Quantum correlations for two-qubit X states through the local quantum uncertainty

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Abstract

Local quantum uncertainty is defined as the minimum amount of uncertainty in measuring a local observable for a bipartite state. It provides a well-defined measure of pairwise quantum correlations in quantum systems and has operational significance in quantum metrology. In this work, we analytically derive the expression of local quantum uncertainty for two-qubit X states which are of paramount importance in various fields of quantum information. As an illustration, we consider two-qubit states extracted from even and odd spin coherent states.

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1 Introduction

Quantum correlations in multipartite systems are a fundamental resource in various protocols of quantum information processing [1, 2, 3, 4]. In this respect, the characterization of the degree of quantumness of correlations between the different parts of a composite system is highly desirable. During the last two decades, several quantifiers were investigated in the literature (for a recent review see [5]). The most familiar ones are the concepts of concurrence, entanglement of formation, quantum discord and its different geometric versions [6, 7, 8, 9, 10, 11]. The interest in quantum discord lies in the existence of nonclassical correlations even in separable states which are not captured by the entanglement [10, 11]. This explains the particular interest and the huge amount of efforts dedicated to the significance and the computation of quantum discord in different quantum systems. However, the derivation of the explicit expression of quantum discord, based on von Neumann entropy, of an arbitrary bipartite quantum system is in general very challenging. Quantum discord can be computed only for some special instances of two-qubit states.

An alternative way to overcome this problem consists in adopting geometric tools to quantify the distance between a bipartite state and its closed one exhibiting only classical correlations [12, 13, 14] (see also [15, 16, 17, 18]). Two variants of geometric quantum discord were introduced in terms of Schatten p-norm: trace norm ($p = 1$) and Hilbert-Schmidt norm ($p = 2$). The geometric quantum discord based on Hilbert-Schmidt norm is contractive under local operations by the unmeasured party and therefore cannot be employed as a faithful quantifier of quantum correlations [19, 20]. The explicit analytical expression of trace discord for two-qubit X states was reported in [21]. The derivation involves lengthy algebraic manipulations and one should recognize that the computability of this quantifier for higher dimensional quantum systems (qudits) is drastically difficult.

Besides these entropic and geometric quantifiers, the notion of local quantum uncertainty, recently reported in [22], constitutes a promising tool in investigating quantum correlations in multipartite systems. This is essentially due to its reliability and its easiness of computability. This quantifier employs the formalism of skew information, introduced in Ref. [23], which determines the uncertainty in the measurement of an observable. More precisely, the local quantum uncertainty is given by the minimum of the skew information over all possible local observables acting on one party of a bipartite system. This minimization can be analytically worked out for any qubit-qudit bipartite system [22]. The local quantum uncertainty is related to the notion of quantum Fisher information [24, 25, 26] and in this sense it is a key ingredient of paramount importance in quantum metrology protocols [22].

In this paper, we give the explicit analytical expression of local quantum uncertainty for a generic family of two qubit X states. This completes the recently obtained results for Bell diagonal, Werner and isotropic states [27, 28]. To illustrate our results, we consider the pairwise quantum correlation in spin coherent states viewed as multipartite symmetric qubit states.

The paper is structured as follows. In section 2, we give a brief review of the concept of local quantum uncertainty. The analytical expression of this discord-like quantifier of quantum correlations is derived for an arbitrary two-qubit X state. We discuss some special two-qubit X states, especially Bell diagonal states and orthogonally invariant two-qubit states for which the local quantum uncertainty was recently derived in [27, 28]. In section 3, we consider the pairwise local quantum uncertainty in balanced superpositions of spin coherent states. We consider two partitioning schemes. The first bipartition lies on the factorization property of SU(2) coherent states. In this picture a j-spin coherent state factorizes as a product of $2j$ identical qubit states ($\frac{1}{2}$-spin coherent states). The second scheme is obtained by a trace procedure over the degree of freedom of $2j-2$ qubits. A special focus is dedicated to even and odd spin coherent states. Concluding remarks close this paper.

2 Local quantum uncertainty in two qubit X states

Closed analytical expression of local quantum uncertainty has been worked out only for certain class of two-qubit states [27, 28, 29]. In this section, we take a step forward and give the method for tackling the calculation of local quantum uncertainty for a generic family of two-qubit X-state which includes diagonal Bell states, Werner states [30] and many others of relevance in several areas of quantum information.

2.1 Local quantum uncertainty: definition

In quantum mechanics, the uncertainty of an observable H in a quantum state ρ is quantified by the variance as

$$\mathcal{V}(\rho, H) = \mathrm{Tr}(\rho H^2) - (\mathrm{Tr}\rho H)^2.$$

For pure states, the variance is of purely quantum nature. But, for mixed states it comprises both classical and quantum contributions. The discrimination between classical and quantum parts is of paramount importance in quantum information theory. In this sense, to deal only with the quantum part of the variance, one employs the formalism of skew information defined as [23, 24]

$$\mathcal{I}(\rho, K) = \mathrm{Tr}(\rho H^2) - \mathrm{Tr}(\sqrt{\rho} H \sqrt{\rho} H).$$

It expresses the information contained in the state ρ that is unaccessible by measuring the observable H. The skew information vanishes only and only when ρ and H commute. The difference $\mathcal{C}(\rho, K) = \mathcal{V}(\rho, K) - \mathcal{I}(\rho, K)$ has the meaning of classical mixing uncertainty. The disentanglement of the variance into classical and quantum parts is behind the relevance of the skew information in quantifying non classical correlations. Indeed, when the state $\rho = \rho_{12}$ describes a two-qubit system and $H = H_1 \otimes \mathbb{I}_2$ is a local observable acting only on the first qubit, the lower bound of the skew information leads to nonclassical correlations of the discord type [22]. In fact, quantum discord quantifies the amount of information in a bipartite system which is accessible by performing local measurements on one part of the global system. In this sense, the local quantum uncertainty is defined by the minimization of the skew information over local observables with fixed non-degenerate spectrum [22]

$$\mathcal{U}(\rho_{12}) \equiv \min_{H_1} \mathcal{I}(\rho_{12}, H_1 \otimes \mathbb{I}_2), \tag{1}$$

