ENTANGLEMENT MEASURE OF PURE SYMMETRIC STATES VIA SPIN COHERENT STATES

Abstract.

1. Introduction

2. Spin Coherent States

Any pure spin-j state $|\psi_j\rangle$ can be expanded in the standard angular momentum basis $\{|j,m\rangle : -j \leq m \leq +j\}$ of joint eigenstates of $\mathbf{J}^2$ and J_z as

$$|\psi_j\rangle = \sum_{m=-j}^{+j} c_m \, |j,m\rangle \tag{2.1}$$

with c_m complex coefficients such that $\sum_m |c_m|^2 = 1$.

Coherent states are eigenstates of $\mathbf{J}.\epsilon$ with eigenvalue j, where ϵ is a unit vector pointing along a given direction. For spin-$\frac{1}{2}$, their general expression reads

$$|\frac{1}{2},\epsilon\rangle = \frac{1}{\sqrt{1+\bar{\epsilon}\epsilon}} \, |\frac{1}{2},\frac{1}{2}\rangle + \frac{\epsilon}{\sqrt{1+\bar{\epsilon}\epsilon}} \, |\frac{1}{2},-\frac{1}{2}\rangle \tag{2.2}$$

More generally, a spin-j coherent state $|j,\epsilon\rangle$ has expansion

$$|j,\epsilon\rangle = \frac{1}{(1+\bar{\epsilon}\epsilon)^j} \sum_{m=-j}^{+j} \epsilon^{j+m} \sqrt{C_{2j}^{j+m}} \, |j,m\rangle \tag{2.3}$$

3. Correspondance Between N-Spinors States And Pure Symmetric States

As early as 1932, Majorana had proposed a connection between N-spinors and pure symmetric state that can be seen geometrically as a set of points in the Bloch sphere as follows[?].

$$|\frac{N}{2}, l-N\rangle = M\sum_{p}\hat{P}\left\{|\epsilon_1, \epsilon_2, \ldots, \epsilon_N\rangle\right\}, \tag{3.1}$$

where

$$|\epsilon_l\rangle = cos(\beta_l/2)e^{-i\alpha_l/2}|0\rangle + sin(\beta_l/2)e^{i\alpha_l/2}|1\rangle, l = 0, 1, 2, \ldots, N. \tag{3.2}$$

and $|\frac{N}{2}, l-N\rangle$ is N-spinor, $\hat{P}$ corresponds to the set of all N! permutations of qubits $\{|\epsilon_l\rangle\}$ and M is a normalization factor.

Every symmetric state of N qubits can be expressed in a unique way over the Dicke basis formed by the N + 1 joined eigenstates $\left\{|\frac{N}{2}, l-N\rangle\right\}$ of the collective operators $\hat{S_Z} = \sum_{i=1}^{N}\hat{\sigma}_i^Z$ and $\hat{S}^2$, where $\hat{S} = \sum_{i=1}^{N}\hat{\sigma}_i$ as follows.

$$|\frac{N}{2}, l-N\rangle = \frac{1}{\sqrt{C_l^N}}\left(|\underbrace{0, 0\ldots, 0}_{l}, \underbrace{1, 1\ldots, 1}_{N-l}\rangle + permutations\right) \tag{3.3}$$

For $N = 2$, the pure symmetric state describe a $spin-1$ particle, $|1, l-2\rangle$, $l = 0, 1, 2$ each l correspond to a state of spin-1 particle $\{|j=1, m=-1\rangle, |j=1, m=0\rangle, |j=1, m=1\rangle\}$ as follows

$$|j=1, m=-1\rangle \longrightarrow |\frac{N}{2} = 1, l-N=0\rangle = |0, 0\rangle$$

$$|j=1, m=0\rangle \longrightarrow |\frac{N}{2} = 1, l-N=-1\rangle = \frac{1}{\sqrt{2}}(|0, 1\rangle + |1, 0\rangle)$$

$$|j=1, m=1\rangle \longrightarrow |\frac{N}{2} = 1, l-N=-2\rangle = |1, 1\rangle$$

An arbitrary pure state of a spin-j can be expressed as.

$$\sum_{m=-j}^{j} c'_m |j, m\rangle = \sum_{l=0}^{N} c_l |\frac{N}{2} = j, l-N\rangle, \tag{3.4}$$

In this way, we can decompose state of a spin-j into $2j$ $spin - \frac{1}{2}$ that can be geometrically represented by $2j$ points in the Bloch sphere.

