Quantum Information and Computation, Vol. 0, No. 0 (2006) 000–000

EVOLUTION FROM ENTANGLEMENT TO DECOHERENCE OF BIPARTITE MIXED "X" STATES

Ting Yu[a] and J. H. Eberly[b]
Department of Physics and Astronomy, University of Rochester
Rochester, NY 14627, USA

Received (received date)
Revised (revised date)

We examine a class of bipartite mixed states which we call X states and report several analytic results related to the occurrence of so-called entanglement sudden death (ESD) under time evolution in the presence of common types of environmental noise. Avoidance of sudden death by application of purely local operations is shown to be feasible in some cases.

Keywords: Entanglement, Decoherence, Mixed States

Communicated by: to be filled by the Editorial

1 Introduction

Inevitably, the entanglement of a multi-partite quantum state becomes degraded with time due to experimental and environmental noise. The influence of noise on bipartite entanglement is a problem in the theory of open systems [1], as well as of practical importance in any application using quantum features of information [2].

The topic of evolution of quantum coherence in the presence of noise sits between two well-investigated problems. One of these is the relaxation toward steady-state of one-body coherence of a simple quantum system (spin, atom, exciton, quantum dot, etc.) in contact with a much larger reservoir [3]. The other is the newer two-body problem where the evolution of the disentanglement of the system from its environment is of interest. It is generally understood that the latter decoherence occurs much more rapidly than the former.

Recently, a practical problem that includes parts of both has drawn attention – the survival of the joint entanglement of two small systems with each other while each is exposed to a local noisy environment. Their rapid disentanglement from their environments is supposed not to be observed, but their disentanglement from each other is considered interesting and potentially important.

We have shown in a specific instance of such bipartite disentanglement of qubits [4, 5] that entanglement is lost in a very different way compared to the usual one-body decoherence

[a]Email: ting@pas.rochester.edu
[b]Email: eberly@pas.rochester.edu

measured by the decay of off-diagonal elements of the density matrix of either qubit system separately. More surprisingly, we have shown [6, 7] that the presence of either pure vacuum noise or even classical noise can cause entanglement to decay to zero in a finite time, an effect that is labelled "entanglement sudden death" (ESD). In the last few years the issue of such entanglement decoherence has been discussed in a number of distinct contexts such as qubit pairs [4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], finite spin chains [21, 22, 23, 24, 25, 26], multipartite systems [27, 28], decoherence dynamics in adiabatic entanglement [29], entanglement transfer [30], and open quantum systems [31, 32, 33, 34], to name a few. In addition, a proposal for the direct measurement of finite-time disentanglement in a cavity QED context has been made recently [35].

In this paper we report several steps that we expect will assist further understanding of this complex and fundamental topic. We focus on the smallest and simplest non-trivial situations, in order to help expose consequences that are dynamically fundamental, as opposed to ones originating simply in one or another kind of complexity. For greatest utility, this more or less mandates that results should be analytic rather than numeric. We will treat two qubits prepared in a mixed state as an information time-evolution question in the presence of noises. For this purpose, solutions of the appropriate Born-Markov-Lindblad master equations can be obtained [36] and we will use a Kraus operator approach throughout the paper [37].

The focus will be maintained strictly on the way information itself evolves by considering the entanglement of two quantum systems A and B exposed to local noises but completely isolated from interacting with each other. We will examine evolution toward complete disentanglement in a class of commonly occurring bipartite density matrices (which we call "X" states) and establish: (a) that X-state character is robust, i.e., an X state remains an X state in its evolution under the most common noise influences, (b) as Werner states are a subclass of X states, we will show that there exisits a critical Werner fidelity below which termination of entanglement must occur in a finite time, and (c) that there are purely local operations that can sometimes be used to alter the survival dynamics of bipartite entanglement. We show that ESD will customarily occur, but that in some cases it can be avoided by applying appropriate local operations initially. We will illustrate all of these in the following sections.

The paper is organized as follows: In Sec. 2, we present two-qubit models where the bipartite system is coupled to external sources of phase-damping and amplitude-damping noises. Explicit time-dependent solutions in terms of Kraus operators are given. Sec. 3 deals with concurrence, the chosen measure of entanglement, and the defining character of an X state. In Sec. 4, the evolution of a Werner state toward decoherence is discussed, as an important example of X-state behavior under the influence of noise. We find a new fidelity boundary below which entanglement sudden death (ESD) must occur for all Werner states. In the following Sec. 5 we derive the ESD that is encountered with depolarizing noise for the X states. In Sec. 6, we show that in some cases it is feasible to transform a short-lived state into a long-lived state by applying specified local operations initially, and we conclude in Sec. 7.

