Local quantum uncertainty for a class of two-qubit X states and quantum correlations dynamics under decoherence

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Abstract

A special emphasis is devoted to the concept of local quantum uncertainty as an indicator of quantum correlations. We study quantum discord for a class of two-qubit states parameterized by two parameters. Quantum discord based on local quantum uncertainty, von Neumann entropy and trace distance (Schatten 1-norm) are explicitly derived and compared. The behavior of quantum correlations, quantified via local quantum uncertainty, under decoherence effects is investigated. We show that the discord-like local quantum uncertainty exhibits the possibility of freezing behavior during its evolution.

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1 Introduction

Characterizing quantum correlations in multipartite quantum systems is one of the most challenging topics in quantum information theory. Various measures to quantify the degree of quantumness in multipartite quantum systems were introduced in the literature. The most familiar ones are the concurrence, the entanglement of formation, the quantum discord and its various geometric versions [1, 2, 3, 4, 5]. The interest in quantum correlations other than entanglement lies in the existence of nonclassical correlations even in separable states [6, 7]. In fact, entanglement does not account for all nonclassical aspects of correlations, especially in mixed states. This yielded many works dedicated to introduce quantum correlation quantifiers beyond entanglement. As the total correlation is the sum of two contributions: a classical part and quantum part, different concepts were considered to develop the best way to distinguish between classical and quantum correlations. In this context, the entropy-based quantum discord [6, 7] is probably the quantifier which has been intensively investigated in the literature for different purposes and from several perspectives (see for instance [5]). However, the analytical evaluation of entropic quantum discord is, from a computational side, difficult to achieve for an arbitrary bipartite system and only partial results were obtained for some special two-qubit states. To overcome such technical difficulties geometric variants of quantum discord were introduced by employing Schatten p-norms. The first geometric formulation of quantum discord was developed in [8] by adopting the Hilbert-Schmidt norm ($p = 2$) to measure the distance between a given state and the set of zero-discord states. This quantum correlations indicator is is easily computable [9, 10, 11, 12] but it suffers from its non-contractibility under trace preserving channels [13]. In fact, the Hilbert-Schmidt based quantum discord can increase under local operations on the unmeasured qubit. The Bures norm (trace norm with $p = 1$) is the only Schatten p-norm which is contractible [14]. This distance was used successfully to describe the quantum correlations in several two-qubit systems [14, 15, 16].

The issue concerning the measures of quantum correlations continues to draw special attention in quantum information science. Recently, the concept of local quantum uncertainty was proposed as an indicator of quantum discord in bipartite systems. It is based on the notion skew information introduced by Wigner to determine the uncertainty in the measurement of an observable [17]. The local quantum uncertainty is defined as the minimum of the skew information over all possible local observables. This measure offers an appropriate tool to evaluate the analytical expressions of quantum correlations encompassed in any qubit-qudit bipartite system [18]. The local quantum uncertainty is related to the quantum Fisher information [19, 20, 21] which is a key ingredient in quantum metrology protocols [22]. Also, it quantifies the speed of the local (unitary) evolution of a bipartite quantum system [18].

In this paper, the analytical derivation of quantum discord is essentially approached in the context of local quantum uncertainty formalism. We consider a special family of rank-2 X states which includes various types of two-qubit states of interest in different models of collective spin systems. One may quote for instance Dicke model [23] and Lipkin-Meshkov-Glick model [24] for which the quantum discord was investigated in relation with their critical properties and quantum phase transitions (see for instance [25, 26, 27, 28]). Remarkably, it has been shown that the quantum discord provides a suitable indicator to understand the role of quantum correlations in characterizing quantum phase transitions [29](see also [30]). We note also that the set of two-qubit under consideration are of special relevance in investigating quantum correlations in bipartite states extracted from multi-qubit Dicke states and their superpositions(e.g., generalized GHZ states, even and odd spin coherent states) [31]. Besides the explicit derivation of the local quantum uncertainty, we also determine the von Neumann entropy-based quantum discord and the trace distance discord for this class of two-qubit states. Another facet of this work concerns the dynamics of the local quantum uncertainty under decoherence effects due to the unavoidable interaction of a quantum system with its environment.

The paper is structured as follows. In section 2, we give the explicit expressions for local quantum uncertainty, the von Neumann entropy-based quantum discord and the trace norm quantum discord for a special class of two-qubit states of particular relevance in investigating bipartite quantum correlations in various collective spin models. In section 3, under four typical quantum decoherence channels (bit flip, phase flip, bit-phase flip and generalized amplitude damping), we give the analytic expressions of local quantum uncertainty. In particular, we show the freezing character of local quantum uncertainty in some particular situations. Concluding remarks close this paper.

2 Local quantum uncertainty, entropic quantum discord and geometric quantum for rank two X states

The two-qubit density matrices which display non-zero entries only along the main- and anti-diagonals are usually called X-states. They generalize several two-qubit states as for instance Bell-diagonal states (see [1]), Werner states [32], isotropic states [33]. Their particular relevance was first identified in investigating the phenomenon of sudden death of entanglement [34] and since then extended to many other contexts of quantum information theory. A generic X-state is parameterized by seven real parameters and the corresponding symmetry is fully characterized by the $su(2) \times su(2) \times u(1)$ subalgebra of the full $su(4)$ algebra describing an arbitrary two-qubit system [35]. This symmetry reduction from $su(4)$ to $su(2) \times su(2) \times u(1)$ renders easy many analytical calculations of concurrence, entanglement of formation, quantum discord and leads to interesting results in studying their properties and their evolution under dissipative processes (see for instance [36, 37]).

In this work, we consider the set of two-qubit density matrices which have the following form

$$\rho = \begin{pmatrix} c_1 & 0 & 0 & \sqrt{c_1 c_2} \\ 0 & \frac{1}{2}(1-c_1-c_2) & \frac{1}{2}(1-c_1-c_2) & 0 \\ 0 & \frac{1}{2}(1-c_1-c_2) & \frac{1}{2}(1-c_1-c_2) & 0 \\ \sqrt{c_1 c_2} & 0 & 0 & c_2 \end{pmatrix} \qquad (1)$$

in the computational basis $\mathcal{B} = \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The parameters c_1 and c_2 satisfy the conditions $0 \leq c_1 \leq 1$, $0 \leq c_2 \leq 1$ and $0 \leq c_1 + c_2 \leq 1$. We assume that all entries of the matrix ρ are positives. In fact, the local unitary transformation, acting on the qubits 1 and 2 of the system,

$$|0\rangle_k \to \exp\left(\frac{i}{2}(\theta_1 + (-)^k\theta_2)\right)|0\rangle_k \quad k = 1, 2$$

eliminates the phase factors of the off diagonal elements and the rank of the density matrix ρ remains unchanged. In the Fano-Bloch representation, the matrix density ρ (1) writes

$$\rho = \frac{1}{4}\sum_{\alpha,\beta} R_{\alpha\beta}\sigma_\alpha \otimes \sigma_\beta$$

where the correlation matrix elements $R_{\alpha\beta}$ are given by $R_{\alpha\beta} = \mathrm{Tr}(\sqrt{\rho}\ \sigma_\alpha \otimes \sigma_\beta)$ with $\alpha, \beta = 0, 1, 2, 3$, σ_i ($i = 1, 2, 3$) are the usual Pauli matrices and σ_0 is 2×2 identity matrix. The non-vanishing correlation matrix elements are given by

$$R_{30} = R_{03} = c_1 - c_2 \quad R_{33} = 2(c_1 + c_2) - 1, \quad R_{11} = 1 - (\sqrt{c_1} - \sqrt{c_2})^2 \quad R_{22} = 1 - (\sqrt{c_1} + \sqrt{c_2})^2. \tag{2}$$

The density matrix (1) is invariant under parity symmetry and exchange transformation (ρ commutes with $\sigma_3 \otimes \sigma_3$ and the permutation operator which exchanges the qubit state $|i, j\rangle$ to $|j, i\rangle$ leaves ρ unchanged). These symmetries simplify considerably the complexity of the analytical evaluations of bipartite correlations. Indeed, from a practical viewpoint, our interest on this type of X states (1) relies upon their simple analytical manipulation in contrast with an arbitrary two-qubit state for which one is forced to resort heavy numerical analysis.

