Quantum Field Theory and Particle Physics (preliminary version 1)

Book · January 2016

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Introduction and References

This book-broject contains my lectures on quantum field theory (QFT) which were delivered during the academic years 2010-2011, 2011-2012 and 2012-2013 at the University of Annaba to first year and second year master students in theoretical physics. Each part of the book covers roughly a semester consisting of 13 weeks of teaching with 2 lectures and 1 recitation per week. In our master program quantum field theory is formally organized as an anual course so either part I and part II can be used as the material for the course or part I and part III. Another possibility is to merge part II and part III in such a way that the content fits within one semester as we will discuss further below.

Part I is essential since we lay in it the foundations and the language of QFT, although I think now the third chapter of this part should be shortened in some fashion. Part II and part III are independent unites so we can do either one in the second semester. Part II deals mainly with the problem of quantization and renormalization of electrodynamics using the canonical approach while part III deals with path integral formulation, gauge theory and the renormalization group. [The last chapter on the renormalization group was not actually covered with the other two chapters of part III in a single semester. In fact it was delivered informally to master and doctoral students].

In my view now a merger of part II and part III in which the last chapter on the renormalization group is completely suppressed (although in my opinion it is the most important chapter of this book), the other two chapters of part III and the last two chapters of part II are shortened considerably may fit within one single semester. Our actual experience has, on the other hand, been as shown on table (1).

The three main and central references of this book were: Strathdee lecture notes for part I and chapter two of part II, Peskin and Schroeder for part II especially the last chapter and the second chapter of part III and Zinn-Justin for the last chapter on the renormalization group of part III. Chapter one of part II on the canonical quantization of the electromagnetic field follows Greiner and Reinhardt. Chapter one of part III on the path integral formulation and the effective action follows Randjbar-Daemi lecture notes. I have also benefited from many other books and reviews; I only mention here A.M.Polyakov and J.Smit books and K.Wilson and J. Kogut review. A far from complete list of references is given in the bibliography.

Year	Spring	Fall
2011	Part I with the exception of section 3.6.	Part II with the exception of sections 6.6, 7.4 and 7.8.
2012	Part I with the exception of section 3.6.	Part III with the exception of section 8.5 and chapter 10.

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Part I

Free Fields, Canonical Quantization and Feynman Diagrams

2

Relativistic Quantum Mechanics

2.1 Special Relativity

2.1.1 Postulates

Classical mechanics obeys the principle of relativity which states that the laws of nature take the same form in all inertial frames. An inertial frame is any frame in which Newton's first law holds. Therefore all other frames which move with a constant velocity with respect to a given inertial frame are also inertial frames.

Any two inertial frames O and $O^{'}$ can be related by a Galilean transformation which is of the general form

$$\begin{aligned} t^{'} &= t+\tau \\ \vec{x}^{'} &= R\vec{x}+\vec{v}t+\vec{d}. \end{aligned} \tag{2.1}$$

In above R is a constant orthogonal matrix, $\vec{d}$ and $\vec{v}$ are constant vectors and τ is a constant scalar. Thus the observer $O^{'}$ sees the coordinates axes of O rotated by R, moving with a velocity $\vec{v}$, translated by $\vec{d}$ and it sees the clock of O running behind by the amount τ. The set of all transformations of the form (2.1) form a 10-parameter group called the Galilean group.

The invariance/covariance of the equations of motion under these transformations which is called Galilean invariance/covariance is the precise statement of the principle of Galilean relativity.

In contrast to the laws of classical mechanics the laws of classical electrodynamics do not obey the Galilean principle of relativity. Before the advent of the theory of special relativity the laws of electrodynamics were thought to hold only in the inertial reference frame which is at rest with respect to an invisible medium filling all space known as the ether. For example electromagnetic waves were thought to propagate through the vacuum at a speed relative to the ether equal to the speed of light $c=1/\sqrt{\mu_0\epsilon_0}=3\times 10^8 m/s$.

The motion of the earth through the ether creates an ether wind. Thus only by measuring the speed of light in the direction of the ether wind we can get the value c whereas measuring it in any other direction will give a different result. In other words we can detect the ether by measuring the speed of light in different directions which is precisely what Michelson and Morley tried to do in their famous experiments. The outcome of these experiments was always negative in the sense that the speed of light was found exactly the same equal to c in all directions.