2 Models

The non-interacting quantum systems A and B and their separate reservoirs labeled a and b are assumed to follow the same evolution route separately. We use the familiar Hamiltonian (for qubit A say):

$$H_{\rm tot}^A = H_{\rm at}^A + H_{\rm res}^a + H_{\rm int}^{\rm Aa}, \tag{1}$$

where:

$$H_{\rm at}^A = \frac{1}{2}\omega_A \sigma_z^A \quad \text{and} \quad H_{\rm res}^a = \sum_k \omega_k a_k^\dagger a_k \tag{2}$$

and for exposure to phase and amplitude noises the interaction Hamiltonians $H_{\rm int}$ are given by

$$H_{\rm ph-int}^{Aa} = \sum_k \sigma_z^A (f_k a_k^\dagger + f_k^* a_k), \tag{3}$$

and

$$H_{\rm am-int}^{Aa} = \sum_k (g_k \sigma_-^A a_k^\dagger + g_k^* \sigma_+^A a_k). \tag{4}$$

Here the a_k are bosonic reservoir coordinates satisfying $[a_k, a_{k'}] = \delta_{k,k'}$, the g_k are broad-band coupling constants, and the σ^As denote the usual Pauli matrices for qubit A. The same forms hold for B, with a set of bosonic reservoir coordinates b_k. $\{A, a\}$ and $\{B, b\}$ are completely independent, and no decoherence-free joint subspaces are available. As remarked, these total Hamiltonians provide well-known solvable qubit-reservoir interactions, but we are interested in the evolution of joint information as a consequence of the completely separate interactions.

We consider qubits A and B prepared in a mixed state. For this purpose, solutions of the appropriate Born-Markov-Lindblad equations can be obtained via several routes, and we find the Kraus operator form [37] convenient for our purpose. Given an initial state ρ (pure or mixed) for two qubits A and B, its evolution can be written compactly as

$$\rho(t) = \sum_\mu K_\mu(t)\rho(0)K_\mu^\dagger(t), \tag{5}$$

where the so-called Kraus operators K_μ satisfy $\sum_\mu K_\mu^\dagger K_\mu = 1$ for all t. Obviously, the Kraus operators contain the complete information about the system's dynamics.

In the case of dephasing noise one has following compact Kraus operators:

$$E_1 = \begin{pmatrix} \gamma_A & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \gamma_B & 0 \\ 0 & 1 \end{pmatrix}, \tag{6}$$

$$E_2 = \begin{pmatrix} \gamma_A & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \omega_B \end{pmatrix}, \tag{7}$$

$$E_3 = \begin{pmatrix} 0 & 0 \\ 0 & \omega_A \end{pmatrix} \otimes \begin{pmatrix} \gamma_B & 0 \\ 0 & 1 \end{pmatrix}, \tag{8}$$

$$E_4 = \begin{pmatrix} 0 & 0 \\ 0 & \omega_A \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & \omega_B \end{pmatrix}, \tag{9}$$

where the time-dependent Kraus matrix elements are

$$\gamma_A(t) = \exp\left(-\Gamma_{\rm ph}^A t/2\right) \quad \text{and} \quad \omega_A(t) = \sqrt{1-\gamma_A^2(t)},$$

where $\Gamma_{\rm ph}^A$ is the phase damping rate of qubit A. We use the similar expressions $\gamma_B(t)$ and $\omega_B(t)$ for qubit B, and will take $\Gamma_{\rm ph}^A = \Gamma_{\rm ph}^B = \Gamma_{\rm ph}$ for greatest simplicity.

