) for $\chi^{(2)}_{\rho_X}$, the closest product state is found to be

$$\pi^{(2)}_{\chi_{\rho_X}} = \frac{1}{4}[\mathbb{1}\otimes\mathbb{1} + y_3\mathbb{1}\otimes\sigma_3]. \qquad (19)$$

For an X state belonging to this case 2 it results, differently from the case 1 above, that the closest product state to ρ_X is different from the product state closest to $\chi^{(2)}_{\rho_X}$: $\pi_{\chi^{(2)}_{\rho_X}} \neq \pi_{\rho_X}$.

VI. GEOMETRIC CORRELATION QUANTIFIERS AND THEIR RELATIONS

After obtaining the relevant closest states to a X state, we are now able to give the explicit expressions of the geometric quantifiers of the various kinds correlations $T_{\rm g}$, $D_{\rm g}$ and $C_{\rm g}$ and to investigate their relations.

From the findings of Sec. IV on the closest product state, it follows that the geometric quantifier of total correlations defined in Eq. (4) for an X state is

$$T_{\rm g}(\rho_X) = \frac{(x_3-\bar{a}_3)^2 + \left(y_3-\bar{b}_3\right)^2 + \left(T_{33}-\bar{a}_3\bar{b}_3\right)^2}{4} + \frac{1}{4}(T_{11}^2+T_{12}^2+T_{21}^2+T_{22}^2), \qquad (20)$$

for both cases 1 and 2. We point out that in general $\mathrm{Tr}(\rho\pi_\rho) \neq \mathrm{Tr}(\pi_\rho^2)$ and therefore $T_{\rm g}(\rho_X) \neq \mathrm{Tr}(\rho_X^2) - \mathrm{Tr}(\pi^2_{\rho_X})$. Thus, total correlations measured by the square norm distance are not expressible as a difference between purities and they are not suitable to be represented by an arrow from a state to another, as instead happens for REB correlation quantifiers.

Concerning the geometric quantifiers of quantum and classical correlations, respectively $D_{\rm g}$ and $C_{\rm g}$, we can obtain their explicit expressions by the results of Sec. V on closest classical state and its closest product state, distinguishing the two cases $k_1 \leq k_3$ (case 1) and $k_1 > k_3$ (case 2).

For X states belonging to the case 1 ($k_1 \leq k_3$), the geometric quantifiers of quantum and classical correlations defined in Eqs. (3) and (4) are

$$\begin{aligned}
D^{(1)}_{\rm g} &= \frac{1}{4}(T_{11}^2+T_{12}^2+T_{21}^2+T_{22}^2) = 2(\rho_{14}^2+\rho_{23}^2), \\
C^{(1)}_{\rm g} &= \frac{(x_3-\bar{a}_3)^2 + \left(y_3-\bar{b}_3\right)^2 + \left(T_{33}-\bar{a}_3\bar{b}_3\right)^2}{4}. \qquad (21)
\end{aligned}$$

Analogously to $T_{\rm g}$, $C_{\rm g} = \|\chi_\rho - \pi_{\chi_\rho}\|^2$ is not expressible as a difference between the purities of the states χ_ρ and π_{χ_ρ} and it is not representable in the same spirit of Eq. (2) by an arrow. Regarding the quantity $L_{\rm g}$ defined in Eq. (5), we know from Sec. V that, in the case 1, $\pi^{(1)}_{\chi_{\rho_X}} = \pi^{(1)}_{\rho_X}$ so that $L^{(1)}_g = 0$. From Eqs. (20) and (21) we then observe that, in this case, among the geometric quantifiers of correlations the additivity relation

$$T_{\rm g} = D^{(1)}_{\rm g} + C^{(1)}_{\rm g} \qquad (22)$$

holds, analogous to the one of Eq. (1) for REB correlation quantifiers with $L = 0$.