The theory of special relativity was the first to accommodate this empirical finding by postulating that the speed of light is the same in all inertial reference frames, i.e. there is no ether. Furthermore it postulates that classical electrodynamics (and physical laws in general) must hold in all inertial reference frames. This is the principle of relativity although now its precise statement can not be given in terms of the invariance/covariance under Galilean transformations but in terms of the invariance/covariance under Lorentz transformations which we will discuss further in the next section.

Einstein's original motivation behind the principle of relativity comes from the physics of the electromotive force. The interaction between a conductor and a magnet in the reference frame where the conductor is moving and the magnet is at rest is known to result in an emotional emf. The charges in the moving conductor will experience a magnetic force given by the Lorentz force law. As a consequence a current will flow in the conductor with an induced motional emf given by the flux rule $\mathcal{E} = -d\Phi/dt$. In the reference frame where the conductor is at rest and the magnet is moving there is no magnetic force acting on the charges. However the moving magnet generates a changing magnetic field which by Faraday's law induces an electric field. As a consequence in the rest frame of the conductor the charges experience an electric force which causes a current to flow with an induced transformer emf given precisely by the flux rule, viz $\mathcal{E} = -d\Phi/dt$.

So in summary although the two observers associated with the states of rest of the conductor and the magnet have different interpretations of the process their predictions are in perfect agreement. This indeed suggests as pointed out first by Einstein that the laws of classical electrodynamics are the same in all inertial reference frames.

The two fundamental postulates of special relativity are therefore:

- The principle of relativity: The laws of physics take the same form in all inertial reference frames.

- The constancy of the speed of light: The speed of light in vacuum is the same in all inertial reference frames.

2.1.2 Relativistic Effects

The gedanken experiments we will discuss here might be called "The train-and-platform thought experiments".

Relativity of Simultaneity We consider an observer $O^{'}$ in the middle of a freight car moving at a speed v with respect to the ground and a second observer O standing on a platform. A light bulb hanging in the center of the car is switched on just as the two observers pass each other.

It is clear that with respect to the observer $O^{'}$ light will reach the front end A and the back end B of the freight car at the same time. The two events "light reaches the front end" and "light reaches the back end" are simultaneous.

According to the second postulate light propagates with the same velocity with respect to the observer O. This observer sees the back end B moving toward the point at which the flash was given off and the front end A moving away from it. Thus light will reach B before it reaches A. In other words with the respect to O the event " light reaches the back end" happens before the event "light reaches the front end".

Time Dilation Let us now ask the question: How long does it take a light ray to travel from the bulb to the floor?

Let us call h the height of the freight car. It is clear that with respect to $O^{'}$ the time spent by the light ray between the bulb and the floor is

$$\Delta t^{'} = \frac{h}{c}. \tag{2.2}$$

The observer O will measure a time Δt during which the freight car moves a horizontal distance $v\Delta t$. The trajectory of the light ray is not given by the vertical distance h but by the hypotenuse of the right triangle with h and vdt as the other two sides. Thus with respect to O the light ray travels a longer distance given by $\sqrt{h^2+v^2\Delta t^2}$ and therefore the time spent is

$$\Delta t = \frac{\sqrt{h^2+v^2\Delta t^2}}{c}. \tag{2.3}$$

Solving for Δt we get

$$\Delta t = \gamma\frac{h}{c} = \gamma\Delta t^{'}. \tag{2.4}$$

The factor γ is known as Lorentz factor and it is given by

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{2.5}$$

Hence we obtain

$$\Delta t^{'} = \sqrt{1-\frac{v^2}{c^2}}\Delta t \leq \Delta t. \tag{2.6}$$

The time measured on the train is shorter than the time measured on the ground. In other words moving clocks run slow. This is called time dilation.

Lorentz Contraction We place now a lamp at the back end B of the freight car and a mirror at the front end A. Then we ask the question: How long does it take a light ray to travel from the lamp to the mirror and back?