Similarly, the Kraus operators for zero-temperature amplitude noise are given by

$$F_1 = \begin{pmatrix} \gamma_A & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \gamma_B & 0 \\ 0 & 1 \end{pmatrix}, \tag{10}$$

$$F_2 = \begin{pmatrix} \gamma_A & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ \omega_B & 0 \end{pmatrix}, \tag{11}$$

$$F_3 = \begin{pmatrix} 0 & 0 \\ \omega_A & 0 \end{pmatrix} \otimes \begin{pmatrix} \gamma_B & 0 \\ 0 & 1 \end{pmatrix}, \tag{12}$$

$$F_4 = \begin{pmatrix} 0 & 0) \\ \omega_A & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ \omega_B & 0 \end{pmatrix}, \tag{13}$$

and the time-dependent Kraus matrix elements are defined similarly as in the dephasing model, e.g., $\gamma_A(t) = \exp\left(-\Gamma_{\text{am}}^A t/2\right)$, etc. With the above explicit solutions of the models, we are able to compute the degree of entanglement of the two qubits in temporal evolution.

3 Decoherence Measure and X States

In order to describe the dynamic evolution of quantum entanglement we use Wootters' concurrence [38]. Any entropy-based measure of entanglement will yield the same conclusion about bipartite separability. Concurrence varies from $C = 0$ for a separable state to $C = 1$ for a maximally entangled state. For any two qubits, the concurrence may be calculated explicitly from their density matrix ρ for qubits A and B:

$$C(\rho) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\ \}, \tag{14}$$

where the quantities λ_i are the eigenvalues in decreasing order of the matrix ζ:

$$\zeta = \rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y), \tag{15}$$

where ρ^* denotes the complex conjugation of ρ in the standard basis $|{+}{+}\rangle, |{+}{-}\rangle, |{-}{+}\rangle, |{-}{-}\rangle$ and σ_y is the Pauli matrix expressed in the same basis as:

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{16}$$

In the following we will examine the evolution of entanglement under noise-induced relaxation of a class of important bipartite density matrices which are defined below. Since a density matrix in this class only contains non-zero elements in an "X" formation, along the main diagonal and anti-diagonal, we call them "X states":

$$\rho^{AB} = \begin{pmatrix} a & 0 & 0 & w \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ w^* & 0 & 0 & d \end{pmatrix}. \tag{17}$$

where $a + b + c + d = 1$.

Such a simple matrix is actually not unusual. Experience shows that this X mixed state arises naturally in a wide variety of physical situations (see [23, 26, 39]). We particularly note that it includes pure Bell states as well as the well-known Werner mixed state [40] as special cases. Unitary transforms of it extend its domain even more widely, as we will explain below.

The mixed states defined here not only are rather common but also have the property that they often retain the X form under noise evolution. This may be expected for phase noise, which can only give time dependence to the off-diagonal matrix elements. The interaction Hamiltonian and Kraus operators for amplitude (e.g., quantum vacuum) noise evolution are different, and evolution under amplitude noise is more elaborate, affecting all six non-zero elements (see [6]), but robust form-invariance during evolution is easy to check. This very simple finding applies to a wide array of realistic noise sources.

For the X state defined in (17), concurrence [38] can be easily computed as

$$C(\rho^{AB}) = 2\max\{0, |z| - \sqrt{ad}, |w| - \sqrt{bc}\}.$$

4 Evolution to Decoherence of the Werner State

Now we examine decoherence evolution under first phase damping and then amplitude damping. Within the set of X matrices, let us focus now on a Werner state [40, 41]:

$$\rho_W = \frac{1-F}{3} I_4 + \frac{4F-1}{3} |\Psi^-\rangle\langle\Psi^-|, \tag{18}$$

whose matrix elements can be matched to those of the X state ρ^{AB} easily. F is termed fidelity, and $1 \geq F \geq \frac{1}{4}$. We will begin by obtaining the time-dependence of entanglement for the Werner state. Under phase noise the only time dependence is in z:

$$z(t) = \frac{1-4F}{6}\gamma^2(t), \quad \text{with} \quad \gamma(t) \equiv e^{-\Gamma_{\rm ph} t/2}. \tag{19}$$

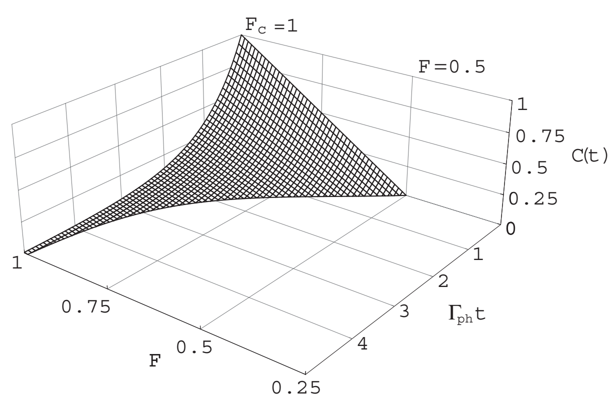

Fig. 1. Phase noise causes ρ_W to disentangle completely in finite time for all Werner states except in the limiting case of a pure Bell state. The graph shows $C(t)$ vs. F and $\Gamma_{\rm ph}t$.