2.1 Local quantum uncertainty: Definition

The concept of local quantum uncertainty is now considered as a promising quantifier of quantum correlation. This is essentially due to its easiness of computability and its reliability. It quantifies the minimal quantum uncertainty in a quantum state due to a measurement of a local observable [18]. For a bipartite quantum state ρ_{12}, the local quantum uncertainty is defined as

$$\mathcal{U}(\rho_{12}) \equiv \min_{K_1} \mathcal{I}(\rho_{12}, K_1 \otimes \mathbb{I}_2), \tag{3}$$

where K_1 is some local observable on the subsystem 1, $\mathbb{I}_2$ is the identity operator leaving unchanged the subsystem 2 and

$$\mathcal{I}(\rho_{12}, K_1 \otimes \mathbb{I}_2) = -\frac{1}{2}\mathrm{Tr}([\sqrt{\rho_{12}}, K_1 \otimes \mathbb{I}_2]^2) \tag{4}$$

is the skew information [17, 19]. The skew information represents the non-commutativity between the state ρ_{12} and the observable K_1. The analytical evaluation the local quantum uncertainty requirers a minimization procedure over the set of all observales acting on the subsystem 1. A closed form for qubit-qudit systems was derived in [18]. Accordingly, for the two-qubit states (1) the expression of the local quantum uncertainty is given by [18]

$$\mathcal{U}(\rho) = 1 - \lambda_{\max}\{W\}, \tag{5}$$

where $\lambda_{\max}\{W\}$ denotes the maximum eigenvalue of the 3×3 matrix W whose matrix elements are defined by

$$\omega_{ij} \equiv \mathrm{Tr}\{\sqrt{\rho}(\sigma_i \otimes \sigma_0)\sqrt{\rho}(\sigma_j \otimes \sigma_0)\}, \tag{6}$$

with $i,j=1,2,3$. The local quantum uncertainty provides an appropriate quantifier of the minimum amount of uncertainty in a bipartite quantum state. For pure bipartite states, it reduces to linear entropy of the reduced densities of the subsystems. Also, it vanishes for classically correlated states. Another interesting property of local quantum uncertainty is its invariance under local unitary operations. This quantum correlations indicator enjoys all required properties to quantify consistently the quantum discord in bipartite systems [18]. Hence, in what follows, we derive the analytical expression of local quantum uncertainty in the two-parameter states (1). This is compared with the geometric quantum discord based on the trace distance [13, 14, 15] and the entropy-based quantum discord originally defined in [6, 7]. To get the explicit form of the matrix elements (6), one needs the squared matrix $\sqrt{\rho}$. It is given by

$$\sqrt{\rho}=\begin{pmatrix} \frac{c_1}{\sqrt{c_1+c_2}} & 0 & 0 & \frac{\sqrt{c_1c_2}}{\sqrt{c_1+c_2}} \\ 0 & \frac{1}{2}\sqrt{1-c_1-c_2} & \frac{1}{2}\sqrt{1-c_1-c_2} & 0 \\ 0 & \frac{1}{2}\sqrt{1-c_1-c_2} & \frac{1}{2}\sqrt{1-c_1-c_2} & 0 \\ \frac{\sqrt{c_1c_2}}{\sqrt{c_1+c_2}} & 0 & 0 & \frac{c_2}{\sqrt{c_1+c_2}} \end{pmatrix} \tag{7}$$

in the computational basis. In Fano-Bloch representation, it rewrites as

$$\sqrt{\rho}=\frac{1}{4}\sum_{\alpha,\beta}\mathcal{R}_{\alpha\beta}\sigma_\alpha\otimes\sigma_\beta$$

with $\mathcal{R}_{\alpha\beta}=\mathrm{Tr}(\sqrt{\rho}\,\sigma_\alpha\otimes\sigma_\beta)$. The non vanishing matrix correlation elements $\mathcal{R}_{\alpha\beta}$ are explicitly given by

$$\mathcal{R}_{00}=\sqrt{c_1+c_2}-\sqrt{1-c_1-c_2},\qquad \mathcal{R}_{03}=\mathcal{R}_{30}=c_1-c_2$$

$$\mathcal{R}_{11}=\sqrt{1-c_1-c_2}+2\frac{\sqrt{c_1c_2}}{\sqrt{c_1+c_2}},\quad \mathcal{R}_{22}=\sqrt{1-c_1-c_2}-2\frac{\sqrt{c_1c_2}}{\sqrt{c_1+c_2}},\quad \mathcal{R}_{33}=\sqrt{c_1+c_2}-\sqrt{1-c_1-c_2}.$$

Reporting the matrix (7) in the equation (6), it is simple to check that the matrix W (6) is diagonal and the diagonal elements are given by

$$\omega_{11}=\sqrt{\frac{1-(c_1+c_2)}{c_1+c_2}}(\sqrt{c_1}+\sqrt{c_2})^2,\quad \omega_{22}=\sqrt{\frac{1-(c_1+c_2)}{c_1+c_2}}(\sqrt{c_1}-\sqrt{c_2})^2,\quad \omega_{33}=\frac{(c_1-c_2)^2}{c_1+c_2}. \tag{8}$$

We note that the eigenvalue ω_{11} is always larger than ω_{22} so that $\omega_{\max}=\max(\omega_{11},\omega_{33})$. It is simply verified that the states for which $\omega_{11}\geq\omega_{33}$ are parameterized by c_1 and c_2 satisfying the following condition

$$\sqrt{(c_1+c_2)(1-(c_1+c_2))}-(\sqrt{c_1}-\sqrt{c_2})^2\geq 0. \tag{9}$$

To examine this condition, the set of states of type (1) is partitioned as

$$\{\rho\equiv\rho_{c_1,c_2},\ \ 0\leq c_1+c_2\leq 1\}=\bigcup_{\alpha\in[0,1]}\{\rho_\alpha\equiv\rho_{c_1,\alpha-c_1},\ \ 0\leq c_1\leq\alpha\}$$

with $c_1+c_2=\alpha$. Therefore, for a fixed value of α and $0\leq c_1\leq\alpha$, the condition (9) becomes

$$2\sqrt{c_1}\,\sqrt{\alpha-c_1}+\sqrt{\alpha}\,(\sqrt{1-\alpha}-\sqrt{\alpha})\geq 0. \tag{10}$$

From this condition, one can see that $(\omega_{11}-\omega_{33})$ is always positive for $\alpha\leq\frac{1}{2}$ and this implies $\omega_{\max}=\omega_{11}$. Conversely, for $\alpha\geq\frac{1}{2}$, one verifies that the condition (10) is satisfied for $\alpha_-\leq c_1\leq\alpha_+$ but not satisfied for $0\leq c_1\leq\alpha_-$ or $\alpha_+\leq c_1\leq\alpha$ where the quantities α_- and α_+ are defined by

$$\alpha_\pm=\frac{\alpha}{2}\pm\frac{1}{2}\sqrt{\alpha\,\sqrt{1-\alpha}\,\left(2\sqrt{\alpha}-\sqrt{1-\alpha}\right)}. \tag{11}$$

Accordingly, the maximum eigenvalue of the local uncertainty matrix W (6) for $\alpha \geq \frac{1}{2}$ writes as

$$\omega_{\max} = \begin{cases} \omega_{33} & \text{for} \quad 0 \leq c_1 \leq \alpha_- \\ \omega_{11} & \text{for} \quad \alpha_- \leq c_1 \leq \alpha_+ \\ \omega_{33} & \text{for} \quad \alpha_+ \leq c_1 \leq \alpha \end{cases} \tag{12}$$