Again with respect to the observer $O^{'}$ the answer is simple. If $\Delta x^{'}$ is the length of the freight car measured by $O^{'}$ then the time spent by the light ray in the round trip between the lamp and the mirror is

$$\Delta t^{'} = 2\frac{\Delta x^{'}}{c}. \tag{2.7}$$

Let Δx be the length of the freight car measured by O and Δt_1 be the time for the light ray to reach the front end A. Then clearly

$$c\Delta t_1 = \Delta x + v\Delta t_1. \tag{2.8}$$

The term $v\Delta t_1$ is the distance traveled by the train during the time Δt_1. Let Δt_2 be the time for the light ray to return to the back end B. Then

$$c\Delta t_2 = \Delta x - v\Delta t_2. \tag{2.9}$$

The time spent by the light ray in the round trip between the lamp and the mirror is therefore

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{\Delta x}{c-v} + \frac{\Delta x}{c+v} = 2\gamma^2 \frac{\Delta x}{c}. \tag{2.10}$$

The time intervals Δt and $\Delta t^{'}$ are related by time dilation, viz

$$\Delta t = \gamma \Delta t^{'}. \tag{2.11}$$

This is equivalent to

$$\Delta x^{'} = \gamma \Delta x \geq \Delta x. \tag{2.12}$$

The length measured on the train is longer than the length measured on the ground. In other words moving objects are shortened. This is called Lorentz contraction.

We point out here that only the length parallel to the direction of motion is contracted while lengths perpendicular to the direction of the motion remain not contracted.

2.1.3 Lorentz Transformations: Boosts

Any physical process consists of a collection of events. Any event takes place at a given point (x, y, z) of space at an instant of time t. Lorentz transformations relate the coordinates (x, y, z, t) of a given event in an inertial reference frame O to the coordinates $(x^{'}, y^{'}, z^{'}, t^{'})$ of the same event in another inertial reference frame $O^{'}$.

Let (x, y, z, t) be the coordinates in O of an event E. The projection of E onto the x axis is given by the point P which has the coordinates $(x, 0, 0, t)$. For simplicity we will assume that the observer $O^{'}$ moves with respect to the observer O at a constant speed v along the x axis. At time $t = 0$ the two observers O and $O^{'}$ coincides. After time t the observer $O^{'}$ moves a distance vt on the x axis. Let d be the distance between $O^{'}$ and P as measured by O. Then clearly

$$x = d + vt. \tag{2.13}$$

Before the theory of special relativity the coordinate $x^{'}$ of the event E in the reference frame $O^{'}$ is taken to be equal to the distance d. We get therefore the transformation laws

$$\begin{aligned} x^{'} &= x - vt \\ y^{'} &= y \\ z^{'} &= z \\ t^{'} &= t. \end{aligned} \tag{2.14}$$

This is a Galilean transformation. Indeed this is a special case of (2.1).

As we have already seen Einstein's postulates lead to Lorentz contraction. In other words the distance between $O^{'}$ and P measured by the observer $O^{'}$ which is precisely the coordinate $x^{'}$ is larger than d. More precisely

$$x^{'} = \gamma d. \tag{2.15}$$

Hence

$$x^{'} = \gamma (x - vt). \tag{2.16}$$

Einstein's postulates lead also to time dilation and relativity of simultaneity. Thus the time of the event E measured by $O^{'}$ is different from t. Since the observer O moves with respect to $O^{'}$ at a speed v in the negative x direction we must have

$$x=\gamma(x^{'}+vt^{'}). \tag{2.17}$$

Thus we get

$$t^{'}=\gamma(t-\frac{v}{c^2}x). \tag{2.18}$$

In summary we get the transformation laws

$$\begin{aligned} x^{'} &= \gamma(x-vt)\\ y^{'} &= y\\ z^{'} &= z\\ t^{'} &= \gamma(t-\frac{v}{c^2}x). \end{aligned} \tag{2.19}$$

This is a special Lorentz transformation which is a boost along the x axis.

Let us look at the clock found at the origin of the reference frame $O^{'}$. We set then $x^{'}=0$ in the above equations. We get immediately the time dilation effect, viz

$$t^{'}=\frac{t}{\gamma}. \tag{2.20}$$

At time $t=0$ the clocks in $O^{'}$ read different times depending on their location since

$$t^{'}=-\gamma\frac{v}{c^2}x. \tag{2.21}$$

Hence moving clocks can not be synchronized.