The results are shown in Fig. 1. In particular we note the occurrence of ESD, in which concurrence non-smoothly goes to zero at a finite time (and remains zero). This is apparent for all initial $F < 1$. It has been noted already for quantum vacuum noise qubit decoherence

[6] and for disentanglement of continuous joint states [31, 32, 33]. The analytic expressions above make it clear why this is so. Since the matrix elements a and d are fixed, as z decays it must become less than $\sqrt{ad}$ at a specific time $\tau_{\rm ph}$, which can be easily determined to be given by

$$\frac{\tau^{\rm ph}}{\tau_0} = \ln\left[\frac{4F-1}{2-2F}\right], \quad 1 > F > \frac{1}{2} \tag{20}$$

where $\tau_0 = 1/\Gamma_{\rm ph}$ marks the $1/e$ point in the purely exponential decay of the underlying individual qubits.

Next we consider Werner state evolution under amplitude noise, and we find from the appropriate Kraus operators given in (10-13) [6] that the following time dependences specify $\rho_W(t)$ at any time:

$$z(t) = \frac{1-4F}{6}\gamma^2, \tag{21}$$

$$a(t) = \frac{1-F}{3}\gamma^4, \tag{22}$$

$$b(t) = \frac{2F+1}{6}\gamma^2 + \frac{1-F}{3}\gamma^2\omega^2, \tag{23}$$

$$c(t) = \frac{2F+1}{6}\gamma^2 + \frac{1-F}{3}\gamma^2\omega^2, \quad \text{and} \tag{24}$$

$$d(t) = \frac{1-F}{3} + \frac{2F+1}{3}\omega^2 + \frac{1-F}{3}\omega^4. \tag{25}$$

In principle the time-dependent γ and ω parameters could be different for qubits A and B, but we again take them identical and write $\gamma = \exp[-\Gamma_{\rm am}t/2]$ and $\omega^2 = 1-\gamma^2$, where we use $\Gamma_{\rm am}$ to denote the upper level decay rate of the qubits.

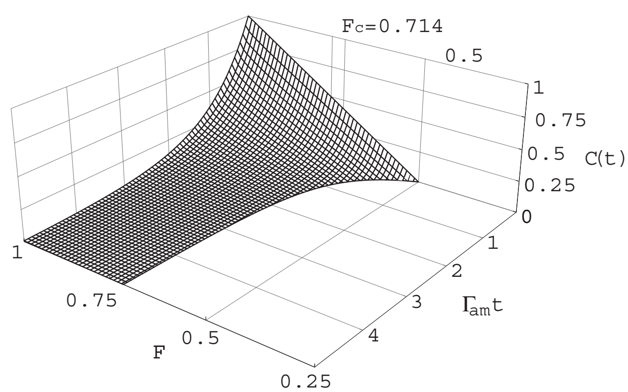

Fig. 2. In the presence of amplitude noise there is long-lived concurrence of Werner states only for sufficiently high fidelity, $F > F_c \simeq 0.714$. The graph shows the critical fidelity boundary in plotting $C(t)$ vs. $\Gamma_{\rm am}t$.

Sudden death of Werner state entanglement appears here also, but with the important added element that sudden death from amplitude noise occurs only for a low range of fidelity values. Our result shows that for all initial F above the critical fidelity $F_c \simeq 0.714$ entanglement remains finite for all time, and has an infinitely long smooth decay, faster than but similar to, the decay of single-qubit coherence. This is shown in the plot in Fig. 2.