The properties, reliability and computability of this discord-like quantifier were reported in [22]. Indeed, the local quantum uncertainty vanishes for the so-called classical-quantum states of the form $\rho_{12} = \sum_i p_i |i\rangle_1\langle i| \otimes \rho_2$, where $\{|i\rangle\}$ is an orthonormal basis. Furthermore, this measure possesses the invariance property under local unitary transformations and does not increase under local quantum transformations on the unmeasured subsystem. In this sense, the local quantum uncertainty provides a reliable discord-like measure. The explicit calculation of this quantum correlations indicator was reported in [22] for a $2 \times d$ bipartite system (qubit-qudit system). In particular, for a two qubit system (spin-$\frac{1}{2}$ particles), the local quantum uncertainty writes [22]

$$\mathcal{U}(\rho_{12}) = 1 - \max(\omega_1, \omega_2, \omega_3), \tag{2}$$

where $\omega_i \ \ (i = 1, 2, 3)$ denote the eigenvalues of the 3×3 matrix W whose matrix elements are given by

$$\omega_{ij} \equiv \mathrm{Tr}(\sqrt{\rho_{12}}\ \sigma_i \otimes \sigma_0\ \sqrt{\rho_{12}}\ \sigma_j \otimes \sigma_0), \tag{3}$$

where σ_0 stands for the identity matrix $\mathbb{I}_2$ and $i, j = 1, 2, 3$. The matrices σ_i $(i = 1, 2, 3)$ are the usual Pauli matrices. The explicit derivation of the local quantum uncertainty (2) gets simplified for two-qubit density matrices with symmetries invariance [22, 27, 28]. Therefore, we shall focus on the local quantum uncertainty for X states which include various types of quantum states usually used in investigating entanglement and quantum correlations in various condensed matter models such ones describing spin collective systems.

2.2 Local quantum uncertainty for X states

In the computational basis of the Hilbert space associated with a two qubit system, the X density matrix have non-zero entries only along the diagonal and anti-diagonal and therefore they are parameterized by seven real parameters [31, 32]. The corresponding symmetry is fully characterized by the $su(2) \times su(2) \times u(1)$ subalgebra of the full $su(4)$ algebra describing an arbitrary two-qubit system [33]. The X states have already found applications in several investigations of concurrence, entanglement of formation, quantum discord [32, 34, 35, 36]. The density matrix for a two-qubit X state writes as

$$\rho = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ \rho_{41} & 0 & 0 & \rho_{44} \end{pmatrix}. \tag{4}$$

in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. The entries are subjected to the normalization property ($\mathrm{Tr}\rho = 1$), the positivity condition ($\rho_{11}\rho_{44} \geq |\rho_{14}|^2$ and $\rho_{22}\rho_{33} \geq |\rho_{23}|^2$) and the complex conjugation requirement ($\rho_{14} = \overline{\rho_{41}}$ and $\rho_{23} = \overline{\rho_{32}}$). The phase factors $e^{i\theta_{14}} = \frac{\rho_{14}}{|\rho_{14}|}$ and $e^{i\theta_{23}} = \frac{\rho_{23}}{|\rho_{23}|}$ of the off diagonal elements can be removed using the local unitary transformations

$$|0\rangle_1 \to \exp\bigg(-\frac{i}{2}(\theta_{14} + \theta_{23})\bigg)|0\rangle_1 \quad |0\rangle_2 \to \exp\bigg(-\frac{i}{2}(\theta_{14} - \theta_{23})\bigg)|0\rangle_2.$$

Hence, the anti-diagonal entries of the density matrix can be made positive. Hereafter, we assume that the elements of the density matrix are non negative. The eigenvalues of the density matrix ρ write

$$\lambda_1 = \frac{1}{2}t_1 + \frac{1}{2}\sqrt{t_1^2 - 4d_1}, \quad \lambda_2 = \frac{1}{2}t_2 + \frac{1}{2}\sqrt{t_2^2 - 4d_2}, \quad \lambda_3 = \frac{1}{2}t_2 - \frac{1}{2}\sqrt{t_2^2 - 4d_2}, \quad \lambda_4 = \frac{1}{2}t_1 - \frac{1}{2}\sqrt{t_1^2 - 4d_1}$$

with $t_1 = \rho_{11} + \rho_{44}$, $t_2 = \rho_{22} + \rho_{33}$, $d_1 = \rho_{11}\rho_{44} - \rho_{14}\rho_{41} = \rho_{11}\rho_{44} - \rho_{14}^2$, and $d_2 = \rho_{22}\rho_{33} - \rho_{32}\rho_{23} = \rho_{22}\rho_{33} - \rho_{32}^2$. The Fano-Bloch decomposition of the state ρ writes as

$$\rho = \frac{1}{4} \sum_{\alpha,\beta} R_{\alpha\beta} \sigma_\alpha \otimes \sigma_\beta \tag{5}$$

where the correlation matrix $R_{\alpha\beta}$ are given by $R_{\alpha\beta} = \mathrm{Tr}(\rho\, \sigma_\alpha \otimes \sigma_\beta)$ with $\alpha, \beta = 0, 1, 2, 3$. Explicitly, they write

$$R_{03} = 1 - 2\rho_{22} - 2\rho_{44}, \quad R_{30} = 1 - 2\rho_{33} - 2\rho_{44}, \quad R_{11} = 2\ (\rho_{32} + \rho_{41}),$$

$$R_{22} = 2\ (\rho_{32} - \rho_{41}), \quad R_{00} = \rho_{11} + \rho_{22} + \rho_{33} + \rho_{44} = 1, \quad R_{33} = 1 - 2\rho_{22} - 2\rho_{33}.$$

For simultaneously non vanishing t_1 (trace of the sub-block matrix 1-4) and t_2 (trace of the sub-block matrix 2-3), the square root of the density matrix ρ writes, in the computational basis, as

$$\sqrt{\rho} = \begin{pmatrix} \frac{\rho_{11}+\sqrt{d_1}}{\sqrt{t_1+2\sqrt{d_1}}} & 0 & 0 & \frac{\rho_{14}}{\sqrt{t_1+2\sqrt{d_1}}} \\ 0 & \frac{\rho_{22}+\sqrt{d_2}}{\sqrt{t_2+2\sqrt{d_2}}} & \frac{\rho_{23}}{\sqrt{t_2+2\sqrt{d_2}}} & 0 \\ 0 & \frac{\rho_{32}}{\sqrt{t_2+2\sqrt{d_2}}} & \frac{\rho_{33}+\sqrt{d_2}}{\sqrt{t_2+2\sqrt{d_2}}} & 0 \\ \frac{\rho_{41}}{\sqrt{t_1+2\sqrt{d_1}}} & 0 & 0 & \frac{\rho_{44}+\sqrt{d_1}}{\sqrt{t_1+2\sqrt{d_1}}} \end{pmatrix}. \tag{6}$$

and the associated eigenvalues $\sqrt{\lambda_1}$, $\sqrt{\lambda_2}$, $\sqrt{\lambda_3}$ and $\sqrt{\lambda_4}$ are given by

$$\sqrt{\lambda_1} = \frac{1}{2}\sqrt{t_1 + 2\sqrt{d_1}} + \frac{1}{2}\sqrt{t_1 - 2\sqrt{d_1}}, \qquad \sqrt{\lambda_2} = \frac{1}{2}\sqrt{t_2 + 2\sqrt{d_2}} + \frac{1}{2}\sqrt{t_2 - 2\sqrt{d_2}}$$

$$\sqrt{\lambda_3} = \frac{1}{2}\sqrt{t_2 + 2\sqrt{d_2}} - \frac{1}{2}\sqrt{t_2 - 2\sqrt{d_2}}, \qquad \sqrt{\lambda_4} = \frac{1}{2}\sqrt{t_1 + 2\sqrt{d_1}} - \frac{1}{2}\sqrt{t_1 - 2\sqrt{d_1}}.$$