To give expression of the local quantum uncertainty measure, the situations where the parameter α is greater or smaller than $\frac{1}{2}$ are treated separately. Hence, for $\alpha \leq \frac{1}{2}$, the local quantum uncertainty takes the form

$$\mathcal{U}(\rho) = 1 - \sqrt{\frac{1-\alpha}{\alpha}}(\sqrt{c_1} + \sqrt{\alpha - c_1})^2 \quad \text{with} \quad 0 \leq c_1 \leq \alpha. \tag{13}$$

For $\alpha \geq \frac{1}{2}$, the local quantum uncertainty is given by two different expressions. Indeed, one obtains

$$\mathcal{U}(\rho) = 1 - \sqrt{\frac{1-\alpha}{\alpha}}(\sqrt{c_1} + \sqrt{\alpha - c_1})^2 \quad \text{for} \quad \alpha_- \leq c_1 \leq \alpha_+, \tag{14}$$

and

$$\mathcal{U}(\rho) = 1 - \frac{(2c_1 - \alpha)^2}{\alpha} \quad \text{for} \quad 0 \leq c_1 \leq \alpha_- \ \text{ and } \ \alpha_+ \leq c_1 \leq \alpha. \tag{15}$$

The quantum discord quantified by local quantum uncertainty in the states ρ (1) is depicted in figure 1 for different values of α. We notice that the local quantum uncertainty is non zero except for $c_1 = c_2 = \frac{1}{4}$ with $\alpha = \frac{1}{2}$. The discord-like local quantum uncertainty goes beyond entanglement. This can be verified by means of Wootters concurrence [38] which is given by the following expression

$$\mathcal{C}_{12}(\rho) = |(\sqrt{c_1} + \sqrt{c_2})^2 - 1|. \tag{16}$$

for the two-qubit states (1). Setting $\alpha = c_1 + c_2$, it is simple to check that for $\alpha \leq \frac{1}{2}$ all two-qubit states of type (1) are entangled except the state with $c_1 = c_2 = \frac{1}{4}$. Similarly, one verifies that for $\alpha > \frac{1}{2}$, the states ρ_α are entangled except those with $(c_1, c_2) = \frac{1}{2}\big(\alpha + \sqrt{2\alpha - 1}, \alpha - \sqrt{2\alpha - 1}\big)$ and $(c_1, c_2) = \frac{1}{2}\big(\alpha - \sqrt{2\alpha - 1}, \alpha + \sqrt{2\alpha - 1}\big)$ which are separable. For this special set of separable states, the local quantum uncertainty is non-zero as it can be verified from the equations (14) and (15). The only state with zero local quantum uncertainty is the separable state with $c_1 = c_2 = \frac{1}{4}$ when $\alpha = \frac{1}{2}$.

The figure 1 shows that for $\alpha \leq \frac{1}{2}$ the local quantum uncertainty is minimal in the states with $c_1 = c_2 = \frac{\alpha}{2}$ that are given by

$$\rho\big(c_1 = \frac{\alpha}{2}, c_2 = \frac{\alpha}{2}\big) = (1-\alpha)|\psi_1\rangle\langle\psi_1| + \alpha|\psi_2\rangle\langle\psi_2| \tag{17}$$

in terms of the Bell states $|\psi_1\rangle$ and $|\psi_2\rangle$ defined by

$$|\psi_1\rangle = \tfrac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \quad , \quad |\psi_2\rangle = \tfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \tag{18}$$

On the other hand, the maximal amount of quantum correlations in the states with $\alpha \leq \frac{1}{2}$ is reached when $(c_1 = 0, c_2 = \alpha)$ or $(c_1 = \alpha, c_2 = 0)$ which are given respectively by

$$\rho(c_1 = 0, c_2 = \alpha) = \alpha|11\rangle\langle 11| + (1-\alpha)|\psi_1\rangle\langle\psi_1| \text{ or } \rho(c_1 = \alpha, c_2 = 0) = \alpha|00\rangle\langle 00| + (1-\alpha)|\psi_1\rangle\langle\psi_1|. \tag{19}$$

It is also important to stress that, for $\alpha \leq \frac{1}{2}$, states with small values of α contain more quantum correlations. This situation is completely different for $\alpha \geq \frac{1}{2}$. In fact, as depicted in the right sub-figure of Fig.1, the local quantum uncertainty increases as the parameter α increases. Also, the local quantum uncertainty starts exhibiting a minimal amount of quantum correlations in the states $(c_1 = \frac{\alpha}{2}, c_2 = \frac{\alpha}{2})$ (17) but as α increases the quantum correlations are minimal for the states $(c_1 = 0, c_2 = \alpha)$ or $(c_1 = \alpha, c_2 = 0)$ (19). This behavior can be seen from Fig.1 by comparing for instance the curves corresponding to $\alpha = 0.6$ and $\alpha = 0.9$. Similarly, the maximal amount of quantum correlations is no longer obtained for states with $(c_1 = 0, c_2 = \alpha)$ or $(c_1 = \alpha, c_2 = 0)$ like in the case where $\alpha \leq \frac{1}{2}$ (see the left sub-figure of Fig.1). For the states with $\alpha \geq \frac{1}{2}$, the maximal local quantum uncertainty is attainable when $(c_1 = \alpha_+, c_2 = \alpha_-)$ or $(c_1 = \alpha_-, c_2 = \alpha_+)$ where the quantities α_+ and α_- are given by (11). It is remarkable that in these particular states encompassing a large amount of quantum correlations, the local quantum uncertainty presents a double sudden change. This is essentially due to the jump between the eigenvalues ω_{11} and ω_{33} of the local quantum uncertainty matrix. This intriguing phenomenon has no analogue with von Neumann based quantum discord as we shall discuss in what follows.

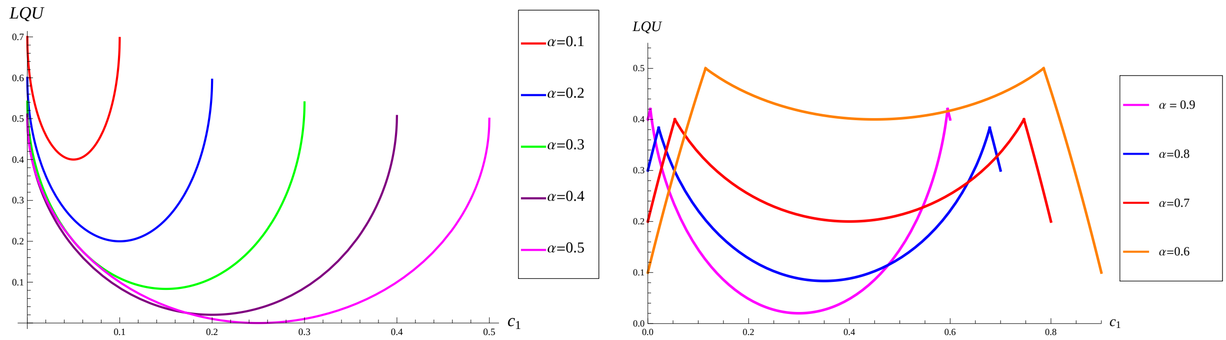

Figure 1. The Local quantum uncertainty $\mathcal{U}$ versus the parameter c_1 for different values of α.