On the other hand, for X states belonging to the case 2 ($k_1 > k_3$) the geometric quantifiers of quantum and classical correlations result to be

$$\begin{aligned}
D^{(2)}_{\rm g} &= (\rho_{14}-\rho_{23})^2 + \frac{1}{2}[(\rho_{11}-\rho_{33})^2 + (\rho_{22}-\rho_{44})^2]\,, \\
C^{(2)}_{\rm g} &= (\rho_{14}+\rho_{23})^2, \qquad (23)
\end{aligned}$$

where the expressions are given in terms of the density matrix elements because they are simpler than those in terms of the Bloch parameters. From Eqs. (20) and (23) we immediately notice that, in this case, an additivity

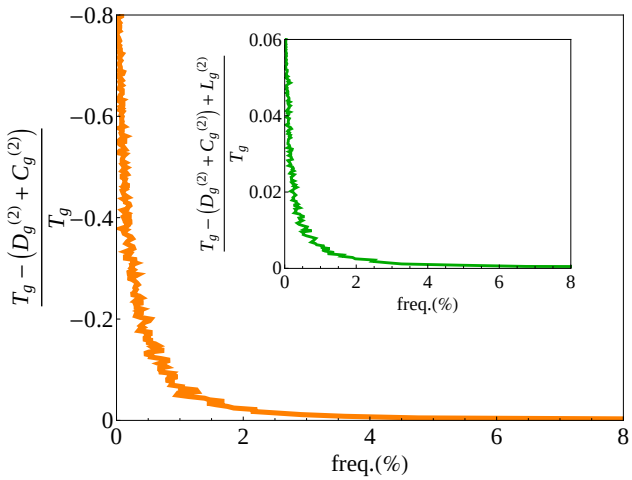

FIG. 2: (Color online). Values of the relative difference $(T_{\rm g} - D_{\rm g}^{(2)} - C_{\rm g}^{(2)})/T_{\rm g}$ of the geometric correlation quantifiers as a function of the probability of their occurrence for a two-qubit X state belonging to the case 2 $k_1 > k_3$. Inset: Values of $(T_{\rm g} + L_{\rm g}^{(2)} - D_{\rm g}^{(2)} - C_{\rm g}^{(2)})/T_{\rm g}$ as a function of the probability of their occurrence for a two-qubit X state belonging to the same case 2. A number of 10^4 random density matrices has been produced.

relation analogous to that of Eq. (1) does not occur. In fact, we have

$$T_{\rm g} - D_{\rm g}^{(2)} - C_{\rm g}^{(2)} = \bar{a}_3^2 \frac{\left(T_{33} - \bar{a}_3\bar{b}_3\right)^2 - \left(1 + \bar{b}_3^2\right)}{4}. \quad (24)$$

It can be shown (the proof is given in appendix B) that in general $T_{\rm g} - D_{\rm g}^{(2)} - C_{\rm g}^{(2)} \leq 0$ and the equality holds if and only if $x_3 + y_3 T_{33} = 0$. In order to estimate the typical value of $T_{\rm g} - D_{\rm g}^{(2)} - C_{\rm g}^{(2)}$, we have generated 10^4 X-shaped random density matrices belonging to the case $k_1 > k_3$, from which we have then numerically obtained $\bar{a}_3$ and $\bar{b}_3$ as solutions of Eq. (13) and finally calculated all the correlations Eqs. (20) and (23). In Fig. 2, we plot the values of $(T_{\rm g} - D_{\rm g}^{(2)} - C_{\rm g}^{(2)})/T_{\rm g}$ versus the (unnormalized) probability to have those amounts for an X state for which $k_1 > k_3$ (case 2). It is possible to see that there exist a nonnegligible amount of states for which this difference is significantly different from zero.

In analogy to what happens for REB correlations in Eq.(1), one may however wonder if the quantity $L_{\rm g}$ of Eq. (5) can be used to close the loop of geometric correlations. We first observe that, as we know from Sec. V, in this case 2 $\pi_{\rho_X} \neq \pi^{(2)}_{\chi_{\rho_X}}$ and then $L_{\rm g}^{(2)} \neq 0$. In particular, it is

$$L_{\rm g}^{(2)} = \bar{a}_3^2 \frac{\left(T_{33} - \bar{a}_3\bar{b}_3\right)^2 + \left(1 + \bar{b}_3^2\right)}{4}, \quad (25)$$

so that

$$T_{\rm g} - D_{\rm g}^{(2)} - C_{\rm g}^{(2)} + L_{\rm g}^{(2)} = \frac{\bar{a}_3^2}{2}\left(T_{33} - \bar{a}_3\bar{b}_3\right)^2. \quad (26)$$

As a consequence, unless $\bar{a}_3 = 0$ which would also imply $L_{\rm g}^{(2)} = 0$, it is impossible to have a closed additive relation among the different kinds of correlations when measured by the square norm. In other words, geometric quantifiers of correlations can be cast in a closed additivity relation only for X states with $L_{\rm g}$ equal to zero. To quantitatively investigate this aspect, in the inset of Fig. 2, for the same 10^4 randomly generated states as before, we plot values of $(T_{\rm g} + L_{\rm g}^{(2)} - D_{\rm g}^{(2)} - C_{\rm g}^{(2)})/T_{\rm g}$ as a function of the probability of their occurrence for a two-qubit X state belonging to the same case 2. Even if the violation is in general small, it is possible to find states for which it is meaningful. An important subclass of the X states is given by the Bell-diagonal states, for which a closed additivity relation holds among the correlation quantifiers, as it is discussed in appendix C.