The Fano-Bloch representation of the matrix (6) writes as

$$\sqrt{\rho} = \frac{1}{4} \sum_{\alpha,\beta} \mathcal{R}_{\alpha\beta} \sigma_\alpha \otimes \sigma_\beta$$

with $\mathcal{R}_{\alpha\beta} = \mathrm{Tr}(\sqrt{\rho}\ \sigma_\alpha \otimes \sigma_\beta)$. The non vanishing matrix correlation elements $\mathcal{R}_{\alpha\beta}$ are explicitly given by

$$\mathcal{R}_{00} = \sqrt{t_1 + 2\sqrt{d_1}} + \sqrt{t_2 + 2\sqrt{d_2}} \qquad \mathcal{R}_{03} = \frac{1}{2}\frac{R_{30} + R_{03}}{\sqrt{t_1 + 2\sqrt{d_1}}} - \frac{1}{2}\frac{R_{30} - R_{03}}{\sqrt{t_2 + 2\sqrt{d_2}}}$$

$$\mathcal{R}_{30} = \frac{1}{2}\frac{R_{30} + R_{03}}{\sqrt{t_1 + 2\sqrt{d_1}}} + \frac{1}{2}\frac{R_{30} - R_{03}}{\sqrt{t_2 + 2\sqrt{d_2}}} \qquad \mathcal{R}_{11} = \frac{1}{2}\frac{R_{11} + R_{22}}{\sqrt{t_2 + 2\sqrt{d_2}}} + \frac{1}{2}\frac{R_{11} - R_{22}}{\sqrt{t_1 + 2\sqrt{d_1}}}$$

$$\mathcal{R}_{22} = \frac{1}{2}\frac{R_{11} + R_{22}}{\sqrt{t_2 + 2\sqrt{d_2}}} - \frac{1}{2}\frac{R_{11} - R_{22}}{\sqrt{t_1 + 2\sqrt{d_1}}} \qquad \mathcal{R}_{33} = \sqrt{t_1 + 2\sqrt{d_1}} - \sqrt{t_2 + 2\sqrt{d_2}}$$

Using the following relations of the Pauli matrices

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} \quad \mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij} \quad \mathrm{Tr}(\sigma_i\sigma_j\sigma_k\sigma_l) = 2(\delta_{ij}\delta_{kl} - \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

one shows that the matrix W (3) is diagonal and the diagonal elements are

$$\omega_{ii} = \frac{1}{4}\left[\sum_\beta \left(\mathcal{R}_{0\beta}^2 - \sum_k \mathcal{R}_{k\beta}^2\right)\right] + \frac{1}{2}\sum_\beta \mathcal{R}_{i\beta}^2 \tag{7}$$

where $i, k = 1, 2, 3$ and $\beta = 0, 1, 2, 3$. They can be cast in the following closed form

$$\omega_{ii} = \frac{1}{4}\eta^{\alpha\beta}(\mathcal{R}\mathcal{R}^t)_{\alpha\beta} + \frac{1}{2}(\mathcal{R}\mathcal{R}^t)_{ii} \tag{8}$$

where the summation over repeated indices is understood, the subscript t stands for transposition transformation, η is the diagonal matrix $\eta = \mathrm{diag}(1,-1,-1,-1)$. The eigenvalues ω_{11}, ω_{22} and ω_{33} (8) involve only the non vanishing Fano-Bloch components of the square root of the density matrix ρ_{12}. Alternatively, they can be expanded as

$$\omega_{11} = \frac{1}{4}(\mathcal{R}_{00}^2 - \mathcal{R}_{33}^2 + \mathcal{R}_{03}^2 - \mathcal{R}_{30}^2 + \mathcal{R}_{11}^2 - \mathcal{R}_{22}^2),$$
$$\omega_{22} = \frac{1}{4}(\mathcal{R}_{00}^2 - \mathcal{R}_{33}^2 + \mathcal{R}_{03}^2 - \mathcal{R}_{30}^2 - \mathcal{R}_{11}^2 + \mathcal{R}_{22}^2),$$
$$\omega_{33} = \frac{1}{4}(\mathcal{R}_{00}^2 + \mathcal{R}_{33}^2 + \mathcal{R}_{03}^2 + \mathcal{R}_{30}^2 - \mathcal{R}_{11}^2 - \mathcal{R}_{22}^2).$$

Using the expressions of Fano-Bloch elements $\mathcal{R}_{\alpha\beta}$ associated with the matrix $\sqrt{\rho_{12}}$, the eigenvalues ω_{ii} $(i = 1,2,3)$ can be also expressed in terms of the correlation matrix elements $R_{\alpha\beta}$ of the state ρ_{12} as

$$\omega_{11} = \sqrt{(t_1 + 2\sqrt{d_1})(t_2 + 2\sqrt{d_2})} \; + \; \frac{1}{4}\frac{(R_{11}^2 - R_{22}^2) + (R_{03}^2 - R_{30}^2)}{\sqrt{(t_1 + 2\sqrt{d_1})(t_2 + 2\sqrt{d_2})}}, \tag{9}$$

$$\omega_{22} = \sqrt{(t_1 + 2\sqrt{d_1})(t_2 + 2\sqrt{d_2})} \; + \; \frac{1}{4}\frac{(R_{22}^2 - R_{11}^2) + (R_{03}^2 - R_{30}^2)}{\sqrt{(t_1 + 2\sqrt{d_1})(t_2 + 2\sqrt{d_2})}}, \tag{10}$$

$$\omega_{33} = \frac{1}{2}\Big(1 + 2(\sqrt{d_1} + \sqrt{d_2})\Big) + \frac{1}{8}\left[\frac{(R_{03} + R_{30})^2 - (R_{11} - R_{22})^2}{t_1 + 2\sqrt{d_1}}\right] + \frac{1}{8}\left[\frac{(R_{03} - R_{30})^2 - (R_{11} + R_{22})^2}{t_2 + 2\sqrt{d_2}}\right], \tag{11}$$

where the quantities t_i and d_i $(i = 1,2)$ are also re-expressed as

$$t_1 = \frac{1}{2}(R_{00} + R_{33}), \;\; t_2 = \frac{1}{2}(R_{00} - R_{33})$$

$$d_1 = \frac{1}{16}\Big[(R_{00} + R_{33})^2 - (R_{30} + R_{03})^2 - (R_{11} - R_{22})^2\Big], d_2 = \frac{1}{16}\Big[(R_{00} - R_{33})^2 - (R_{30} - R_{03})^2 - (R_{11} + R_{22})^2\Big],$$

in terms of the Fano-Bloch components $R_{\alpha\beta}$. We observe that, for two-qubit X states with positive entries, R_{11} is always larger than R_{22}. This implies $\omega_{11} \geq \omega_{22}$ and the local quantum uncertainty for the states (4) writes simply as

$$\mathcal{U}(\rho_{12}) = 1 - \max(\omega_{11}, \omega_{33}). \tag{12}$$

Therefore, only two distinct situations have to be separately treated, that is $\omega_{11} \geq \omega_{33}$ and $\omega_{11} \leq \omega_{33}$.