2.2 Entropy-based quantum discord

The quantum discord in two-qubit states is defined as the difference between the mutual information and the classical correlations in a bipartite quantum system [6, 7]

$$D(\rho) = I(\rho) - C(\rho). \tag{20}$$

The total correlation is usually quantified by the mutual information I given by

$$I(\rho) = S(\rho_1) + S(\rho_2) - S(\rho), \tag{21}$$

where ρ is the state of a bipartite quantum system formed by two qubits labeled as 1 and 2, the operator $\rho_{1(2)} = \mathrm{Tr}_{1(2)}(\rho)$ is the reduced state of 1(2) and $S(\rho)$ is the von Neumann entropy of the quantum state ρ. The non vanishing eigenvalues of the density matrix ρ (1) are $\lambda_1 = c_1 + c_2$ and $\lambda_2 = 1 - (c_1 + c_2)$ and the joint entropy writes as

$$S(\rho) = H(c_1 + c_2) \tag{22}$$

where $H(x) = -x \log_2 x - (1-x)\log_2(1-x)$ is the binary entropy function. The eigenvalues of the reduced density matrices ρ_1 and ρ_2 are identical. They are given by $\frac{1}{2}(1+c_1-c_2)$ and $\frac{1}{2}(1-c_1+c_2)$ so that the marginal entropy for ρ_1 and ρ_2 are given by

$$S(\rho_1) = S(\rho_2) = H\left(\frac{1 + c_1 - c_2}{2}\right). \tag{23}$$

Reporting (22) and (23) in the definition (21), the mutual information writes as

$$I(\rho) = 2H\left(\frac{1 + c_1 - c_2}{2}\right) - H(c_1 + c_2). \tag{24}$$

To determine the classical correlations $C(\rho)$, one adopts the method developed in [39] for rank-2 two-qubit states. Indeed, this method simplifies considerably the analytical derivation of the entropic quantum discord. It consists in purifying the two-qubit system by a third qubit describing the environment and making use of the Koashi-Winter theorem [40]. This theorem constitutes the key ingredient to get the quantum discord in two-qubit systems described by density matrices of rank 2. Moreover, it establishes a nice connection between the quantum discord and the entanglement of formation (for more details see the references [41, 42, 43]). In this approach, the classical correlation $C(\rho)$ is expressed in term of the entanglement of formation between the second qubit and the third qubit representing the environment. To apply this approach for the class of states of interest in this work and to employ the Koashi-Winter theorem, we first consider the purification of the states of type (1) that we rewtite as follows

$$\rho = \lambda_1 |\Phi_1\rangle\langle\Phi_1| + \lambda_2 |\Phi_2\rangle\langle\Phi_2|$$

where λ_1 and λ_2 are the eigenvalues of ρ and $|\Phi_1\rangle$ and $|\Phi_2\rangle$ are the corresponding eigenstates:

$$|\Phi_1\rangle = \frac{\sqrt{c_1}}{\sqrt{c_1 + c_2}}|0,0\rangle + \frac{\sqrt{c_2}}{\sqrt{c_1 + c_2}}|1,1\rangle, \quad |\Phi_2\rangle = \frac{1}{\sqrt{2}}|0,1\rangle + \frac{1}{\sqrt{2}}|1,0\rangle. \tag{25}$$

Attaching a qubit 3 to the two-qubit system $1 - 2$, we write the purification of ρ as

$$|\Phi\rangle_{123} = \sqrt{\lambda_1}|\Phi_1\rangle \otimes |0\rangle_3 + \sqrt{\lambda_2}|\Phi_2\rangle \otimes |1\rangle_3 \tag{26}$$

such that the whole system 123 is described by the pure state $\rho_{123} = |\Phi\rangle_{123}\langle\Phi|$ from which one extracts the density matrix $\rho_{23} = \mathrm{Tr}_1\rho_{123}$ associated to the subsystem $2 - 3$. As we are dealing with two-qubit states of rank 2, the analytical expression of the classical correlations can be obtained by means of the Koashi-Winter theorem [40, 39, 19, 42]. Indeed, one has

$$C(\rho) = S(\rho_2) - E(\rho_{23}). \tag{27}$$

where $S(\rho_2)$ is given by (23) and the entanglement of formation $E(\rho_{23})$ is defined by

$$E(\rho_{23}) = H(\frac{1}{2} + \frac{1}{2}\sqrt{1 - |C_{23}|^2}). \tag{28}$$

Using the Wootters formula [38], the concurrence C_{23} for the bipartite state ρ_{23} writes

$$|C_{23}|^2 = 2(1 - c_1 - c_2)(\sqrt{c_1} - \sqrt{c_2})^2. \tag{29}$$

Reporting the Koashi-Winter relation (27) in the definition (20) and using the equation (21), the quantum discord in the states ρ takes the simple form

$$D(\rho) = S(\rho_1) + \ E(\rho_{23}) - S(\rho), \tag{30}$$

which can be rewritten, using the expressions (22), (23) and (28), as

$$D(\rho) = H\left(\frac{1+c_1-c_2}{2}\right) + H\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-2(1-c_1-c_2)(\sqrt{c_1}-\sqrt{c_2})^2}\right) - H(c_1+c_2), \tag{31}$$

in terms of the parameters c_1 and c_2. Setting $\alpha = c_1 + c_2$, the quantum discord in the states ρ_α, with a fixed value of α, is given by

$$D(\rho_\alpha) = H\left(\frac{1-\alpha}{2}+c_1\right) + H\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-2(1-\alpha)(\alpha-2\sqrt{c_1}\sqrt{\alpha-c_1})}\right) - H(\alpha). \tag{32}$$

Figure 2 shows that the entropic quantum discord $D(\rho_\alpha)$ exhibits similar behavior as the quantum discord based on local quantum uncertainty for $\alpha \leq \frac{1}{2}$. The minimal amount of quantum correlations is obtained in the states ρ_α with ($c_1 = c_2 = \frac{\alpha}{2}$) (17) and the maximally correlated states are the states with ($c_1 = 0, c_2 = \alpha$) and ($c_1 = \alpha, c_2 = 0$) given by (19). However, for the states ρ_α with $\alpha \geq \frac{1}{2}$, we observe that the entropic quantum discord $D(\rho_\alpha)$ and quantum discord based on local quantum uncertainty $\mathcal{U}(\rho_\alpha)$ behave differently. Indeed, for the particular values $\alpha = 0.6$ or $\alpha = 0.7$, one can see that the maximal amount of entropic quantum discord is reached for ($c_1 = 0, c_2 = \alpha$) and ($c_1 = \alpha, c_2 = 0$) contrarily to quantum correlations based on local quantum uncertainty for which the maximum is attained for the states ($c_1 = \alpha_+, c_2 = \alpha_-$) or ($c_1 = \alpha_-, c_2 = \alpha_+$) where α_+ and α_- are given by (11). Furthermore, the entropic quantum discord for the states ρ_α with $\alpha = 0.8$ or $\alpha = 0.9$ is maximal when ($c_1 = c_2 = \frac{\alpha}{2}$) while the maximum of the quantum discord based on local quantum uncertainty $\mathcal{U}(\rho_\alpha)$ is obtained when ($c_1 = \alpha_+, c_2 = \alpha_-$) or ($c_1 = \alpha_-, c_2 = \alpha_+$). Also, from figure 2, it can be inferred that the minimal value of the quantum discord $D(\rho_\alpha)$ for $\alpha = 0.8$ or $\alpha = 0.9$ is not obtained in the states with ($c_1 = 0, c_2 = \alpha$) and ($c_1 = \alpha, c_2 = 0$) as it is the case with local quantum uncertainty (see figure 1).