We consider now two events A and B with coordinates (x_A,t_A) and (x_B,t_B) in O and coordinates $(x_A^{'},t_A^{'})$ and $(x_B^{'},t_B^{'})$ in $O^{'}$. We can immediately compute

$$\Delta t^{'}=\gamma(\Delta t-\frac{v}{c^2}\Delta x). \tag{2.22}$$

Thus if the two events are simultaneous with respect to O, i.e. $\Delta t=0$ they are not simultaneous with respect to $O^{'}$ since

$$\Delta t^{'}=-\gamma\frac{v}{c^2}\Delta x. \tag{2.23}$$

2.1.4 Spacetime

The above Lorentz boost transformation can be rewritten as

$$\begin{aligned} x^{0^{'}} &= \gamma(x^0-\beta x^1)\\ x^{1^{'}} &= \gamma(x^1-\beta x^0)\\ x^{2^{'}} &= x^2\\ x^{3^{'}} &= x^3. \end{aligned} \tag{2.24}$$

In the above equation

$$x^0=ct\ ,\ x^1=x\ ,\ x^2=y\ ,\ x^3=z. \tag{2.25}$$

$$\beta = \frac{v}{c} \ , \ \gamma = \sqrt{1-\beta^2}. \tag{2.26}$$

This can also be rewritten as

$$x^{\mu'} = \sum_{\nu=0}^{4} \Lambda^{\mu}_{\nu} x^{\nu}. \tag{2.27}$$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.28}$$

The matrix Λ is the Lorentz boost transformation matrix. A general Lorentz boost transformation can be obtained if the relative motion of the two inertial reference frames O and $O^{'}$ is along an arbitrary direction in space. The transformation law of the coordinates x^{μ} will still be given by (2.27) with a more complicated matrix Λ. A general Lorentz transformation can be written as a product of a rotation and a boost along a direction $\hat{n}$ given by

$$\begin{aligned} x^{'0} &= x^0 \cosh\alpha - \hat{n}\vec{x}\sinh\alpha \\ \vec{x}^{'} &= \vec{x} + \hat{n}\bigg((\cosh\alpha - 1)\hat{n}\vec{x} - x^0 \sinh\alpha\bigg). \end{aligned} \tag{2.29}$$

$$\frac{\vec{v}}{c} = \tanh\alpha \ \hat{n}. \tag{2.30}$$

Indeed the set of all Lorentz transformations contains rotations as a subset.

The set of coordinates (x^0, x^1, x^2, x^3) which transforms under Lorentz transformations as $x^{\mu'} = \Lambda^{\mu}_{\nu} x^{\nu}$ will be called a 4−vector in analogy with the set of coordinates (x^1, x^2, x^3) which is called a vector because it transforms under rotations as $x^{a'} = R^a_b x^b$. Thus in general a 4−vector a is any set of numbers (a^0, a^1, a^2, a^3) which transforms as (x^0, x^1, x^2, x^3) under Lorentz transformations, viz

$$a^{\mu'} = \sum_{\nu=0}^{4} \Lambda^{\mu}_{\nu} a^{\nu}. \tag{2.31}$$

For the particular Lorentz transformation (2.28) we have

$$\begin{aligned} a^{0'} &= \gamma(a^0 - \beta a^1) \\ a^{1'} &= \gamma(a^1 - \beta a^0) \\ a^{2'} &= a^2 \\ a^{3'} &= a^3. \end{aligned} \tag{2.32}$$

The numbers a^{μ} are called the contravariant components of the 4−vector a. We define the covariant components a_{μ} by

$$a_0 = a^0 \ , \ a_1 = -a^1 \ , \ a_2 = -a^2 \ , \ a_3 = -a^3. \tag{2.33}$$

By using the Lorentz transformation (2.32) we verify given any two 4−vectors a and b the identity

$$a^{0'}b^{0'} - a^{1'}b^{1'} - a^{2'}b^{2'} - a^{3'}b^{3'} = a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3. \tag{2.34}$$

In fact we can show that this identity holds for all Lorentz transformations. We recall that under rotations the scalar product $\vec{a}\vec{b}$ of any two vectors $\vec{a}$ and $\vec{b}$ is invariant, i.e.

$$a^{1'}b^{1'} + a^{2'}b^{2'} + a^{3'}b^{3'} = a^1b^1 + a^2b^2 + a^3b^3. \tag{2.35}$$

The 4-dimensional scalar product must therefore be defined by the Lorentz invariant combination $a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3$, namely

$$\begin{aligned} ab &= a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3 \\ &= \sum_{\mu=0}^{3} a_\mu b^\mu \\ &= a_\mu b^\mu. \end{aligned} \tag{2.36}$$

In the last equation we have employed the so-called Einstein summation convention, i.e. a repeated index is summed over.