2.3 Particular cases

The X density matrices split in two 2×2 block matrices corresponding to decoupling Hilbert subspaces $(1-4)$ and $(2-3)$. In deriving the result (12), we have assumed that the trace t_1 of the sub-matrix $(1-4)$ and the trace t_2 of the sub-matrix $(2-3)$ are non zero. Now, we consider the special situations where $t_1 = 0$ or $t_2 = 0$. We note that when $t_1 = 0$, the trace condition of the density matrix ρ_{12} imposes $t_2 = 1$ and vice-versa. We note also t_1 vanishes if and only if $\rho_{11} = \rho_{44} = 0$ and the positivity condition of the density matrix ρ_{12} (4) implies $\rho_{14} = \rho_{41} = 0$. In this case, we have $d_1 = 0$. Similarly, $t_2 = 0$ implies $\rho_{22} = \rho_{33} = 0$ and subsequently one has $\rho_{23} = \rho_{32} = 0$ and $d_2 = 0$.

In the case where the density matrix (4) is restricted to the block matrix $(2-3)$, the correlation matrix elements of the matrix $\sqrt{\rho}$ write simply as

$$\mathcal{R}_{00} = -\mathcal{R}_{33} = \sqrt{1+2\sqrt{\rho_{22}\rho_{33}-\rho_{23}^2}}, \quad \mathcal{R}_{11} = \mathcal{R}_{22} = \frac{2\rho_{23}}{\sqrt{1+2\sqrt{\rho_{22}\rho_{33}-\rho_{23}^2}}}, \quad \mathcal{R}_{30} = -\mathcal{R}_{03} = \frac{\rho_{22}-\rho_{33}}{\sqrt{1+2\sqrt{\rho_{22}\rho_{33}-\rho_{23}^2}}}$$

Therefore, for $t_1 = 0$ the equation (8) gives $\omega_{11} = 0$, $\omega_{22} = 0$ and

$$\omega_{33} = \frac{1}{2}\left[1+2\sqrt{\rho_{22}\rho_{33}-\rho_{23}^2}\right] + \frac{1}{2}\left[\frac{(\rho_{22}-\rho_{33})^2-4\rho_{23}^2}{1+2\sqrt{\rho_{22}\rho_{33}-\rho_{23}^2}}\right] \tag{13}$$

in terms of the matrix elements of the density ρ_{12}. For this special two-qubit state, the local quantum uncertainty is given by $1-\omega_{33}$.

Similarly, in the special case where $t_2 = 0$ (or equivalently $\rho_{22} = \rho_{33} = \rho_{23} = \rho_{32} = 0$), the Fano-Bloch elements of the matrix $\sqrt{\rho}$ are given by

$$\mathcal{R}_{00} = \mathcal{R}_{33} = \sqrt{1+2\sqrt{\rho_{11}\rho_{44}-\rho_{14}^2}}, \quad \mathcal{R}_{11} = -\mathcal{R}_{22} = \frac{2\rho_{14}}{\sqrt{1+2\sqrt{\rho_{11}\rho_{44}-\rho_{14}^2}}}, \quad \mathcal{R}_{03} = \mathcal{R}_{30} = \frac{\rho_{11}-\rho_{44}}{\sqrt{1+2\sqrt{\rho_{11}\rho_{44}-\rho_{14}^2}}}$$

It follows that for the X states with $t_2 = 0$, one gets $\omega_{11} = 0$, $\omega_{22} = 0$ and

$$\omega_{33} = \frac{1}{2}\left[1+2\sqrt{\rho_{11}\rho_{44}-\rho_{14}^2}\right] + \frac{1}{2}\left[\frac{(\rho_{11}-\rho_{44})^2-4\rho_{14}^2}{1+2\sqrt{\rho_{11}\rho_{44}-\rho_{14}^2}}\right] \tag{14}$$

in terms of the non vanishing density matrix elements. In this case, the local quantum uncertainty reads as $1-\omega_{33}$.

2.4 Local quantum uncertainty for some special two qubit X states

To exemplify our method for calculating the local quantum uncertainty, we discuss now on some special two-qubit states for which the quantum correlations were evaluated recently in the literature using this quantifier. We shall consider three special X states: (i) Werner states, (ii) Bell-diagonal states and (iii) orthogonal invariant two-qubit states. These three types of two-qubit states are X shaped states with correlation elements verifying (i) $R_{11} = R_{22} = R_{33}$ and $R_{30} = R_{03} = 0$, (ii) $R_{11} \neq R_{22} \neq R_{33}$ and $R_{30} = R_{03} = 0$, (iii) $R_{ii} \neq R_{i+1\ i+1} = R_{i+2\ i+2}$ and $R_{30} = R_{03} = 0$ with $i = 1,2,3$ (mod 3).

2.4.1 Werner states

The two-qubit Werner states given by [30]

$$\rho_{\rm W} = \frac{1-f}{3}\,\sigma_0\otimes\sigma_0 + \frac{4f-1}{3}|\psi^-\rangle\langle\psi^-| \tag{15}$$

are the mixtures of maximally chaotic state and the maximally entangled state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. f denotes the fidelity which characterizes the overlap between Bell state and isotropic state $(0 \leq f \leq 1)$. The Werner states are separable for $f \leq \frac{1}{2}$ and entangled for $\frac{1}{2} < f \leq 1$. This is easily verified from the concurrence formula given by $\mathcal{C}(\rho_{\rm W}) = \max(0, 2f-1)$. In the Fano-Bloch representation, the states $\rho_{\rm W}$ write

$$\rho_{\rm W} = \frac{1}{4}\biggl(\sigma_0\otimes\sigma_0 + \frac{1-4f}{3}\sum_{i=1}^{3}\sigma_i\otimes\sigma_i\biggr). \tag{16}$$

Using the results (9), (10) and (11), one gets

$$\omega_{11} = \omega_{22} = \omega_{33} = \frac{2}{3}(1-f) + \frac{2}{\sqrt{3}}\sqrt{f(1-f)} \tag{17}$$

and the local quantum uncertainty is simply given by

$$\mathcal{U}(\rho_{\rm W}) = 1 - \frac{2}{3}(1-f) + \frac{2}{\sqrt{3}}\sqrt{f(1-f)} \tag{18}$$

which coincides with the result derived in [29].