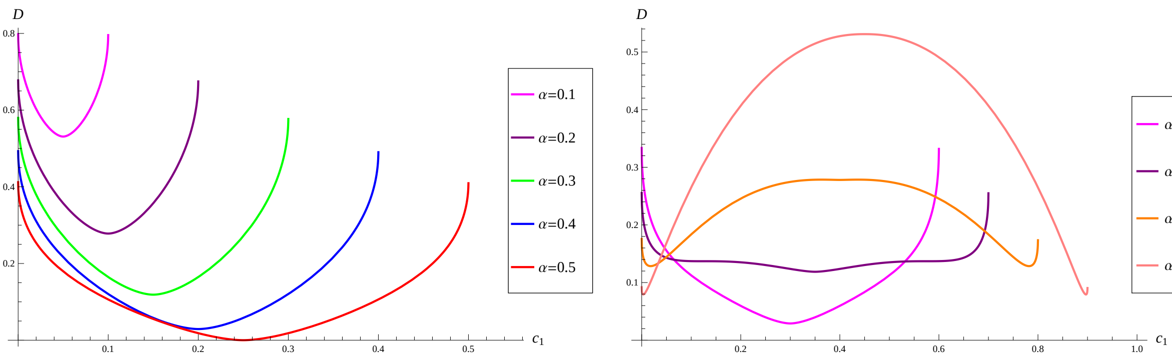

Figure 2. The quantum discord $D(\rho_\alpha)$ versus the parameter c_1 for different values of α.

2.3 Geometric quantum discord

The lack of a closed-form expression of quantum discord based on von Neumann entropy motivated the introduction of the geometric quantifiers of quantum correlations. The first geometric variant of quantum discord was introduced in [8] by means of Hilbert-Schmidt norm. However, as pointed out in [13], the Hilbert-Schmidt quantum discord cannot be considered as a good indicator of the quantumness of correlations. In fact, the Hilbert-Schmidt quantum discord can increase under local operations on unmeasured qubit [13, 44]. To overcome this drawback, the trace norm (or 1-norm) was employed as a reliable geometric quantifier of quantum discord [14]. The expressions of trace distance quantum discord have been analytically derived for general Bell-diagonal states [14, 45] and for an arbitrary two-qubit X state [46]. The trace distance quantum discord for a two-qubit state ρ is defined by

$$D_{\rm g}(\rho) = \frac{1}{2} \min_{\chi \in \Omega} ||\rho - \chi||_1, \tag{33}$$

where the trace distance is defined by $||\rho - \chi||_1 = {\rm Tr}\sqrt{(\rho - \chi)^\dagger(\rho - \chi)}$. It measures the distance between the state ρ and the classical-quantum state χ belonging to the set Ω of classical-quantum states. A generic state $\chi \in \Omega$ is of the form $\chi = \sum_k p_k \, \Pi_{k,1} \otimes \rho_{k,2}$ where $\{p_k\}$ is a probability distribution, $\Pi_{k,1}$ are the orthogonal projector associated with the qubit 1 and $\rho_{k,2}$ is density matrix associated with the second qubit. For two-qubit X states, the minimization in (33) was analytically worked out to get the explicit expression of trace distance quantum discord in [46]. Thus, for the states ρ under consideration (1), it writes as

$$D_{\rm g}(\rho) = \frac{1}{2}\sqrt{\frac{R_{11}^2 \max\{R_{33}^2, R_{22}^2 + R_{03}^2\} - R_{22}^2 \min\{R_{11}^2, R_{33}^2\}}{\max\{R_{33}^2, R_{22}^2 + R_{03}^2\} - \min\{R_{11}^2, R_{33}^2\} + R_{11}^2 - R_{22}^2}}, \tag{34}$$

where the correlation matrix elements are given by (2). Using the expressions (2), one verifies that

$$R_{22}^2 - R_{33}^2 + R_{03}^2 = 2(1 - (c_1 + c_2))(\sqrt{c_1} - \sqrt{c_2})^2,$$

and $\max\{R_{33}^2, R_{22}^2 + R_{03}^2\} = R_{22}^2 + R_{03}^2$. It follows that the geometric quantum discord (34) rewrites as

$$D_{\rm g}(\rho) = \frac{1}{2}\bigg[\Theta(|R_{33}| - |R_{11}|)\; |R_{11}| + \Theta(|R_{11}| - |R_{33}|)\; \sqrt{\frac{R_{11}^2(R_{22}^2 + R_{03}^2) - R_{22}^2 R_{33}^2}{R_{11}^2 - R_{33}^2 + R_{03}^2}}\bigg] \tag{35}$$

where $\Theta(.)$ is the usual Heaviside function. Clearly, one has to treat separately the states with $|R_{33}| \leq |R_{11}|$ and $|R_{11}| \leq |R_{33}|$. As previously, we set $c_1 + c_2 = \alpha$. It is simple to verify that for $0 \leq \alpha \leq \frac{2}{3}$, $|R_{11}|$ is always larger than $|R_{33}|$. For the situation where $\frac{2}{3} \leq \alpha \leq 1$, the two-qubit states ρ_α satisfy the conditions $|R_{33}| \leq |R_{11}|$ when $c_1 \in [c_-, c_+]$ and $|R_{11}| \leq |R_{33}|$ when $c_1 \in [0, c_-] \cup [c_+, \alpha]$ with

$$c_\pm = \frac{\alpha}{2} \pm \sqrt{(1 - \alpha)(2\alpha - 1)}. \tag{36}$$

are the solutions of the equation $|R_{11}| = |R_{33}|$ for a fixed value of α. As by-product, the expression (35) becomes

$$D_{\rm g}(\rho_\alpha) = \frac{1}{2}\sqrt{\Big(1 - (\sqrt{c_1} + \sqrt{\alpha - c_1})^2\Big)^2 + 4\sqrt{c_1(\alpha - c_1)}(\sqrt{c_1} - \sqrt{\alpha - c_1})^2} \quad \text{for} \quad 0 \leq \alpha \leq \frac{2}{3} \tag{37}$$

and for the states ρ_α with $\frac{2}{3} \leq \alpha \leq 1$, one gets

$$D_{\rm g}(\rho_\alpha) = \begin{cases} \frac{1}{2}\Big(1 - (\sqrt{c_1} - \sqrt{\alpha - c_1})^2\Big) & \text{for} \;\; 0 \leq c_1 \leq c_- \\ \frac{1}{2}\sqrt{\Big(1 - (\sqrt{c_1} + \sqrt{\alpha - c_1})^2\Big)^2 + 4\sqrt{c_1(\alpha - c_1)}(\sqrt{c_1} - \sqrt{\alpha - c_1})^2} & \text{for} \;\; c_- \leq c_1 \leq c_+ \\ \frac{1}{2}\Big(1 - (\sqrt{c_1} - \sqrt{\alpha - c_1})^2\Big) & \text{for} \;\; c_+ \leq c_1 \leq \alpha \end{cases} \tag{38}$$

In figure 3, the 1-norm geometric quantum discord shows a quasi similar behavior as the quantum discord based on local quantum uncertainty plotted in figure 1. Indeed, for $\alpha < \frac{1}{2}$ the trace distance quantum correlations is maximal in the states ρ_α with $(c_1 = 0, c_2 = \alpha)$ and $(c_1 = \alpha, c_2 = 0)$ given by (19). Also, the amount of quantum correlations is minimal in the state with $(c_1 = c_2 = \frac{\alpha}{2})$ given by (17). This agrees with results obtained with the local quantum uncertainty. The behaviors of these two quantifiers become sensibly different for $\alpha \geq \frac{1}{2}$. Indeed, for $\alpha = \frac{1}{2}$ the variation of the geometric discord, with respect to the parameter c_1, is quasi linear. The geometric discord exhibits also a double sudden change when $(c_1 = c_+, c_2 = c_-)$ and $(c_1 = c_-, c_2 = c_+)$ ($c_\pm$ are given by (36)) which are different from the two-qubit states corresponding to $(c_1 = \alpha_+, c_2 = \alpha_-)$ and $(c_1 = \alpha_-, c_2 = \alpha_+)$ ($\alpha_\pm$ are given by (11)) where the double sudden change occurs with the local quantum uncertainty. This dissimilitude poses a serious challenge and particularly when one needs to employ the sudden change of quantum correlations in a multipartite system to understand the quantum phase transitions.