There is a simple way to express the coefficients c_l in terms of the Majorana spinor orientations (α_l, β_l)[?], it may be first identified that an identical rotation $R \otimes R \ldots \otimes R$ on the symmetric state transforms it into another symmetric state. Choosing $R^{-1}(\alpha_s, \beta_s) \otimes R^{-1}(\alpha_s, \beta_s) \otimes \ldots \otimes R^{-1}(\alpha_s, \beta_s)$, where α_s, β_s correspond to the orientation of any one of the spinors

$$\langle 1, 1, \ldots, 1| R^{-1}(\alpha_s, \beta_s) \otimes R^{-1}(\alpha_s, \beta_s) \otimes \ldots \otimes R^{-1}(\alpha_s, \beta_s)|\Psi_{sym}\rangle = 0. \tag{3.5}$$

This is because the rotation $R_s^{-1} \otimes R_s^{-1} \otimes \ldots \otimes R_s^{-1}$ takes one of the spinors $|\epsilon_s\rangle$ with the orientation angles (α_s, β_s) to $|0\rangle$. There exist N rotations $R_s^{-1}, s = 1, 2, \ldots, N$ which lead to the same result

$$\langle 1, 1, \ldots, 1| \, R_s^{-1} \otimes R_s^{-1} \otimes \ldots \otimes R_s^{-1} \left\{ \sum_{l=0}^{N} c_l \, |\frac{N}{2}, l - N\rangle \right\} = 0. \tag{3.6}$$

or

$$\sum_{l=0}^{N} c_l \, \langle \frac{N}{2}, -\frac{N}{2}| \, R_s^{-1} \, |\frac{N}{2}, l - \frac{N}{2}\rangle = 0, \tag{3.7}$$

with $R_s^{-1} \equiv R_s^{-1} \otimes R_s^{-1} \otimes \ldots \otimes R_s^{-1}$. This quantity can be expressed as follow

$$\sum_{l=0}^{N} c_l \, \langle \frac{N}{2}, -\frac{N}{2}| \, R_s^{-1} \, |\frac{N}{2}, l - \frac{N}{2}\rangle = \sum_{l=0}^{N} c_l \, (-1)^l \, \sqrt{C_l^N} \left(cos\left(\frac{\beta_s}{2}\right)\right)^{N-l} \left(sin\left(\frac{\beta_s}{2}\right)\right)^{l} e^{i\left(l-\frac{N}{2}\right)\alpha_s} = 0 \tag{3.8}$$

and simplifying, we obtain

$$A \sum_{l=0}^{N} c_l \, (-1)^l \, \sqrt{C_l^N} \epsilon^l = 0, \tag{3.9}$$

where $\epsilon = tan\left(\frac{\beta_s}{2}\right) e^{(i\alpha_s)}$ and $A = \left(cos\left(\frac{\beta_s}{2}\right)\right)^N e^{-i\alpha_s \frac{N}{2}}$. In other words, given the parameters c_l, the N-roots ϵ_s determine the orientations (α_s, β_s) of the spinors constituting the N-qubit symmetric state.

The case whene $c_l = \eta^l \sqrt{C_l^N}$, the equation (3.9) become

$$(\epsilon\eta - 1)^N = 0, \tag{3.10}$$

with one solution $\epsilon = \frac{1}{\eta}$ that means, all the stars in the Bloch sphere coincide in a single point, this case represents a coherent state whene all the $|\epsilon_i\rangle$ are equal then we have

$$|j, \epsilon\rangle = |\underbrace{\epsilon, \epsilon, \ldots, \epsilon}_{2j}\rangle = \frac{1}{(1 + \bar{\epsilon}\epsilon)^j} \sum_{m=-j}^{+j} \epsilon^{j+m} \sqrt{C_{2j}^{j+m}} \, |j, m\rangle \tag{3.11}$$

4. Entanglement Classes For Tripartite Symmetric States

In the case of tripartite states, several classes can be defined, apart from fully separable states and fully entangled states, a new concept appears is the partially separable states. There is also others classes that we call $|W\rangle$ and $|GHZ\rangle$ states.

It is known that two pure states $|\psi\rangle$ and $|\phi\rangle$ are in the same class if they are related by SLOCC operator with non-zero probability.

Tripartite symmetric states can generally expressed as follow

$$|\frac{N}{2}, l-N\rangle = M\sum_{p}\hat{P}\left\{|\epsilon_1, \epsilon_2, \epsilon_3\rangle\right\}, \tag{4.1}$$

In the case when all the three qubits $\{|\epsilon_i\rangle\}$ are equal, we have separable states which are in this case coherent states. When two of the three qubits are equal $M\sum_p \hat{P}\left\{|\epsilon_1, \epsilon_1, \epsilon_2\rangle\right\}$ this state can be related by a SLOCC operator with the state $|W\rangle = |001\rangle + |010\rangle + |100\rangle$ so we can define the **W** class as :$|W\rangle = M\sum_p \hat{P}\left\{|\epsilon_1, \epsilon_1, \epsilon_2\rangle\right\}$ that can be represented in the Bloch sphere as 2 points one of theme is degenerate 2 times. The third class is the **GHZ** that can be defined as follow $|GHZ\rangle = M\sum_p \hat{P}\left\{|\epsilon_1, \epsilon_2, \epsilon_3\rangle\right\} with : \epsilon_1 \neq \epsilon_2 \neq \epsilon_3$, geometrically can be seen as 3 distinct points in the Bloch sphere.