5 Bistability Decoherence

In this section, we discuss the entanglement decoherence of a representative X matrix under bistability noise, by which we mean noise that induces incoherent random transfer back and forth between the two qubit states when they are energetically degenerate. Examples occur in bistable systems of all kinds, for example in semiconductor junctions or double-well electron potentials in photonic crystals. A physically different example is polarization of photons in optical fiber with indeterminately random local birefringence. In all of these cases we can speak of the effect as arising from exposure to an infinite-temperature reservoir. In that case the population-equalizing up-transfer and down-transfer rates will be denoted $\Gamma_{\rm eq}$ and taken the same for the two qubits. Given these remarks, our basic model with Hamiltonians (1) and (4) still allows a useful Kraus representation and the Kraus matrices are given by:

$$G_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} \gamma(t) & 0 \\ 0 & 1 \end{pmatrix}, \tag{26}$$

$$G_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ \omega(t) & 0 \end{pmatrix}, \tag{27}$$

$$G_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & \gamma(t) \end{pmatrix}, \tag{28}$$

$$G_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \omega(t) \\ 0 & 0 \end{pmatrix}, \tag{29}$$

where now

$$\gamma(t) = \exp\left(-\Gamma_{\rm eq} t/2\right) \quad \text{and} \quad \omega(t) = \sqrt{1-\gamma^2(t)}.$$

State equalization presents the extreme opposite case from vacuum noise, in the sense that strong equalization treats both qubit states equally incoherently, whereas vacuum noise induces incoherent decay into just the energetically lower of the two states.

One finds that the X form of the density matrix (17) is still preserved under state equalizing noise and so at time t it retains the X form (we set $w = 0$ for simplicity):

$$\rho(t) = \begin{pmatrix} a(t) & 0 & 0 & 0 \\ 0 & b(t) & z(t) & 0 \\ 0 & z(t) & c(t) & 0 \\ 0 & 0 & 0 & d(t) \end{pmatrix}. \tag{30}$$

We assume that the two qubits are affected by two identical local depolarization noises, and in this case the time-dependent matrix elements are given by the following:

$$\begin{aligned} 4a(t) &= \gamma^4 a + a + \omega^2(b+c) + \omega^4 d \\ &\quad + 2\gamma^2 a + \gamma^2\omega^2(b+c), \end{aligned} \tag{31}$$

$$\begin{aligned} 4b(t) &= 2\gamma^2 b + \gamma^2\omega^2(a+d) \\ &\quad + b + \gamma^4 b + \omega^2(a+d) + \omega^4 c, \end{aligned} \tag{32}$$

$$\begin{aligned} 4c(t) &= 2\gamma^2 c + \gamma^2\omega^2(a+d) \\ &\quad +c + \omega^2(d+a) + \omega^4 b + \gamma^4 c, \qquad (33) \\ 4d(t) &= d + \omega^2(b+c) + \omega^4 a + \gamma^4 d \\ &\quad +2\gamma^2 d + \gamma^2\omega^2(b+c), \qquad (34) \\ z(t) &= \gamma^2 z. \qquad (35) \end{aligned}$$

Some algebraic examination shows that this result also leads to ESD, i.e., bistable equalization leads all entangled X states (17) to become separable states in a finite time.

It is easy to check that when $t \to \infty$,

$$z \to z(\infty) = 0, \tag{36}$$

$$\{a, b, c, d\} \to \{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\}, \tag{37}$$

and the asymptotic separability arising from the equivalence of all diagonal elements is just a special case of the general theorem by Zyczkowski, et al.[43].

6 Fragile and Robust Initial Entangled States

It is known that entangled states evolve differently under different environmental noise influences if special symmetries exist. For example, in the case of collective dephasing noise (see, e.g., [4]), there may exist decoherence-free subspaces in which the entangled states are well protected against interaction with the noise. For the models presented here the noises influence each qubit independently, so there are no decoherence-free subspaces and there is no such protection from ESD available. However, we now show that it is still possible to avoid sudden death by using appropriate local initial preparations. To illustrate this, we consider another mixed state within the category of the X matrix defined in (17):

$$\tilde{\rho}_W = \frac{1-F}{3} I_4 + \frac{4F-1}{3} |\Phi^-\rangle\langle\Phi^-|, \tag{38}$$

where $|\Phi^-\rangle = (|++\rangle - |--\rangle)/\sqrt{2}$, is a Bell state. In matrix form at any $t > 0$ we can write

$$\tilde{\rho}_W(t) = \begin{pmatrix} a(t) & 0 & 0 & w(t) \\ 0 & b(t) & 0 & 0 \\ 0 & 0 & c(t) & 0 \\ w^*(t) & 0 & 0 & d(t) \end{pmatrix}. \tag{39}$$

It is easy to compute the concurrence of this mixed state: $C = 2\max\{0, |w| - \sqrt{bc}\}$.