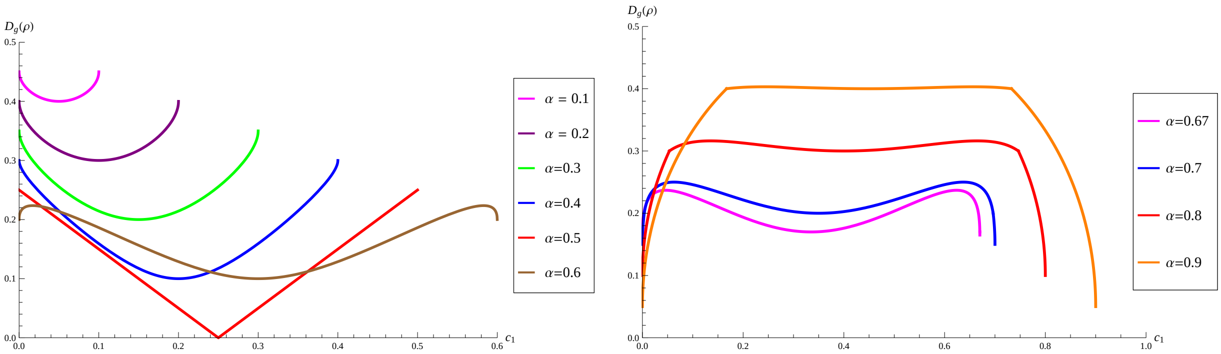

Figure 3. The geometric quantum discord $D_{\mathrm{g}}(\rho_\alpha)$ versus the parameter c_1 for $\alpha \leq \frac{2}{3}$ and $\alpha \geq \frac{2}{3}$.

3 Local quantum uncertainty under decoherence

Quantum correlations dynamics under decohering effects has received a great deal of attention and various decoherence scenarios (Markovian or non-Markovian) were investigated [47]-[54]. In particular, it has been shown that the entanglement suffers from sudden death [55]-[60] and the entropic quantum discord is more robust than entanglement [61]. In fact, when a two-qubit state is under the influence of a local noisy environment, the entanglement can suddenly disappear while the quantum discord shows more resilience against the decoherence effects. Dynamics of trace distance quantum discord was also studied for some two-qubit states. In particular, it has been shown that this quantumness indicator exhibits in Bell diagonal states the so-called freezing phenomenon; the quantum correlations remain constant during the evolution of the system [62]. In this section, we focus on the dynamics of quantum discord quantified by local quantum uncertainty. To simplify our purpose, we restrict the set two-qubit states given (1) to ones of Bell type by setting $c_1 = c_2 = c$. They are given by

$$\rho(c_1 = c_2 = c) = \frac{1}{4}\Big(\sigma_0 \otimes \sigma_0 + \sigma_1 \otimes \sigma_1 + (1 - 4c)\sigma_2 \otimes \sigma_2 - \sigma_3 \otimes \sigma_3\Big) \tag{39}$$

where $0 \leq c \leq \frac{1}{2}$. For open quantum systems, the Markovian dynamics can be entirely specified by a quantum channel $\mathcal{E} : \rho \longrightarrow \mathcal{E}(\rho)$ whose action on the state can be completely characterized as follows

$$\mathcal{E}(\rho) = \sum_{ij} (E_i \otimes E_j)\rho(E_i \otimes E_j)^\dagger \tag{40}$$

where E_i denotes the Kraus operators describing the decohering process of a single qubit. The Kraus operators satisfy the trace-preserving condition $\sum_i (E_i)^\dagger E_i = \mathbb{I}$. For several decoherence scenarios, the action of the decoherence channel is in general parameterized by a time dependent decoherence probability p . In what follows, we will consider the dynamics of the local quantum uncertainty in the states (39) for certain noise channels (i.e., phase flip, bit flip, and bit-phase flip and generalized amplitude damping)

3.1 The depolarizing quantum channel

The depolarizing channel is a decohering process used to describe three different types of errors: (i) bit flip error, (ii) phase flip error or (iii) both [1].

(i) Bit flip error: For bit flip quantum channel, the Kraus operators are

$$E_0 = \sqrt{1-p/2}, \quad \sigma_0 \qquad E_1 = \sqrt{p/2}\, \sigma_1. \tag{41}$$

Under the local action of the bit flip channel, the density matrix (39) evolves as

$$\rho_{\rm BF} = \frac{1}{4}\big(\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} R_{ii}^{\rm BF} \sigma_i \otimes \sigma_i\big) \tag{42}$$

where

$$R_{11}^{\rm BF} = 1, \qquad R_{22}^{\rm BF} = (1-4c)(1-p)^2, \qquad R_{33}^{\rm BF} = (4c-1)(1-p)^2.$$

the explicit derivation of the local quantum uncertainty in the states (42) requires the square root of the density matrix 42 and the expressions of the matrix elements ω_{ij} given by (6). Lengthy but straightforward calculation gives

$$\omega_{11}^{\rm BF} = \sqrt{1-(1-p)^4(1-4c)^2}, \qquad \omega_{22}^{\rm BF} = 0, \qquad \omega_{33}^{\rm BF} = 0, \tag{43}$$

Clearly, for $p = 0$, one recovers the results (8) with $c_1 = c_2 = c$. The local quantum uncertainty is then given by

$$\mathcal{U}(\rho_{\rm BF}) = 1 - \sqrt{1-(1-p)^4(1-4c)^2}. \tag{44}$$

This reduces, for $p = 0$, to $\mathcal{U}(\rho_{\rm BF}) = 1 - \sqrt{1-(1-4c)^2}$ which can be obtained from the equations (13) and (14). In the asymptotic limit $p \longrightarrow 1$, the quantum correlations are completely transferred to the environment.

(ii) Phase flip error : The phase flip channel describes the quantum noise process with loss of quantum information without loss of energy. In the operator-sum representation formalism, the Kraus operators for single qubit phase flip write

$$E_0 = \sqrt{1-p/2}\sigma_0, \qquad E_1 = \sqrt{p/2}\sigma_3. \tag{45}$$

Under phase flip channel, the evolved quantum state writes as

$$\rho_{\rm PF} = \frac{1}{4}\big(\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} R_{ii}^{\rm PF} \sigma_i \otimes \sigma_i\big) \tag{46}$$

where the correlation elements are given by

$$R_{11}^{\rm PF} = (1-p)^2 \quad R_{22}^{\rm PF} = (1-4c)(1-p)^2 \quad R_{33}^{\rm PF} = (4c-1).$$

In this decohering scenario, the matrix elements (6) take the form

$$\omega_{11}^{\rm PF} = 2\sqrt{2}\sqrt{c(1-2c)} \qquad \omega_{22}^{\rm PF} = 2\sqrt{2}\sqrt{c(1-2c)}\sqrt{1-(1-p)^4} \qquad \omega_{33}^{\rm PF} = \sqrt{1-(1-p)^4}. \tag{47}$$

We note that $\omega_{22}^{\rm PF} \leq \omega_{11}^{\rm PF}$ and $\omega_{22}^{\rm PF} \leq \omega_{33}^{\rm PF}$. This implies that $\omega_{\max}^{\rm PF}$ is given by $\omega_{11}^{\rm PF}$ or $\omega_{33}^{\rm PF}$. For a given value of c, the condition $\omega_{11}^{\rm PF} \geq \omega_{33}^{\rm PF}$ is satisfied when the probability p is such that $0 \leq p \leq 1 - \sqrt{|4c-1|}$. It is remarkable that in this case, the local quantum uncertainty,

$$\mathcal{U}(\rho_{\rm PF}) = 1 - 2\sqrt{2}\sqrt{c(1-2c)}, \tag{48}$$

remains constant (i.e, time independent). In this interval, the quantum correlations are unaffected by the noisy environment and the local quantum uncertainty exhibits a freezing behavior. This reflects the fact that the local quantum uncertainty is robust against the phase flip errors. This freezing behavior is followed by a sudden change at the critical point $p_c = 1 - \sqrt{|4c-1|}$. Hence for $1 - \sqrt{|4c-1|} \leq p \leq 1$, the local quantum uncertainty is given by

$$\mathcal{U}(\rho_{\rm PF}) = 1 - \sqrt{1-(1-p)^4} \tag{49}$$

and decays monotonically to disappear completely when $p \longrightarrow 1$.