We define the separation 4−vector Δx between two events A and B occurring at the points $(x_A^0, x_A^1, x_A^2, x_A^3)$ and $(x_B^0, x_B^1, x_B^2, x_B^3)$ by the components

$$\Delta x^\mu = x_A^\mu - x_B^\mu. \tag{2.37}$$

The distance squared between the two events A and B which is called the interval between A and B is defined by

$$\Delta s^2 = \Delta x_\mu \Delta x^\mu = c^2\Delta t^2 - \Delta\vec{x}^2. \tag{2.38}$$

This is a Lorentz invariant quantity. However it could be positive, negative or zero.

In the case $\Delta s^2 > 0$ the interval is called timelike. There exists an inertial reference frame in which the two events occur at the same place and are only separated temporally.

In the case $\Delta s^2 < 0$ the interval is called spacelike. There exists an inertial reference frame in which the two events occur at the same time and are only separated in space.

In the case $\Delta s^2 = 0$ the interval is called lightlike. The two events are connected by a signal traveling at the speed of light.

2.1.5 Metric

The interval ds^2 between two infinitesimally close events A and B in spacetime with position 4−vectors x_A^μ and $x_B^\mu = x_A^\mu + dx^\mu$ is given by

$$\begin{aligned} ds^2 &= \sum_{\mu=0}^{3}(x_A - x_B)_\mu (x_A - x_B)^\mu \\ &= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \\ &= c^2(dt)^2 - (d\vec{x})^2. \end{aligned} \tag{2.39}$$

We can also write this interval as (using also Einstein's summation convention)

$$\begin{aligned} ds^2 &= \sum_{\mu,\nu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} dx^\mu dx^\nu \\ &= \sum_{\mu,\nu=0}^{3} \eta^{\mu\nu} dx_\mu dx_\nu = \eta^{\mu\nu} dx_\mu dx_\nu. \end{aligned} \tag{2.40}$$

The 4×4 matrix η is called the metric tensor and it is given by

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.41}$$

Clearly we can also write

$$ds^2 \quad = \quad \sum_{\mu,\nu=0}^{3} \eta_\mu^\nu dx^\mu dx_\nu = \eta_\mu^\nu dx^\mu dx_\nu. \tag{2.42}$$

In this case

$$\eta_\mu^\nu = \delta_\mu^\nu. \tag{2.43}$$

The metric η is used to lower and raise Lorentz indices, viz

$$x_\mu = \eta_{\mu\nu} x^\nu. \tag{2.44}$$

The interval ds^2 is invariant under Poincare transformations which combine translations a with Lorentz transformations Λ:

$$x^\mu \longrightarrow x^{'\mu} = \Lambda^\mu_\nu x^\nu + a^\mu. \tag{2.45}$$

We compute

$$ds^2 = \eta_{\mu\nu} dx^{'\mu} dx^{'\nu} = \eta_{\mu\nu} dx^\mu dx^\nu. \tag{2.46}$$

This leads to the condition

$$\eta_{\mu\nu} \Lambda^\mu_\rho \Lambda^\nu_\sigma = \eta_{\rho\sigma} \Leftrightarrow \Lambda^T \eta \Lambda = \eta. \tag{2.47}$$

2.2 Klein-Gordon Equation

The non-relativistic energy-momentum relation reads

$$E = \frac{\vec{p}^2}{2m} + V. \tag{2.48}$$

The correspondence principle is

$$E \longrightarrow i\hbar \frac{\partial}{\partial t} \ , \ \vec{p} \longrightarrow \frac{\hbar}{i} \vec{\nabla}. \tag{2.49}$$

This yields immediately the Schrodinger equation

$$\bigg(-\frac{\hbar^2}{2m} \nabla^2 + V \bigg) \psi = i\hbar \frac{\partial \psi}{\partial t}. \tag{2.50}$$

We will only consider the free case,i.e. $V = 0$. We have then

$$-\frac{\hbar^2}{2m} \nabla^2 \psi = i\hbar \frac{\partial \psi}{\partial t}. \tag{2.51}$$