Consider the time dependences for $\tilde{\rho}_W$ obtained from the amplitude-noise Kraus operators as before. It is easy to check that the sudden death condition for $\tilde{\rho}_W$'s concurrence is now:

$$\frac{4F-1}{6}\gamma^2 = \frac{1-F}{3}\gamma^2 + \frac{2F+1}{6}\gamma^2\omega^2, \tag{40}$$

which is satisfied at a finite t for any value of fidelity F. That is, here there is no range of "protected" fidelity values under amplitude noise, as was the case for the other form of X state in Fig. 2. Indefinite survival is impossible in this case, similar to what the plot in Fig. 1 shows for phase noise.

However, this result has important implications related to survival. One easily shows that $\tilde{\rho}_W$ is closely related to the earlier Werner state ρ_W, which does have a range of protected fidelity values. In fact ρ_W and $\tilde{\rho}_W$ are unitary transforms of each other under a *purely local* transformation operator: $U = i\sigma_x^A \otimes I_B$. This shows that survival against noise of initial mixed state entanglement can in a wide range of situations be dramatically improved by a simple local unitary operation (here changing $\tilde{\rho}_W$ into ρ_W), even while the degree of entanglement is not changed. Intuitively, it is easy to see that the noise influence represented by the Kraus operators varies for different matrix elements of a bipartite density matrix. Although local operations cannot change the degree of entanglememt, it is possible that local operations can rearrange the matrix elements of the X states such that the resulting density matrix is more robust (or fragile) than the original one. Therefore, ESD may be manipulated by preparatory transformation that is purely local.

7 Concluding Remarks

In summary, we have examined quantitatively via fully analytic expressions the non-local decoherence properties of a wide range of mixed states. We have used relatively simple Kraus operators to do this. We have also shown that three well understood physical noise types (phase noise, amplitude noise, and state-equalizing noise) do not alter the form of the X mixed state during evolution, although entanglement survival may be long or short. In particular, we have established that Werner states are subject to the sudden death effect, and have specified a new critical fidelity boundary below which sudden death must occur. Surprisingly, we have found that Werner states are more robust under pure amplitude noise (e.g., spontaneous emission) than under pure phase noise even though amplitude noise is in a sense more disruptive than dephasing, as the former causes diagonal and off-diagonal relaxation and the latter off-diagonal relaxation alone. Moreover, we have shown that bistability decoherence can cause all the X states to disentangle completely in finite time. In addition, we have shown that in some cases an initial mixed state's entanglement can be preserved under subsequent noisy evolution, i.e., sudden death avoided, by an initial local unitary operation. Such local operations may offer a useful tool in entanglement control when the duration of entangled states is crucial in the processes of quantum state storage and preparation.

Finally, we note that while we have found interesting and unexpected features of Werner and other X mixed states in the presence of common noise sources, our Kraus operators treated all of them as white (Markovian) noises. It will be an important theoretical challenge to extend these results to the case of non-Markovian environmental influences [14, 44, 45].

Acknowledgements

We acknowledge financial support from NSF Grant PHY-0456952 and ARO Grant W911NF-05-1-0543. We also thank C. Broadbent for assistance with Figs. 1 and 2.

References

1. T. Yu and J.H. Eberly, Phys. Rev. Lett. **97**, 140403 (2006).
2. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge, England, 2000).