(iii) Bit-phase flip error: The corresponding Kraus operators are given by

$$E_0 = \sqrt{1-p/2}\,\sigma_0 \qquad E_1 = \sqrt{p/2}\,\sigma_2, \tag{50}$$

and their action on a state of type (39) leads to the following density matrix

$$\rho_{\rm BPF} = \frac{1}{4}\big(\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} R_{ii}^{\rm BPF} \sigma_i \otimes \sigma_i\big) \tag{51}$$

where the Fano-Bloch components are

$$R_{11}^{\rm BPF} = (1-p)^2, \quad R_{22}^{\rm BPF} = (1-4c), \quad R_{33}^{\rm BPF} = (4c-1)(1-p)^2.$$

The square root of the state $\rho_{\rm BPF}$ can be esaily calcualted and from the expressions of the matrix elements (6) one obtains

$$\omega_{11}^{\rm BPF} = 2\sqrt{2}\sqrt{c(1-2c)}, \qquad \omega_{22}^{\rm BPF} = \sqrt{1-(1-p)^4}, \qquad \omega_{33}^{\rm BPF} = 2\sqrt{2}\sqrt{c(1-2c)}\sqrt{1-(1-p)^4}. \tag{52}$$

The local quantum uncertainty can be derived similarly to the phase flip process by exchanging $\omega_{22}^{\rm PF}$ by $\omega_{33}^{\rm BPF}$ and $\omega_{33}^{\rm PF}$ by $\omega_{22}^{\rm BPF}$. This gives

$$\mathcal{U}(\rho_{\rm BPF}) = 1 - 2\sqrt{2}\sqrt{c(1-2c)} \quad \text{for} \quad 0 \leq p \leq p_c \tag{53}$$

and

$$\mathcal{U}(\rho_{\rm BPF}) = 1 - \sqrt{1-(1-p)^4} \quad \text{for} \quad p_c \leq p \leq 1, \tag{54}$$

with $p_c = 1 - \sqrt{|4c-1|}$. As with the phase flip channel, the local quantum uncertainty exhibits also a freezing behavior in the interval $[0, p_c]$. This behavior is essentially due to the phase flip errors since when the bit flip error alone acts on the system, the local quantum uncertainty is monotonically decreasing. Furthermore, it is remarkable that for both phase flip and Bit-phase flip, the freezing phenomenon occurs during the same period. To investigate the duration of local quantum uncertainty freezing, we consider the variation of the critical point p_c with respect to the parameter c. We treat separately the situations where $0 \leq c \leq \frac{1}{4}$ and $\frac{1}{4} \leq c \leq \frac{1}{2}$. For

$0 \le c \le \frac{1}{4}$, the critical point p_c increases as the parameter c increases. As it can be verified from the equations (48) and (53), increasing the parameter c, to get a large freezing interval, is accompanied by a diminution of the amount of quantum correlations in the system. Similarly, for the states with $\frac{1}{4} \le c \le \frac{1}{2}$, large freezing intervals are also obtained for states with less quantum correlations.

3.2 Generalized amplitude damping

Now we consider the dynamics of the states (39) under the effect of an amplitude-damping channel which describes the dissipative interaction between the system and the environment. This process is modeled by treating the environment as a large collection of independent harmonic oscillators interacting weakly with the system. In the operator-sum representation formalism, the evolution of the system is described by the following four Kraus operators

$$E_0 = \frac{\sqrt{p}}{2}\big[(1+\sqrt{1-\gamma})\sigma_0 + (1-\sqrt{1-\gamma})\sigma_3\big], \quad E_1 = \sqrt{p\gamma}\ \sigma_+,$$
$$E_2 = \frac{\sqrt{1-p}}{2}\big[(\sqrt{1-\gamma}+1)\sigma_0 + (\sqrt{1-\gamma}-1)\sigma_3\big], \quad E_3 = \sqrt{(1-p)\gamma}\ \sigma_- \tag{55}$$

where $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$, p and γ are the decoherence probabilities [1]. To simplify the calculation of the local quantum uncertainty, we fix $p = \frac{1}{2}$. In this special situation, the states (39) evolve as

$$\rho_{\rm GAD} = \frac{1}{4}\big(\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} R_{ii}^{\rm GAD}\sigma_i \otimes \sigma_i\big) \tag{56}$$

where

$$R_{11}^{\rm GAD} = (1-\gamma), \quad R_{22}^{\rm GAD} = (1-4c)(1-\gamma), \quad R_{33}^{\rm GAD} = (4c-1)(1-\gamma)^2.$$

After some lengthy but feasible algebraic manipulations of the matrix elements (6), one gets

$$\omega_{11}^{\rm GAD} = \sqrt{1-(1-\gamma)^2(1-4c)^2}, \quad \omega_{22}^{\rm GAD} = \sqrt{\gamma(2-\gamma)}, \quad \omega_{33}^{\rm GAD} = \sqrt{\gamma(2-\gamma)}\sqrt{1-(1-\gamma)^2(1-4c)^2}. \tag{57}$$

In this case, we have $\omega_{33}^{\rm GAD} \le \omega_{22}^{\rm GAD} \le \omega_{11}^{\rm GAD}$ and the local quantum uncertainty is given by

$$\mathcal{U}(\rho_{\rm GAD}) = 1 - \sqrt{1-(1-\gamma)^2(1-4c)^2}. \tag{58}$$

The quantum discord disappears in the asymptotic regime ($p \longrightarrow 1$). No freezing behavior can be observed under this decohering process contrarily to phase flip or bit-phase flip channels which destroy the information encoded in the phase relations without any exchange of energy.

4 Concluding remarks

The local quantum uncertainty constitutes presumably an efficient tool to characterize the quantum correlations in bipartite quantum systems. This is mainly due to its reliability and easiness to use from a computational viewpoint. In this spirit, we have derived the quantum discord for a special class of two-qubit states by employing the formalism of local quantum uncertainty. It has been shown that this indicator of quantumness might exhibits a double sudden change in some particular circumstances. We believe that this behavior can relevant in investigating the role of quantum correlations in quantum phase transitions.

We have also derived, for this class of two-qubit states, the expressions of the quantum discord based on von Neumann entropy and the geometric quantum discord defined by means of trace distance (Schatten one-norm). The amount of quantum correlations measured with these three kinds of the quantum discords are compared. Despite many concordances, they are not only quantitatively but also qualitatively different (see the right sub-figures in figures 1, 2 and 3). To focus exclusively on the singular behaviors, the comparison revealed that the local quantum uncertainty and the trace distance quantum discords exhibits double sudden change but for different critical points. Such behavior cannot occur for the entropic quantum discord. We have also notified some differences between the amount of quantum correlations quantified by local quantum uncertainty and entropic quantum discord (see the right sub-figures in figures 1 and 2).

Quantum systems are inevitably subject to decoherence effects, hence it is crucial to find the situations where the quantum correlations are not affected by environmental noises during their evolutions. In addressing this issue for a special class of Bell-diagonal states subjected to depolarizing channels, we have observed the freezing phenomenon of discord-like local quantum uncertainty which originates from the robustness of quantum discord [61]. We note that a similar freezing phenomenon has been shown for one-norm quantum discord [14, 62]. To guarantee a durable physical exploitation of the coherence, it is necessary to increase the freezing duration. This can be achieved but with two-qubit states containing less amount of quantum correlations. Thus, the price to pay for long-time freezing is the missing of quantum correlations.

References

[1] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge Univ. Press, Cambridge, 2000).

[2] V. Vedral, Rev. Mod. Phys. **74** (2002) 197.

[3] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Rev. Mod. Phys. **81** (2009) 865.