2.4.2 Two qubit Bell-diagonal states

The subset of X states which are diagonal in the Bell basis is parameterized by three parameters. The corresponding density matrices are of the form

$$\rho_{\rm B} = \frac{1}{4}(\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i). \tag{19}$$

Using the results (9), (10) and (11), the eigenvalues of the matrix W (cf. equation (3)) write as

$$\omega_{11} = \frac{1}{2}\bigg(\sqrt{(1-c_1)^2 - (c_2+c_3)^2} + \sqrt{(1+c_1)^2 - (c_2-c_3)^2}\bigg) \tag{20}$$

$$\omega_{22} = \frac{1}{2}\bigg(\sqrt{(1-c_2)^2 - (c_3+c_1)^2} + \sqrt{(1+c_2)^2 - (c_3-c_1)^2}\bigg) \tag{21}$$

$$\omega_{33} = \frac{1}{2}\bigg(\sqrt{(1-c_3)^2 - (c_1+c_2)^2} + \sqrt{(1+c_3)^2 - (c_1-c_2)^2}\bigg) \tag{22}$$

in terms of the correlation elements c_1, c_2 and c_3. This is exactly the result derived in [27]. For $c_1 = c_2 = c_3 = \frac{1-4f}{3}$, the Bell diagonal states become of Werner type and in this case the expressions (20), (21) and (22) reduce to the eigenvalues given by (17) and one recovers the local quantum uncertainty of Werner states (18).

2.4.3 Orthogonal invariant two-qubit states

Any two qubit state invariant under the operation $\mathcal{O} \otimes \mathcal{O}$ (with $\mathcal{O}$ an arbitrary orthogonal matrix) can be expanded in terms of the three generators $\mathbb{I}_4$, $\mathcal{F}_1$ and $\mathcal{F}_2$ as [28]

$$\rho_{\rm O} = a\mathbb{I}_4 + b\mathcal{F}_1 + c\mathcal{F}_2 \tag{23}$$

where the real parameters a, b and c are positive and satisfy $4a+2b+2c = 1$ (trace condition), $\mathbb{I}_4$ is the identity and the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined by

$$\mathcal{F}_1 = \sum_{ij} |ij\rangle\langle ji| \quad \mathcal{F}_2 = \sum_{ij} |ii\rangle\langle jj|$$

in the computational basis. The density matrix (23) is X shaped

$$\rho_{\rm O} = \begin{pmatrix} a+b+c & 0 & 0 & c \\ 0 & a & b & 0 \\ 0 & b & a & 0 \\ c & 0 & 0 & a+b+c \end{pmatrix}. \tag{24}$$

From the general expressions (9), (10) and (11), one obtains

$$\omega_{11} = \omega_{33} = 2\bigg(\sqrt{(a+b)(a+b+2c)} + \sqrt{a^2-b^2}\bigg) \quad \omega_{22} = 2\bigg(\sqrt{(a-b)(a+b+2c)} + (a+b)\bigg)$$

which reproduces the result obtained in [28].

3 Pairwise local quantum uncertainty in spin coherent states systems

For completeness, we shall briefly recall the most useful relations for $SU(2)$ coherent states. The corresponding Lie algebra is spanned by three generators J_+ , J_- and J_z satisfying the structure relations

$$[J_\pm, J_3] = \pm J_\pm \quad [J_+, J_-] = 2J_3$$

The associated unitary representations are labeled by the angular momentum quantum number j (integer or half integer) and the representation space is $\mathcal{H}_j = \{|j,m\rangle, m = -j, -j+1, \cdots, j-1, j\}$. As it is well known, the spin coherent states can be constructed from the lowest-weight state $|j,-j\rangle$ as follows

$$|j,\eta\rangle = \exp(\xi J_+ - \xi^* J_-)|j,-j\rangle \quad \xi \in \mathbb{C}. \tag{25}$$

Using the disentangling theorem for angular momentum operators, one gets

$$|j,\eta\rangle = D_j(\xi)|j,-j\rangle = \exp(\xi J_+ - \xi^* J_-)|j,-j\rangle = (1+|\eta|^2)^{-j}\exp(\eta J_+)|j,-j\rangle \ , \tag{26}$$

where $\eta = (\xi/|\xi|)\tan|\xi|$. Using the standard action of the rasing operator J_+ on $\mathcal{H}_j$ given by

$$|j,m\rangle = \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}}\frac{(J_+)^{j+m}}{(j+m)!}|j,-j\rangle,$$

the states (26) can be expanded as

$$|j,\eta\rangle = (1+|\eta|^2)^{-j}\sum_{m=-j}^{j}\left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{1/2}\eta^{j+m}|j,m\rangle, \tag{27}$$

in the angular momentum basis $\{|j,m\rangle\}$. The coherent states $|j,\eta\rangle$ satisfy the over-completion property given by

$$\int d\mu(j,\eta)|j,\eta\rangle\langle j,\eta| = I \ , \qquad d\mu(j,\eta) = \frac{2j+1}{\pi}\frac{d^2\eta}{(1+|\eta|^2)^2} \ . \tag{28}$$

They are not orthogonal to each other and the overlap between two distinct $SU(2)$ coherent states is non zero:

$$\langle j,\eta_1|j,\eta_2\rangle = (1+|\eta_1|^2)^{-j}(1+|\eta_2|^2)^{-j}(1+\eta_1^*\eta_2)^{2j} \ . \tag{29}$$

For $j=\frac{1}{2}$, the spin coherent states (27) reduce to

$$|\eta\rangle = \frac{1}{\sqrt{1+\bar{\eta}\eta}}|\downarrow\rangle + \frac{\eta}{\sqrt{1+\bar{\eta}\eta}}|\uparrow\rangle. \tag{30}$$

Here and in the following $|\eta\rangle$ is short for the spin-$\frac{1}{2}$ coherent state $|\frac{1}{2},\eta\rangle$ with $|\uparrow\rangle \equiv |\frac{1}{2},\frac{1}{2}\rangle$ and $|\downarrow\rangle \equiv |\frac{1}{2},-\frac{1}{2}\rangle$) or equivalently $|0\rangle \equiv |\frac{1}{2},-\frac{1}{2}\rangle$ and $|1\rangle \equiv |\frac{1}{2},+\frac{1}{2}\rangle$. It is also remarkable that the tensorial product of two different spin coherent states $|j_1,\eta\rangle$ and $|j_2,\eta\rangle$ gives a j_1+j_2-spin coherent state labeled by the same parameter. This factorization property writes as

$$|j_1,\eta\rangle\otimes|j_2,\eta\rangle = (D_{j_1}\otimes D_{j_2})\left(|j_1,j_1\rangle\otimes|j_2,j_2\rangle\right) = D_{j_1+j_2}\,|j_1+j_2, j_1+j_2\rangle \ = \ |j_1+j_2,\eta\rangle. \tag{31}$$