We here point out that the numerical results above are useful to further highlight the qualitative conceptual aspect on the impossibility to find in general a closed relation among the geometric correlation quantifiers. For example, there could be states, outside the class of X states, for which the differences $T_{\rm g} - D_{\rm g}^{(2)} - C_{\rm g}^{(2)}$ and $T_{\rm g} + L_{\rm g}^{(2)} - D_{\rm g}^{(2)} - C_{\rm g}^{(2)}$ are larger than those we numerically find here for X states. Moreover, as discussed before, another critical point arising when using square norm distance measure to quantify correlations is the absence of a "direction" in the correlation measures. While REB correlation quantifiers can be represented by arrows with precise directions from a state to another one, as Eqs. 2 and Fig. 1 clearly indicate, this is no longer true when geometric quantifiers of correlations are taken into account.

VII. PHYSICAL MODEL

In this section we consider a specific physical model where the two state space zones $k_1 \leq k_3$ and $k_1 > k_3$ above investigated can be dynamically connected. In particular, we take two noninteracting qubits, A and B, embedded in separated cavities and subject to a non-Markovian dynamics, as already reported in Ref. [40]. Each qubit interacts only and independently with its local environment, so that the total Hamiltonian is $H_{\rm tot} = H_A + H_B$. The single "qubit+reservoir" Hamiltonian H_S $(S = A, B)$ is given by $(\hbar = 1)$

$$H_S = \omega_0^S \sigma_+^S \sigma_-^S + \sum_k \omega_k b_k^{S\,\dagger} b_k^S + \sum_k (g_k^S \sigma_+^S b_k^S + g_k^{S\,*} \sigma_-^S b_k^{S\,\dagger}), \quad (27)$$

where ω_0^S is the transition frequency of the two-level system (qubit) S, $\sigma_\pm^S$ are the system raising and lowering operators, the index k labels the field modes of the reservoir S with frequencies ω_k, $b_k^{S\,\dagger}$, b_k^S are the modes creation and annihilation operators and g_k^S the coupling constants. We will consider the case where both Hamiltonians H_A and H_B have the same parameters. The Hamiltonian of Eq. (27) may represent a qubit made by the excited and ground electronic states of a two-level atom interacting with a reservoir given by the quantized modes

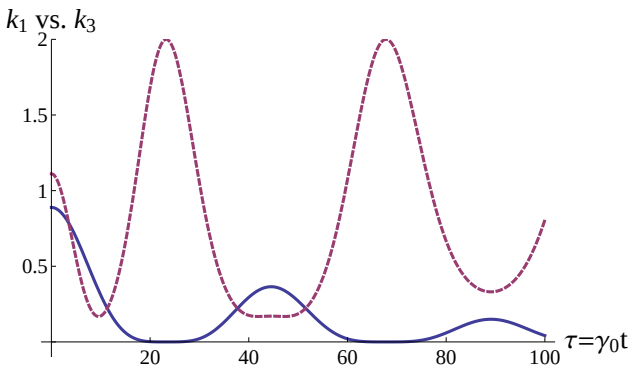

FIG. 3: (Color online). k_1 (blue solid line) versus k_3 (purple dashed line) as a function of the dimensionless time $\gamma_0 t$ for $\lambda = 0.01\gamma_0$ and starting from the state $|\Psi\rangle = \sqrt{1/3}|00\rangle + \sqrt{2/3}|11\rangle$.