[4] O. Gühne and G. Tóth, Phys. Rep. **474** (2009) 1.

[5] K. Modi, A. Brodutch, H. Cable, T. Paterek and V. Vedral, Rev. Mod. Phys. **84** (2012) 1655.

[6] H. Ollivier and W. H. Zurek, Phys. Rev. Lett. **88** (2001) 017901.

[7] L. Henderson and V. Vedral, J. Phys. A **34** (2001) 6899.

[8] B. Dakić, V. Vedral, and Č. Brukner, Phys. Rev. Lett. **105** (2010) 190502

[9] B. Bellomo, R. Lo Franco and G. Compagno, Phys. Rev. A **86** (2012) 012312.

[10] B. Bellomo, G.L. Giorgi, F. Galve, R. Lo Franco, G. Compagno and R. Zambrini, Phys. Rev. A **85** (2012) 032104.

[11] M. Daoud and R. Ahl Laamara, Phys. Lett. A **376** (2012) 2361.

[12] M. Daoud and R. Ahl Laamara, Int. J. Quantum Inf. **10** (2012) 1250060.

[13] M. Piani, Phys. Rev. A **86** (2012) 034101.

[14] F. M. Paula, T. R. de Oliveira, and M. S. Sarandy, Phys. Rev. A **87** (2013) 064101; F.M. Paula, J.D. Montealegre, A. Saguia, T.R. de Oliveira and M.S. Sarandy, EPL, **103** (2013) 50008.

[15] T.R. Bromley, M. Cianciaruso, R. Lo Franco, G. Adesso, J. Phys. A: Math. Theor. **47** (2014) 405302

[16] F. Ciccarello, T. Tufarelli and V. Giovannetti, New J. Phys. **16** (2014) 013038.

[17] E. P. Wigner and M. M. Yanasse, Proc. Nat. Acad. Sci. USA **49** (1963) 910.

[18] D. Girolami, T. Tufarelli, and G. Adesso, Phys. Rev. Lett. **110** (2013) 240402.

[19] S. Luo, Phys. Rev. Lett. **91** (2003) 180403.

[20] D. Petz and C. Ghinea in *Quantum Probability and Related Topics*, edited by R. Rebolledo and M. Orszag, World Scientific, 2011.

[21] S. Luo, Proc. Amer. Math. Soc. **132** (2003) 885.

[22] D. Girolami, T. Tufarelli and G. Adesso, Phys. Rev. Lett. **110** (2013) 240402.

[23] R. H. Dicke, Phys. Rev. **93** (1954) 99.

[24] H. J. Lipkin, N. Meshkov and A. J. Glick, Nucl. Phys. **62** (1965) 188.

[25] L. Amico, R. Fazio, A. Osterloh and V. Vedral, Rev. Mod. Phys. **80** (2008) 517 .

[26] T. J. Osborne and M.A. Nielsen, Phys. Rev. A **66** (2002) 032110 .

[27] R. Dillenschneider, Phys. Rev. B **78** (2008) 224413

[28] M.S. Sarandy, Phys. Rev. A **80** (2009) 022108

[29] J. Maziero, L. C. Céleri, R. M. Serra and M. Sarandy, Phys. Lett. A **376** (2012) 1540 .

[30] T. Werlang and G. Rigolin, Phys. Rev. A **81** (2010) 044101.

[31] Chen Wang, Yu-Yu Zhang and Qing-Hu Chen, Phys. Rev. A **85** (2012) 052112

[32] R.F. Werner, Phys. Rev. A **40** (1989) 4277.

[33] M. Horodecki and P. Horodecki, Phys. Rev. A **59** (1999) 4206.

[34] T. Yu and J.H. Eberly, Quant. Inf. Comput. **7** (2007) 459.

[35] A.R.P. Rau, J. Phys. A: Math. Theor. **42** (2009) 412002

[36] L. Jakobczyk and A. Jamroz, Phys. Lett. A **333** (2004) 35; M. Franca Santos, P. Milman, L. Davidovich and N. Zagury, Phys. Rev. A **73** (2006) 040305; M. Ikram, F.L. Li and M. S. Zubairy, Phys. Rev. A **75** (2007) 062336; A. Al-Qasimi and D.F.V. James, Phys. Rev. A **77** (2007) 012117.

[37] A.R.P. Rau, M. Ali and G. Alber, EPL **82** (2008) 40002; X. Cao and H. Zheng, Phys. Rev. A **77** (2008) 022320; C.E. Lopez, G. Romero, F. Castra, E. Solano and J.C. Retamal, Phys. Rev. Lett. **101** (2008) 080503; M. Ali, G. Alber and A.R.P. Rau, J. Phys. B: At. Mol. Opt. Phys. **42** (2009) 025501.

[38] W. K. Wootters, Phys. Rev. Lett. **80** (1998) 2245.

[39] M. Shi, W. Yang, F. Jiang and J. Du, J. Phys. A: Math. Theor. **44** (2011) 415304.

[40] M. Koashi and A. Winter, Phys. Rev. A **69** (2004) 022309.

[41] S. Luo, Phys. Rev. A **77** (2008) 042303.

[42] F.F. Fanchini, M.C. de Oliveira, L.K. Castelano and M.F. Cornelio, Phys. Rev. A **87** (2013) 032317.

[43] M. Daoud and R. Ahl Laamara, J. Phys. A: Math. Theor. **45** (2012) 325302.

[44] X. Hu, H. Fan, D. L. Zhou and W.-M. Liu, Phys. Rev. A **87** (2013) 032340.

[45] T. Nakano, M. Piani and G. Adesso, Phys. Rev. A **88** (2013) 012117.

[46] F. Ciccarello, T. Tufarelli and V. Giovannetti, New Journal of Physics **16** (2014) 013038.

[47] L. Mazzola, J. Piilo and S. Maniscalco, Phys. Rev. Lett. **104** (2010) 200401.

[48] J. Maziero, T. Werlang , F.F. Fanchini, L.C. Céleri and R. M. Serra, Phys. Rev. A **81** (2010) 022116.

[49] J. Maziero, L.C. Céleri, R.M. Serra and V. Vedral, Phys. Rev. A **80** (2009) 044102.

[50] T. Werlang, S. Souza, F.F. Fanchini and C.J. Villas Boas, Phys. Rev. A **80** (2009) 024103.

[51] B. Wang, Z.-Y Xu, Z.-Q. Chen and M. Feng, Phys. Rev. A **81** (2010) 014101.

[52] F.F. Fanchini, T. Werlang, C.A. Brasil, L.G.E. Arruda and A.O. Caldeira, Phys. Rev. A **81** (2010) 052107.

[53] J.-S. Xu, X.-Y. Xu, C.-F. Li, C.-J. Zhang, X.-B. Zou and G.-C Guo, Nat. Commun. **1** (2010) 7.

[54] A. Ferraro, L. Aolita, D. Cavalcanti, F.M. Cucchietti and A. Acìn, Phys. Rev. A **81** (2010) 052318.

[55] T. Yu and J. H. Eberly, Phys. Rev. Lett. **93** (2004) 140404.

[56] J.H. Eberly and T. Yu, Science **316** (2007) 555.

[57] T. Yu and J.H. Eberly, Science **323** (2009) 598.

[58] T. Yu and J.H. Eberly, J. Mod. Opt. **54** (2007) 2289.

[59] D. Zhou and R. Joynt, Phys. Rev. A **81** (2010) 010103(R).

[60] D. Zhou, G.-W. Chern, J. Fei, and R. Joynt, Int. J. Mod. Phys. B **26** (2012) 1250054.

[61] T. Werlang, S. Souza, F.F. Fanchini, and C.J. Villas Boas, Phys. Rev. A **80** (2009) 024103.

[62] M. Cianciaruso, T.R. Bromley, W. Roga, R. Lo Franco and G. Adesso, Scientific Reports **5** (2015) 10177