This property is very interesting in our context. Indeed, a j-spin coherent state may be viewed as multipartite state describing $2j$ qubits:

$$|j,\eta\rangle = (|\eta\rangle)^{\otimes 2j} = \left(\frac{1}{\sqrt{1+\bar{\eta}\eta}}|\downarrow\rangle + \frac{\eta}{\sqrt{1+\bar{\eta}\eta}}|\uparrow\rangle\right)^{\otimes 2j} = (1+\bar{\eta}\eta)^{-j}\sum_{m=-j}^{+j}\begin{pmatrix}2j\\ j+m\end{pmatrix}^{\frac{1}{2}}\eta^{j+m}|j,m\rangle.$$

In this section, we study the pairwise quantum correlations in balanced superpositions of spin coherent states using the concept of local quantum uncertainty. A special attention is devoted to two-qubit states extracted from even and odd spin coherent states defined by

$$|j,\eta,m\rangle = \mathcal{N}_m(|j,\eta\rangle + e^{im\pi}|j,-\eta\rangle) \tag{32}$$

where the integer $m \in \mathbb{Z}$ takes the values $m = 0 \pmod 2$ and $m = 1 \pmod 2$. The normalization factor $\mathcal{N}_m$ is

$$\mathcal{N}_m = \left[2 + 2p^{2j}\cos m\pi\right]^{-1/2}$$

where p denotes the scalar product between the states $|\eta\rangle$ and $|-\eta\rangle$:

$$p = \langle\eta|-\eta\rangle = \frac{1-\bar{\eta}\eta}{1+\bar{\eta}\eta}. \tag{33}$$

Clearly for $j = \frac{1}{2}$ the $SU(2)$ coherent states coincides with qubits and the even and odd coherent states coincide with $|\uparrow\rangle$ and $|\downarrow\rangle$. It follows that the states $|j,\eta,m\rangle$ can be treated as multi-qubit (or multi-fermion) states comprising $2j$ qubits (or fermions)

$$|j,\eta,m\rangle = \mathcal{N}_m((|\eta\rangle)^{\otimes 2j} + e^{im\pi}\,(|-\eta\rangle)^{\otimes 2j}). \tag{34}$$

This key property provides a scheme to study with pairwise correlations in even and odd spin coherent states considered not as single entity but viewed as multi-qubit systems. Thus, to derive the local quantum uncertainty in such states we consider in what follows two different partitioning schemes.

3.1 Pure bipartite spin coherent states

We begin with a pure bipartite partition of the following balanced superposition of spin coherent states

$$|j,\eta,\theta\rangle = \mathcal{N}_\theta(|j,\eta\rangle + e^{i\theta}|j,-\eta\rangle) \tag{35}$$

where the normalization factor is given by $|\mathcal{N}_\theta|^{-2} = 2 + 2p^{2j}\cos\theta$. Using the factorization property of spin coherent states (31), the states (35) can be also expressed as

$$|j,\eta,\theta\rangle = \mathcal{N}_\theta(|j_1,\eta\rangle\otimes|j_2,\eta\rangle + e^{i\theta}|j_1,-\eta\rangle\otimes|j_2,-\eta\rangle) \tag{36}$$

with $j = j_1 + j_2$ so that the spin j system splits into two subsystems of spin j_1 and j_2. The resulting bipartite states can be also rewritten as two qubit states in the basis

$$|j_i,\eta,0\rangle \longrightarrow |0\rangle_{j_i} \qquad |j_i,\eta,\pi\rangle \longrightarrow |1\rangle_{j_i}, \quad i = 1,2.$$

defined by means of odd and even spin coherent associated with the angular momenta j_1 and j_2. Indeed, for each subsystem, an orthogonal basis $\{|0\rangle_l, |1\rangle_l\}$, with $l = j_1$ or j_2, can be defined as

$$|0\rangle_l = \frac{|l,\eta\rangle + |l,-\eta\rangle}{\sqrt{2(1+p^{2l})}} \qquad |1\rangle_l = \frac{|l,\eta\rangle - |l,-\eta\rangle}{\sqrt{2(1-p^{2l})}}. \tag{37}$$

The bipartite density state $\rho = |j,\eta,\theta\rangle\langle j,\eta,\theta|$ is pure. The concurrence in this pure bipartite system writes

$$\mathcal{C}_{j_1,j_2}(\theta) = \frac{\sqrt{1-p^{4j_1}}\sqrt{1-p^{4j_2}}}{1+p^{2j}\cos\theta}. \tag{38}$$

Using the Schmidt decomposition, the state (36) can be written as

$$|j,\eta,\theta\rangle = \sqrt{\lambda_+}|+\rangle_1\otimes|+\rangle_2 + \sqrt{\lambda_-}|-\rangle_1\otimes|-\rangle_2 \tag{39}$$

where $\lambda_\pm$ denote the eigenvalues of the reduced density of the first subsystem $\rho_{j_1} = \mathrm{Tr}_{j_2}(\rho)$ obtained by tracing out the spin j_2 . They write as

$$\lambda_\pm = \frac{1}{2}\bigg(1\pm\sqrt{1-\mathcal{C}^2}\bigg). \tag{40}$$

in terms of the concurrence $\mathcal{C}\equiv\mathcal{C}_{j_1,j_2}(\theta)$ given by (38). In the basis $\{|+\rangle_1\otimes|+\rangle_2, |+\rangle_1\otimes|-\rangle_2, |-\rangle_1\otimes|+\rangle_2, |-\rangle_1\otimes|-\rangle_2\}$, the density matrix $\rho_{j_1,j_2}(\theta) = |j,\eta,\theta\rangle\langle j,\eta,\theta|$ takes the form

$$\rho_{j_1,j_2}(\theta) = \begin{pmatrix} \lambda_+ & 0 & 0 & \sqrt{\lambda_+\lambda_-} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\lambda_+\lambda_-} & 0 & 0 & \lambda_- \end{pmatrix} \tag{41}$$

Using the result (7), one verifies that $\omega_{11} = 0$, $\omega_{22} = 0$ and $\omega_{33} = 1-4\lambda_+\lambda_-$. It follows that the local quantum uncertainty coincides with the squared concurrence (38)

$$\mathcal{U}(\rho_{j_1,j_2}(\theta)) = \mathcal{C}^2_{j_1,j_2}(\theta). \tag{42}$$