We can also classify the tripartite symmetric states, based on the concept of tangle $\tau_{ABC} = \tau_{A(BC)} - \tau_{AB} - \tau_{AC}$. The tangle can be seen as the discriminant of the polynomial $P(\epsilon) = c_0 - c_1\sqrt{3}\epsilon + c_2\sqrt{3}\epsilon^2 - c_3\epsilon^3$.

$$\tau_{ABC} \propto (\epsilon_1 - \epsilon_2)^2(\epsilon_1 - \epsilon_3)^2(\epsilon_2 - \epsilon_3)^2$$

It is clear that $\tau_{ABC} \neq 0$, $\epsilon_1 \neq \epsilon_2 \neq \epsilon_3$ correspond to **GHZ** class. The vanishing of the tangle indicate the **W** or separable class,

5. Entanglement Measure Of W States

The case when N=2 in symmetric states (*Bipartite qubit states*) we have $|\psi\rangle = c_0\,|0,0\rangle + c_1/\sqrt{2}(|0,1\rangle + |1,0\rangle) + c_2\,|1,1\rangle$. to write $|\psi\rangle$ as a symmetric state of two qubits we must solve the equation $\sum_{l=0}^{N} c_l\,(-1)^l\,\sqrt{C_l^N}\,\epsilon^l = 0$, in the case N=2

$$c_0 - c_1\sqrt{2}\epsilon + c_2\epsilon^2 = 0 \Longrightarrow \begin{array}{l} \epsilon_1 = \frac{c_1\sqrt{2}-\sqrt{2c_1^2-4c_0c_2}}{2c_2} \\ \epsilon_2 = \frac{c_1\sqrt{2}+\sqrt{2c_1^2-4c_0c_2}}{2c_2} \end{array} \tag{5.1}$$

We have separability when $\epsilon_1 = \epsilon_2 \Leftrightarrow |2c_1^2 - 4c_0c_2| = 0$. This quantity cas be seen also as concurrence, which can be defined as a distance separate the two points on the Bloch sphere.
This quantity in terms of ϵ_1 and ϵ_2 can be written as follow:

$$|(\epsilon_1 - \epsilon_2)^2| = 0 \tag{5.2}$$

It is clear that the concurrence represents a distance between the two points on the Bloch sphere.

In the case N=3, we have three points on the Bloch sphere, the measure via the distance beetwen theme is not evident to be a measure of entanglement but in the case of **W** class, we have just two points on the Bloch sphere and the distance beetwen them is a good candidate to be a measure of entanglement of **W** class.

In order to have a condition of separability in case of tripartite symmetric state, this equation $c_0 - c_1\sqrt{3}\epsilon + c_2\sqrt{3}\epsilon^2 - c_3\epsilon^3 = 0$ must have one solution. The condition to have it, is as follow:

$$\begin{array}{ccccc} P(\epsilon) & = & c_0 - c_1\sqrt{3}\epsilon + c_2\sqrt{3}\epsilon^2 - c_3\epsilon^3 & = & 0 \\ P'(\epsilon) & = & -c_1\sqrt{3} + 2c_2\sqrt{3}\epsilon - 3c_3\epsilon^2 & = & 0 \\ P''(\epsilon) & = & 2c_2\sqrt{3} - 6c_3\epsilon & = & 0 \end{array} \tag{5.3}$$

Then we can have the separability criterion in more simple way:

$$\begin{array}{ccc} \left(\frac{c_2}{\sqrt{3}}\right)^2 - \frac{c_1}{\sqrt{3}}c_3 & = & 0 \\ c_0c_3 - \frac{c_1}{\sqrt{3}}\frac{c_2}{\sqrt{3}} & = & 0 \end{array} \tag{5.4}$$

In the case of **W** class ($|W\rangle = M\sum_p \hat{P}\,\{|\epsilon_1, \epsilon_1, \epsilon_2\rangle\}$), if we write these conditions in terms of ϵ_1,ϵ_2, we will find the same condition as the concurrence $|(\epsilon_1 - \epsilon_2)^2| = 0$. That can be seen as minimum distance between our **W** state and the separable state which can be a measure of entanglement.

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