of a high-Q cavity [41] and it can also be implemented by superconducting Josephson qubits in the framework of circuit QED [42] and by entangled polarization photons in an all-optical setup [43]. If the two-qubit state has initially an X structure, this is maintained during the dynamics locally governed by the Hamiltonian of Eq. (27) and the density matrix elements at time t, in the same basis as in Eq. (7), are [40]

$$\begin{aligned}
\rho_{11}(t) &= \rho_{11}(0)P_t^2, \\
\rho_{22}(t) &= \rho_{22}(0)P_t + \rho_{11}(0)P_t(1-P_t), \\
\rho_{33}(t) &= \rho_{33}(0)P_t + \rho_{11}P_t(1-P_t), \\
\rho_{44}(t) &= 1 - [\rho_{11}(t) + \rho_{22}(t) + \rho_{33}(t)], \\
|\rho_{14}(t)| &= |\rho_{14}(0)|P_t, \quad |\rho_{23}(t)| = |\rho_{23}(0)|P_t, \quad (28)
\end{aligned}$$

where $P_t = \mathrm{e}^{-\lambda t}\left[\cos\left(\frac{dt}{2}\right) + \frac{\lambda}{d}\sin\left(\frac{dt}{2}\right)\right]^2$, with $d = \sqrt{2\gamma_0\lambda - \lambda^2}$. The parameter λ represents the spectral width of the coupling while γ_0 is the spontaneous emission rate of the qubit (atom). Under this evolution the two-qubit states may cross the two zones $k_1 \leq k_3$ and $k_1 > k_3$ of Sec. V. This can be explicitly seen by choosing, for example, the initial state $|\Psi\rangle = \sqrt{1/3}|00\rangle + \sqrt{2/3}|11\rangle$ and $\lambda = 0.01\gamma_0$ (strong coupling regime). In Fig. 3 we plot k_1 and k_3 as functions of the dimensionless time $\gamma_0 t$. The plot clearly displays that there are time regions when k_1 is larger than k_3 and vice versa.

VIII. CONCLUSIONS

The distance between a quantum state and its closest states without certain properties has been employed in the literature to quantify in a unified view the various kinds of correlations present in an arbitrary multipartite quantum system. Using relative entropy as distance measure, an additivity relation among total T, quantum D and classical C correlations holds [12]. This additivity relation is of the kind $T = D + C - L$ and contains a quantity L defined as the relative entropy-based (REB) distance between two particular closest product states linked to the system state [12].

In this paper we have investigated how a unified view of various kinds of correlations works in terms of a different distance measure. Among the possible suitable distance measures, we have considered the square norm (Hilbert-Schmidt) distance, inspired by the fact that it has been already exploited to define geometric quantum discord $D_{\rm g}$ to quantify quantum correlations present in a state [28]. Using the square norm, we have then considered the geometric quantifiers for classical $C_{\rm g}$ and total $T_{\rm g}$ correlations, recently introduced [32], and defined a quantity $L_{\rm g}$, analogous to the REB quantity L. We have given explicit expressions of the geometric correlation quantifiers for the class of two-qubit X states ρ_X, by firstly finding the relevant closest states linked to ρ_X. We have then analyzed the relationship among the various correlation quantifiers and we have shown that there exist a subclass of X states for which it is not possible to find a closed additivity relation of the kind $T_{\rm g} = D_{\rm g} + C_{\rm g} - L_{\rm g}$. The additivity relation holding when relative entropy is used as distance measure is therefore not preserved when one quantifies in a unified way the different kinds of correlations present in a quantum state by using the square norm. Moreover we have explored numerically the abundance and importance of the nonadditivity of geometric correlations and found that there is a nonnegligible amount of X states with a meaningful deviation from additivity. Therefore, both the analytical proof given for a subclass of X states and the numerically evaluated occurrence among X-states show that for a two-qubit random state, a closed relation among geometric correlation quantifiers needs not to be satisfied.

The results of this paper seem to confirm that different distance measures cannot serve just as well to quantify the various kinds of correlations present in a quantum state. In this sense, in Ref. [32] it had been shown that, when one compares the dynamics of correlation quantifiers based on relative entropy with that of correlation quantifiers based on square norm distance, different qualitative behaviors may occur. These findings recall the known result in the entanglement theory that different entanglement measures may induce different orderings in the state space [44]. The present results give a clear indication that different distance measures (in particular relative entropy and square norm) may not share important general qualitative properties, like the occurrence of closed additivity relations among the various correlation quantifiers.

Appendix A

In this appendix we show that the function F of Eq. (10), measuring the distance between a two-qubit state ρ and the arbitrary product state $\rho_A \otimes \rho_B$ of Eq. (9), has an absolute minimum for $a_1 = a_2 = b_1 = b_2 = 0$ and for $a_3 = \bar{a}_3$, $b_3 = \bar{b}_3$ solutions of Eqs. (13) when an X

state is considered.