For $\theta = m\pi$ $(m\in\mathbb{Z})$, the logical qubits $|j,\eta,m=0\rangle$ and $|j,\eta,m=1\rangle$ coincide with even and odd spin coherent states. They behave like a multipartite state of Greenberger-Horne-Zeilinger (GHZ) type [37] in the limit $p\to 0$. Indeed, in this limit the states $|\eta\rangle$ and $|-\eta\rangle$ approach orthogonality and an orthogonal basis can be defined such that $|\mathbf{0}\rangle\equiv|\eta\rangle$ and $|\mathbf{1}\rangle\equiv|-\eta\rangle$. Thus, the state $|j,\eta,m\rangle$ becomes of GHZ-type

$$|j,\eta,m\rangle \sim |\mathrm{GHZ}\rangle_{2j} = \frac{1}{\sqrt{2}}(|\mathbf{0}\rangle\otimes|\mathbf{0}\rangle\otimes\cdots\otimes|\mathbf{0}\rangle + e^{im\pi}|\mathbf{1}\rangle\otimes|\mathbf{1}\rangle\otimes\cdots\otimes|\mathbf{1}\rangle) \tag{43}$$

which is maximally entangled and the bipartite local quantum uncertainty is $\mathcal{U}(\rho_{j_1,j_2}(\theta = m\pi)) = 1$.

Another interesting limiting case concerns the situation where $p^2\to 1$ (or $\eta\to 0$). In this case the state $|j,\eta,m = 0 \pmod 2\rangle$ (34) reduces to ground state of a collection of $2j$ fermions

$$|j,0,0 \pmod 2\rangle \sim |\downarrow\rangle\otimes|\downarrow\rangle\otimes\cdots\otimes|\downarrow\rangle, \tag{44}$$

which is completely separable and

$$\mathcal{U}(\rho_{j_1,j_2}(\theta = m\pi) = 0.$$

The odd spin coherent state $|j,\eta,1 \pmod 2\rangle$ becomes a multipartite state of W type [38]

$$|j,0,1 \pmod 2\rangle \sim |\mathrm{W}\rangle_{2j} = \frac{1}{\sqrt{2j}}(|\uparrow\rangle\otimes|\downarrow\rangle\otimes\cdots\otimes|\downarrow\rangle + |\downarrow\rangle\otimes|\uparrow\rangle\otimes\ldots\otimes|\downarrow\rangle + \cdots + |\downarrow\rangle\otimes|\downarrow\rangle\otimes\cdots\otimes|\uparrow\rangle)\,. \tag{45}$$

and, in this limiting situation, the local quantum uncertainty is given by

$$\mathcal{U}(\rho_{j_1,j_2}(\theta = \pi)) = 4\frac{j_1j_2}{j_1+j_2}.$$

3.2 Mixed bipartite states

Now, we consider bipartite mixed density matrices ρ_{ij} obtained, from the even and odd spin coherent states, by a trace procedure consisting in removing all degrees of freedom of all $\frac{1}{2}$-spins except two labeled by i and j. The whole system is invariant under exchange symmetry. Thus, the trace procedure leads to $2j(2j-1)/2$ identical density matrices that we denote in what follows by ρ_{12}. After some algebra, one obtains

$$\rho_{12} = \mathcal{N}^2(|\eta,\eta\rangle\langle\eta,\eta| + |-\eta,-\eta\rangle\langle-\eta,-\eta| + e^{im\pi}q|-\eta,-\eta\rangle\langle\eta,\eta| + e^{-im\pi}q|\eta,\eta\rangle\langle-\eta,-\eta|), \tag{46}$$

where q is defined by

$$q = p^{2j-2}.$$

Setting $\eta = e^{i\phi}\sqrt{\frac{1-p}{1+p}}$, the density matrix takes the form

$$\rho_{12} = \frac{1}{4(1+p^{2j}\cos m\pi)}\begin{pmatrix} (1+p)^2(1+q\cos m\pi) & 0 & 0 & e^{-2i\phi}(1-p^2)(1+q\cos m\pi) \\ 0 & (1-p^2)(1-q\cos m\pi) & (1-p^2)(1-q\cos m\pi) & 0 \\ 0 & (1-p^2)(1-q\cos m\pi) & (1-p^2)(1-q\cos m\pi) & 0 \\ e^{2i\phi}(1-p^2)(1+q\cos m\pi) & 0 & 0 & (1-p)^2(1+q\cos m\pi) \end{pmatrix} \tag{47}$$

in the computational basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$. The phase factor ϕ is irrelevant for our purpose and will be taken equal to zero. More precisely, the phase factor can be removed by a local transformation and the local quantum uncertainty remains unchanged as we discussed in the previous section. The bipartite mixed density ρ_{12} (47) writes in Fano-Bloch representation as

$$\rho_{12} = \sum_{\alpha\beta} R_{\alpha\beta}\sigma_\alpha \otimes \sigma_\beta \tag{48}$$

where the non vanishing correlations elements $R_{\alpha\beta}$ $(\alpha,\beta = 0,1,2,3)$ are given by

$$R_{00} = 1, \quad R_{11} = \frac{1-p^2}{1+p^{2j}\cos m\pi}, \quad R_{22} = \frac{(p^2-1)\, p^{2j-2}\cos m\pi}{1+p^{2j}\cos m\pi},$$

$$R_{33} = \frac{p^2+p^{2j-2}\cos m\pi}{1+p^{2j}\cos m\pi}, \quad R_{03} = R_{30} = \frac{p+p^{2j-1}\cos m\pi}{1+p^{2j}\cos m\pi}.$$

From the equations (9), (10) and (11), one obtains

$$\omega_{11} = \sqrt{\frac{1-p^2}{1+p^2}}\,\frac{\sqrt{1-p^{4j-4}}}{1+p^{2j}\cos m\pi}, \quad \omega_{22} = p^2\sqrt{\frac{1-p^2}{1+p^2}}\,\frac{\sqrt{1-p^{4j-4}}}{1+p^{2j}\cos m\pi}, \quad \omega_{33} = \frac{2p^2}{1+p^2}\,\frac{1+p^{2j-2}\cos m\pi}{1+p^{2j}\cos m\pi}. \tag{49}$$

We observe that ω_{11} is always larger than ω_{22} and to determine the expression of local quantum expression, one should find the conditions under which $\omega_{11} \leq \omega_{33}$ or $\omega_{33} \leq \omega_{11}$. To do this, we study the sign of the difference between ω_{11} and ω_{33}. From (49), one shows that

$$\mathrm{sign}(\omega_{11} - \omega_{33}) = \mathrm{sign}\bigg(2(1-p^4) - (1+3p^4)(1+p^{2(j-1)}\cos m\pi)\bigg).$$