3. See, for example, *Principles of Magnetic Resonance*, C.P. Slichter (Springer-Verlag, Berlin, 1978), 2nd ed.
4. T. Yu and J.H. Eberly, Phys. Rev. B **66**, 193306 (2002).
5. T. Yu and J.H. Eberly, Phys. Rev. B **68**, 165322 (2003).
6. T. Yu and J. H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004).
7. T. Yu and J.H. Eberly, Opt. Comm. **264**, 393 (2006) and arXiv: quant-ph/0503089.
8. M. Lucamarini, S. Paganelli and S. Mancini, Phys. Rev. A **69**, 062308 (2004).
9. L. Jakobczyk and A. Jamroz, Phys. Lett. A **333**, 35 (2004).
10. D. Tolkunov, V. Privman, and P. K. Aravind, Phys. Rev. A **71**, 060308 (2005).
11. M. Ban, J. Phys. A **39**, 1927 (2006).
12. M. Ban and F. Shibata, Phys. Lett. A **345**, 35 (2006).
13. V. S. Malinovsky and I. R. Sola, Phys. Rev. Lett. **96**, 050502 (2006).
14. I. Glendinning, M. Jakob and M. Noelle, arXiv: quant-ph/0604030.
15. J.-H. An, S.-J. Wang, and H.-G. Luo, arXiv: quant-ph/0604020.
16. X.T. Liang, Phys. Lett. A **349**, 98 (2006).
17. K. Roszak and P. Machnikowski, Phys. Rev. A **73**, 022313 (2006).
18. A. Jamroz, J. Phys. A **39**, 7727 (2006).
19. Z. Ficek and R. Tanas, Phys. Rev. A **74**, 024304 (2006).
20. D. Solenov, D. Tolkunov and V. Privman, Phys. Lett. **359**, 81 (2006).
21. J. S. Pratt and J. H. Eberly, Phys. Rev. B **64**, 195314 (2001).
22. G. Lagmago Kamta and A.F. Starace, Phys. Rev. Lett. **88**, 107901(2002).
23. J. Wang, H. Batelaan, J. Podany, A. F. Starace, J. Phys. B **39**, 4343 (2006).
24. D. V. Khveshchenko, Phys. Rev. B **68**, 193307 (2003).
25. I. A. Grigorenko and D. V. Khveshchenko, Phys. Rev. Lett. **94**, 040506 (2005).
26. J. S. Pratt, Phys. Rev. Lett. **93**, 237205 (2004).
27. A. R. R. Carvalho, F. Mintert and A. Buchleitner, Phys. Rev. Lett. **93**, 230501 (2004).
28. A. R. R. Carvalho, F. Mintert, S. Palzer and A. Buchleitner, arXiv: quant-ph/0508114.
29. C.P. Sun, et al., Eur. Phys. J. D **13**, 145 (2001).
30. L. Lamata, J. Leon and D. Salgado, Phys. Rev. A **73**, 052325 (2006).
31. L. Diosi, in *Irreversible Quantum Dynamics*, edited by F. Benatti and R. Floreanini (Springer, New York, 2003), pp. 157-163.
32. P.J. Dodd and J.J. Halliwell, Phys. Rev. A **69**, 052105 (2004)
33. P.J. Dodd, Phys. Rev. A **69**, 052106 (2004).
34. K. Zyczkowski, P. Horodecki, M. Horodecki and R. Horodecki, Phys. Rev. A **65**, 012101 (2002).
35. M. F. Santos, P. Milman, L. Davidovich and N. Zagury, Phys. Rev. A **73**, 022313 (2006).
36. The master equations can be derived from many different routes, for example, via quantum trajectories, see T. Yu, L. Diosi, N. Gisin and W. Strunz, Phy. Rev. A **61**, 91 (1999) and W. Strunz and T. Yu, Phys. Rev. A **69**, 052115 (2004).
37. K. Kraus, *States, Effect, and Operations: Fundamental Notions in Quantum Theory* (Springer-Verlag, Berlin, 1983).
38. See W.K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998); S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).
39. S. Bose, I. Fuentes-Guridi, P. L. Knight, and V. Vedral, Phys. Rev. Lett. **87**, 050401 (2001).
40. R. Werner, Phys. Rev. A **40**, 4277 (1989).
41. Werner states are a mixture of maximal entanglement via $|\Psi^-\rangle$, the singlet Bell state, and maximal incoherence via the two-particle identity I_4 [40] and are entangled only for $1 \geq F \geq \frac{1}{2}$. They violate Bell inequalities for a still smaller range of fidelities, but can nevertheless be used for teleportation outside this zone. They have been reported in parametric down conversion, for example: M. Barbieri, F. De Martini, G. Di Depi and P. Mataloni, Phys. Rev. Lett. **92**, 177901 (2004).
42. T. Yu and J. H. Eberly, in preparation.
43. K. Zyczkowski, P. Horodecki, A. Sanpera and M. Lewenstein, Phys. Rev. A **58**, 883 (1998).

44. M. Yönac, T. Yu and J. H. Eberly, J. Phys. B **39**, S621 (2006) and arXiv: quant-ph/0602206.
45. C. Anastopoulos and B. L. Hu, Phys. Rev. A **62**, 033821 (2000); S. Shresta, C. Anastopoulos, A. Dragulescu and B. L. Hu, Phys. Rev. A **71**, 022109 (2005).