Let us consider the function F of Eq. (10), in the case of X states. It can be divided into two parts. We write $F = F_1 + F_2$, where

$$\begin{aligned} F_1 &= \frac{1}{4}[a_1^2 + b_1^2 + a_2^2 + b_2^2 \\ &+(T_{11} - a_1 b_1)^2 + (T_{12} - a_1 b_2)^2 \\ &+(T_{21} - a_2 b_1)^2 + (T_{22} - a_2 b_1)^2 \\ &+a_1^2 b_3^2 + a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2], \end{aligned} \tag{A1}$$

and

$$F_2 = \frac{1}{4}[(x_3 - a_3)^2 + (y_3 - b_3)^3 + (T_{33} - a_3 b_3)^2]. \tag{A2}$$

We observe that F_2 only depends on a_3, b_3 and also that, if $a_1 = b_1 = a_2 = b_2 = 0$ gives a minimum for F_1, this occurs irrespectively of the value of a_3 and b_3. Therefore, the absolute minimum of F is obtained for the values of variables giving the absolute minimum of F_1 and F_2 separately. The absolute minimum of the convex function F_2 is just obtained in the values $a_3 = \bar{a}_3$ and $b_3 = \bar{b}_3$ that are solutions of Eqs. (13). In order to verify that F_1 actually has a minimum in $a_1 = b_1 = a_2 = b_2 = 0$, it is enough to consider the part of function F_1 non-including a_3, b_3 (the remaining part is equal to zero for $a_1 = b_1 = a_2 = b_2 = 0$). Thus we focus on the function $f = f(a_1, b_1, a_2, b_2)$ given by

$$\begin{aligned} f &= a_1^2 + b_1^2 + a_2^2 + b_2^2 + (T_{11} - a_1 b_1)^2 + (T_{12} - a_1 b_2)^2 \\ &+(T_{21} - a_2 b_1)^2 + (T_{22} - a_2 b_1)^2. \end{aligned} \tag{A3}$$

In order to show that f has an absolute minimum in $a_1 = b_1 = a_2 = b_2 = 0$, we consider the difference $\Delta f = f(a_1, b_1, a_2, b_2) - f(0, 0, 0, 0)$ between the value of f in any possible point and its value in $\{0, 0, 0, 0\}$, that is

$$\begin{aligned} \Delta f &= a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_1^2 b_1^2 + a_2^2 b_2^2 + a_2^2 b_1^2 + a_1^2 b_2^2 \\ &-(T_{11} a_1 b_1 + T_{12} a_1 b_2 + T_{21} a_2 b_1 + T_{22} a_2 b_2). \end{aligned} \tag{A4}$$

The last term of Eq. (A4) admits as a minimum value $-(|T_{11} a_1 b_1| + |T_{12} a_1 b_2| + |T_{21} a_2 b_1| + |T_{22} a_2 b_2|)$. Furthermore, since $|T_{ij}| \leq 1$, the lower bound of this expression is $-(|a_1 b_1| + |a_1 b_2| + |a_2 b_1| + |a_2 b_2|)$, so that

$$\begin{aligned} \Delta f &\geq a_1^2 + b_1^2 + a_2^2 + b_2^2 + a_1^2 b_1^2 + a_2^2 b_2^2 + a_2^2 b_1^2 + a_1^2 b_2^2 \\ &-(|a_1 b_1| + |a_1 b_2| + |a_2 b_1| + |a_2 b_2|). \end{aligned} \tag{A5}$$

We now observe that

$$\begin{aligned} a_1^2 + b_1^2 + a_2^2 + b_2^2 &= (|a_1| - |b_1|)^2 + (|a_2| - |b_2|)^2 \\ &+2|a_1 b_1| + 2|a_2 b_2| \end{aligned} \tag{A6}$$

and also

$$\begin{aligned} a_1^2 + b_1^2 + a_2^2 + b_2^2 &= (|a_1| - |b_2|)^2 + (|a_2| - |b_1|)^2 \\ &+2|a_1 b_2| + 2|a_2 b_1| \end{aligned} \tag{A7}$$

Putting these expressions in Eq. (A5), we obtain

$$\begin{aligned} \Delta f &\geq \frac{1}{2}[(|a_1| - |b_1|)^2 + (|a_2| - |b_2|)^2 + (|a_1| - |b_2|)^2 \\ &+(|a_2| - |b_1|)^2] + a_1^2 b_1^2 + a_2^2 b_2^2 + a_2^2 b_1^2 + a_1^2 b_2^2 \end{aligned} \tag{A8}$$

Therefore Δf is always greater than zero unless $a_1 = b_1 = a_2 = b_2 = 0$. **QED**.