This condition is analyzed for some particular values of j (see the figures 1 and 2). It is important to note that for $j = 1$, the state (47) is pure and coincides with the state $\rho_{j_1,j_2}(\theta)$ (41) when $j_1 = j_2 = 1/2$ and $\theta = m\pi$. In this case, from the results (49) one verifies that $(\omega_{11} = 0, \omega_{22} = 0, \omega_{33} = \frac{4p^2}{(1+p^2)^2})$ for $m = 0$ and $(\omega_{11} = 0, \omega_{22} = 0, \omega_{33} = 0)$ for $m = 1$. Then, the local quantum uncertainty is given by

$$\mathcal{U}(\rho_{1/2,1/2}) = \frac{(1-p^2)^2}{(1+p^2)^2} \quad \text{for} \quad m = 0$$

and

$$\mathcal{U}(\rho_{1/2,1/2}) = 1 \quad \text{for} \quad m = 1.$$

This result can be also obtained from (42). Another interesting limiting case concerns the situation where $j > 1$ and $p \longrightarrow 0$. In this limit, the even and odd spin coherent states become of GHZ type. From (49) one obtains

$$\omega_{11} = 1, \quad \omega_{22} = 0 \quad \omega_{33} = 0$$

and the pairwise local quantum uncertainty is identically vanishing (see the figures 1 and 2). Similarly, for $j > 1$ and $p^2 \longrightarrow 1$, it is simple to check that the local quantum uncertainty vanishes for even spin coherent states ($m = 0$) (see figure 1). This agrees with the fact that the even spin coherent states become separable in this limiting situation (see (44)). However, for $m = 1$ and $p^2 \longrightarrow 1$, the expressions of ω_{11}, ω_{22} and ω_{33} given by the equations (49) become

$$\omega_{11} = \frac{j-2}{\sqrt{2}j}, \quad \omega_{22} = \frac{j-2}{\sqrt{2}j}, \quad \omega_{33} = \frac{j-1}{j}.$$

In this case, the local quantum uncertainty behaves like

$$\mathcal{U}(\rho_{12}(m=1)) \sim \frac{1}{j}$$

as it is depicted in figure 2. This result gives the pairwise local quantum correlations for two-qubit states extracted from the multi-qubit states W_{2j} of W type (45).

Finally, we note the sudden change in the behavior of local quantum uncertainty (see the figures). In fact, it is commonly accepted that the quantum discord quantified by geometric or entropic measures might experience sudden transition for certain two-qubit systems. This intriguing phenomenon, exhibited also by local quantum uncertainty, might be useful in the understanding the role of quantum correlations in quantum phase transitions [39].

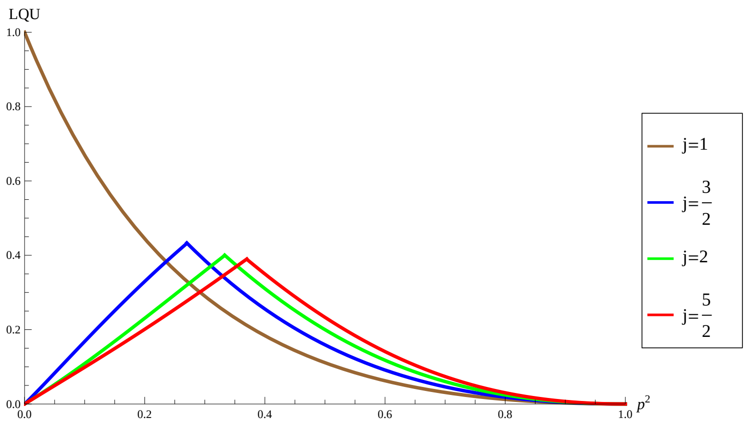

Figure 1. The pairwise local quantum uncertainty in even spin coherent states ($m = 0$) for different values of j.

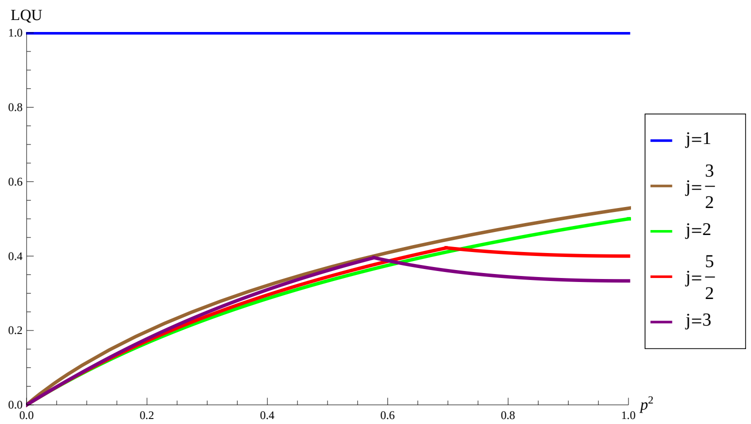

Figure 2. The pairwise local quantum uncertainty in odd spin coherent states $(m = 1)$ for different values of j.

4 Concluding remarks

To conclude, we summarize the main points developed in this paper. The first result concerns the derivation of the analytical expression of local quantum uncertainty for two-qubit in a generic X-states. This quantum correlations quantifier provides an efficient and computable way to characterize the nature of correlations present in a multi-partite quantum system. Moreover, the analytical expression of local quantum uncertainty obtained in this paper reproduces the results recently reported in the literature for some special class of two-qubit states such as Bell diagonal states [27], Werner states [29] and orthogonally invariant two-qubit states [28]. To illustrate the obtained results, we computed the pairwise local quantum uncertainty in even and odd j-spin coherent states viewed as multipartite systems comprising $2j$ qubits. Two partitioning schemes were considered. For the pure bi-partitions, the local quantum uncertainty is given by the concurrence. The second partitioning picture deals with two-qubit states obtained from even and odd j-spin coherent states by tracing out the degrees of freedom of $2j - 2$ qubits. The explicit form of pairwise local quantum uncertainty is determined and analyzed for some specific values of spin. Our interest in X states is mainly motivated by their relevance in various collective spin systems exhibiting quantum phase transitions such as Dicke model [40] and Lipkin-Meshkov-Glick [41] model. The role of quantum correlations in connection with quantum phase transition was considered in several works using either entropic or geometric quantum discord. In this sense, the analytical form of the local quantum uncertainty obtained here provides a reliable quantum correlations indicator in such systems. We hope to report on this issue in the future. We notice also that the derivation of local quantum uncertainty discussed throughout this work is easily adapted to two-qubits states of the form [33]

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & 0 & 0 \\ \rho_{21} & \rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & \rho_{34} \\ 0 & 0 & \rho_{43} & \rho_{44} \end{pmatrix} \tag{50}$$

where the coupling is between the subspaces $(1 - 2)$ and $(3 - 4)$. Finally, it will be important to examine the dynamics of the discord-like local quantum uncertainty under decoherence effects due the inevitable interaction of quantum systems with their environment.

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