The energy-momentum 4−vector is given by

$$p^\mu=(p^0,p^1,p^2,p^3)=(\frac{E}{c},\vec{p}). \quad (2.52)$$

The relativistic momentum and energy are defined by

$$\vec{p}=\frac{m\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}\ ,\ E=\frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}. \quad (2.53)$$

The energy-momentum 4−vector satisfies

$$p^\mu p_\mu=\frac{E^2}{c^2}-\vec{p}^2=m^2c^2. \quad (2.54)$$

The relativistic energy-momentum relation is therefore given by

$$\vec{p}^2c^2+m^2c^4=E^2. \quad (2.55)$$

Thus the free Schrodinger equation will be replaced by the relativistic wave equation

$$(-\hbar^2c^2\nabla^2+m^2c^4)\phi=-\hbar^2\frac{\partial^2\phi}{\partial t^2}. \quad (2.56)$$

This can also be rewritten as

$$\bigg(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2-\frac{m^2c^2}{\hbar^2}\bigg)\phi=0. \quad (2.57)$$

This is Klein-Gordon equation. In contrast with the Schrodinger equation the Klein-Gordon equation is a second-order differential equation. In relativistic notation we have

$$E\longrightarrow i\hbar\frac{\partial}{\partial t}\Leftrightarrow p_0\longrightarrow i\hbar\partial_0\ ,\ \partial_0=\frac{\partial}{\partial x^0}=\frac{1}{c}\frac{\partial}{\partial t}. \quad (2.58)$$

$$\vec{p}\longrightarrow\frac{\hbar}{i}\vec{\nabla}\Leftrightarrow p_i\longrightarrow i\hbar\partial_i\ ,\ \partial_i=\frac{\partial}{\partial x^i}. \quad (2.59)$$

In other words

$$p_\mu\longrightarrow i\hbar\partial_\mu\ ,\ \partial_\mu=\frac{\partial}{\partial x^\mu}. \quad (2.60)$$

$$p_\mu p^\mu\longrightarrow-\hbar^2\partial_\mu\partial^\mu=\hbar^2\bigg(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\bigg). \quad (2.61)$$

The covariant form of the Klein-Gordon equation is

$$\bigg(\partial_\mu\partial^\mu+\frac{m^2c^2}{\hbar^2}\bigg)\phi=0. \quad (2.62)$$

Free solutions are of the form

$$\phi(t,\vec{x})=e^{-\frac{i}{\hbar}px}\ ,\ px=p_\mu x^\mu=Et-\vec{p}\vec{x}. \quad (2.63)$$

Indeed we compute

$$\partial_\mu \partial^\mu \phi(t,\vec{x}) = -\frac{1}{c^2\hbar^2}(E^2 - \vec{p}^2c^2)\phi(t,\vec{x}). \tag{2.64}$$

Thus we must have

$$E^2 - \vec{p}^2c^2 = m^2c^4. \tag{2.65}$$

In other words

$$E^2 = \pm\sqrt{\vec{p}^2c^2 + m^2c^4}. \tag{2.66}$$

There exists therefore negative-energy solutions. The energy gap is $2mc^2$. As it stands the existence of negative-energy solutions means that the spectrum is not bounded from below and as a consequence an arbitrarily large amount of energy can be extracted. This is a severe problem for a single-particle wave equation. However these negative-energy solutions, as we will see shortly, will be related to antiparticles.

From the two equations

$$\phi^*\left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right)\phi = 0, \tag{2.67}$$

$$\phi\left(\partial_\mu\partial^\mu + \frac{m^2c^2}{\hbar^2}\right)\phi^* = 0, \tag{2.68}$$

we get the continuity equation

$$\partial^\mu J_\mu = 0, \tag{2.69}$$

where

$$J_\mu = \frac{i\hbar}{2m}[\phi^*\partial_\mu\phi - \phi\partial_\mu\phi^*]. \tag{2.70}$$

We have included the factor $i\hbar/2m$ in order that the zero component J_0 has the dimension of a probability density. The continuity equation can also be put in the form

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\vec{J} = 0, \tag{2.71}$$

where

$$\rho = \frac{J_0}{c} = \frac{i\hbar}{2mc^2}[\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}]. \tag{2.72}$$

$$\vec{J} = -\frac{i\hbar}{2mc}[\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*]. \tag{2.73}$$

Clearly the zero component J_0 is not positive definite and hence it can be a probability density. This is due to the fact that the Klein-Gordon equation is second-order.