Appendix B

In this appendix we want to prove that the quantity $h = \bar{a}_3^2 \left[\left(T_{33} - \bar{a}_3 \bar{b}_3\right)^2 - \left(1 + \bar{b}_3^2\right) \right]$, where $\bar{a}_3, \bar{b}_3$ are solutions of Eqs. (13), that quantifies the difference $T_{\rm g} - D_{\rm g}^{(2)} - C_{\rm g}^{(2)} = h/4$ given in Eq. (24), satisfies the inequality $h \leq 0$ with the equality verified if and only if $x_3 + y_3 T_{33} = 0$. To this aim, being $1 + \bar{b}_3^2 \geq 1$, it will be sufficient to prove that $\left|T_{33} - \bar{a}_3 \bar{b}_3\right| \leq 1$ and successively to find the condition for which the upper bound 1 is achieved.

Let us start noticing that $\bar{a}_3, \bar{b}_3$ give the minimum of the function F_2 of Eq. (A2). Thus, $F_2(a_3, b_3) \geq F_2(\bar{a}_3, \bar{b}_3) \geq (T_{33} - \bar{a}_3 \bar{b}_3)^2$. Then, if in Eq. (A2) we replace a_3, b_3 with x_3, y_3, the inequality

$$(T_{33} - x_3 y_3)^2 \geq (T_{33} - \bar{a}_3 \bar{b}_3)^2 \tag{B1}$$

holds. Using Eqs. (8) of an X state, we have

$$T_{33} - x_3 y_3 = p_1 - t_1^2 - (p_2 - t_2^2), \tag{B2}$$

where $p_1 = \rho_{11} + \rho_{44}$, $p_2 = 1 - p_1 = \rho_{22} + \rho_{33}$, $t_1 = \rho_{11} - \rho_{44}$, and $t_2 = \rho_{22} - \rho_{33}$. $T_{33} - x_3 y_3$ reaches its maximum value for $t_1 = 0$ and $t_2 = p_2$, while its minimum value is found for $t_1 = p_1$ and $t_2 = 0$. In any case $(T_{33} - x_3 y_3)_{\max} = p_1 + p_2^2 - p_2 \leq 1 - p_2$ and $(T_{33} - x_3 y_3)_{\min} = p_1 - p_1^2 - p_2 \geq -1 + p_1$. Then $\left|T_{33} - x_3 y_3\right| \leq 1$ and, according to inequality (B1), $\left|T_{33} - \bar{a}_3 \bar{b}_3\right| \leq 1$, as we wanted to prove.

The cases such that $h = \bar{a}_3^2[\left(T_{33} - \bar{a}_3 \bar{b}_3\right)^2 - \left(1 + \bar{b}_3^2\right)] = 0$ correspond to i) $\bar{a}_3 = 0$ or to ii) $\bar{b}_3 = 0$ and $|T_{33}| = 1$. From Eqs. (13), we can notice that case (i) implies $b_3 = y_3$ and then $x_3 + y_3 T_{33} = 0$; case (ii) instead implies that $\bar{a}_3 = x_3 = \pm y_3$ where $\pm y_3$ correspond to $T_{33} = \mp 1$ respectively, so that again $x_3 + y_3 T_{33} = 0$.

Appendix C

In this appendix we discuss the properties of a particular subclass of X states, namely the Bell-diagonal states, also called states with maximally mixed marginals [24]. Looking at the matrix form of an X state given in Eq. (7), Bell-diagonal states have diagonal elements $\rho_{11} = \rho_{44}$, $\rho_{22} = \rho_{33}$ and non-diagonal density matrix elements real, that is $\gamma_{14}, \gamma_{23} = 0, \pi$. Therefore, the parameters of their Bloch representation are $x_3 =$

$y_3 = T_{12} = T_{21} = 0$, $T_{11} = 2e^{i\gamma_{14}}[\rho_{14} + e^{i(\gamma_{14}-\gamma_{23})}\rho_{23}]$, $T_{22} = 2e^{i\gamma_{14}}[e^{i(\gamma_{14}-\gamma_{23})}\rho_{23} - \rho_{14}]$ and $T_{33} = 2(\rho_{11} - \rho_{22})$, giving a Bloch representation [24]

$$\rho^{\rm B} = [\mathbb{1} \otimes \mathbb{1} + \sum_{i=1}^{3} T_{ii}\sigma_i \otimes \sigma_i]/4. \qquad \text{(C1)}$$

Bell-diagonal states have the peculiar property that the quantity L of Eq. (2) is zero ($L = 0$), so that $T = D + C$ for REB correlation quantifiers [12]. Indeed, these states present the same closed additivity relation even for the geometric correlation quantifiers of Eqs. (3) and (4), that is $T_{\rm g} = D_{\rm g} + C_{\rm g}$ [32] with $L_{\rm g} = 0$.

Bell-diagonal states $\rho^{\rm B}$ have $x_3 = y_3 = 0$ and it is possible to show that the corresponding solutions of Eqs. (13) are $\bar{a}_3 = \bar{b}_3 = 0$ [32], so that the closest product state using the square norm, in this case, reduces to the product of the marginals $\pi_{\rho^B} = (\mathbb{1}/2) \otimes (\mathbb{1}/2)$.

Regarding the closest classical state and its closest product state (see Sec. V), for Bell-diagonal states it occurs that, if $\gamma_{14} = \gamma_{23}$ then $k_{ii} = T_{ii}^2$, while if $\gamma_{14} = \pi - \gamma_{23}$ one has $k_{11} = T_{22}^2$, $k_{22} = T_{11}^2$ and always $k_{33} = T_{33}^2$. The two cases $k_1 \leq k_3$ and $k_1 > k_3$, obtained from Eq. (15), thus involve direct comparisons among absolute values of the components T_{11}, T_{22}, T_{33} of the correlation tensor. Then, if a Bell-diagonal state belongs to the case 1 ($k_1 \leq k_3$), being $x_3 = y_3 = 0$, the closest classical state of Eq. (16) reduces to $\chi^{(1)}_{\rho^{\rm B}} = \frac{1}{4}[\mathbb{1} \otimes \mathbb{1} + T_{33}\sigma_3 \otimes \sigma_3]$ with $T_{33}^2 = k_3$. On the other hand, for a Bell-diagonal state belonging to the case 2 ($k_1 > k_3$), being $y_3 = 0$ and $\gamma_{14}, \gamma_{23} = 0, \pi$, the closest classical state of Eq. (17) reduces either to $\chi^{(2)}_{\rho^{\rm B}} = \frac{1}{4}[\mathbb{1} \otimes \mathbb{1} + T_{11}\sigma_1 \otimes \sigma_1]$ if $\gamma_{14} = \gamma_{23}$ with $T_{11}^2 = k_1$, or to $\chi^{(2)}_{\rho^{\rm B}} = \frac{1}{4}[\mathbb{1} \otimes \mathbb{1} + T_{22}\sigma_2 \otimes \sigma_2]$ if $\gamma_{14} = \pi - \gamma_{23}$ with $T_{22}^2 = k_1$. Since we have already seen that the closest product state to $\rho^{\rm B}$ is given by the product of its marginals, the same happens for the product state closest to the closest classical state $\chi^{(1)}_{\rho^{\rm B}}$ or $\chi^{(2)}_{\rho^{\rm B}}$.

For a Bell-diagonal state the explicit expression of geometric quantum discord can be written as $D_{\rm g}(\rho^{\rm B}) = \frac{1}{4}[T_{11}^2 + T_{22}^2 + T_{33}^2 - T^2]$ [28], where $T \equiv \max\{|T_{11}|, |T_{22}|, |T_{33}|\}$. The geometric quantifier of total correlations of Eq. (20) reduces to $T_{\rm g}(\rho^{\rm B}) = (T_{11}^2 + T_{22}^2 + T_{33}^2)/4$ with the geometric quantifier of classical correlations given by $C_{\rm g}(\rho^{\rm B}) = T^2/4$ [32], so that it is always $T_{\rm g} = D_{\rm g} + C_{\rm g}$. In particular, for Bell-diagonal states belonging to the case 2, the second member of Eq. (24) is zero because $x_3 + y_3 T_{33} = 0$, being $x_3 = y_3 = 0$.

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