The Dirac equation is a relativistic wave equation which is a first-order differential equation. The corresponding probability density will therefore be positive definite. However negative-energy solutions will still be present.

2.3 Dirac Equation

Dirac equation is a first-order differential equation of the same form as the Schrodinger equation, viz

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi. \tag{2.74}$$

In order to derive the form of the Hamiltonian H we go back to the relativistic energy-momentum relation

$$p_\mu p^\mu - m^2c^2 = 0. \tag{2.75}$$

The only requirement on H is that it must be linear in spatial derivatives since we want space and time to be on equal footing. We thus factor out the above equation as follows

$$\begin{aligned} p_\mu p^\mu - m^2c^2 &= (\gamma^\mu p_\mu + mc)(\beta^\nu p_\nu - mc) \\ &= \gamma^\mu\beta^\nu p_\mu p_\nu - mc(\gamma^\mu - \beta^\mu)p_\mu - m^2c^2. \end{aligned} \tag{2.76}$$

We must therefore have $\beta^\mu = \gamma^\mu$, i.e.

$$p_\mu p^\mu = \gamma^\mu\gamma^\nu p_\mu p_\nu. \tag{2.77}$$

This is equivalent to

$$\begin{aligned} p_0^2 - p_1^2 - p_2^2 - p_3^2 &= (\gamma^0)^2p_0^2 + (\gamma^1)^2p_1^2 + (\gamma^2)^2p_2^2 + (\gamma^3)^2p_3^2 \\ &+ (\gamma^1\gamma^2 + \gamma^2\gamma^1)p_1p_2 + (\gamma^1\gamma^3 + \gamma^3\gamma^1)p_1p_3 + (\gamma^2\gamma^3 + \gamma^3\gamma^2)p_2p_3 \\ &+ (\gamma^1\gamma^0 + \gamma^0\gamma^1)p_1p_0 + (\gamma^2\gamma^0 + \gamma^0\gamma^2)p_2p_0 + (\gamma^3\gamma^0 + \gamma^0\gamma^3)p_3p_0. \end{aligned} \tag{2.78}$$

Clearly the objects γ^μ can not be complex numbers since we must have

$$\begin{aligned} &(\gamma^0)^2 = 1 \ , \ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1 \\ &\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0. \end{aligned} \tag{2.79}$$

These conditions can be rewritten in a compact form as

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}. \tag{2.80}$$

This algebra is an example of a Clifford algebra and the solutions are matrices γ^μ which are called Dirac matrices. In four-dimensional Minkowski space the smallest Dirac matrices must be 4×4 matrices. All 4×4 representations are unitarily equivalent. We choose the so-called Weyl or chiral representation given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix} \ , \ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{2.81}$$

The Pauli matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.82}$$

Remark that

$$(\gamma^0)^+ = \gamma^0 \ , \ (\gamma^i)^+ = -\gamma^i \Leftrightarrow (\gamma^\mu)^+ = \gamma^0\gamma^\mu\gamma^0. \tag{2.83}$$

The relativistic energy-momentum relation becomes

$$p_\mu p^\mu - m^2c^2 \quad = \quad (\gamma^\mu p_\mu + mc)(\gamma^\nu p_\nu - mc) = 0. \tag{2.84}$$

Thus either $\gamma^\mu p_\mu + mc = 0$ or $\gamma^\mu p_\mu - mc = 0$. The convention is to take

$$\gamma^\mu p_\mu - mc = 0. \tag{2.85}$$

By applying the correspondence principle $p_\mu \longrightarrow i\hbar\partial_\mu$ we obtain the relativistic wave equation

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0. \tag{2.86}$$

This is the Dirac equation in a covariant form. Let us introduce the Feynamn "slash" defined by

$$\partial\!\!\!/ = \gamma^\mu\partial_\mu. \tag{2.87}$$

$$(i\hbar\partial\!\!\!/ - mc)\psi = 0. \tag{2.88}$$

Since the γ matrices are 4×4 the wave function ψ must be a four-component object which we call a Dirac spinor. Thus we have

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}. \tag{2.89}$$

The Hermitian conjugate of the Dirac equation (2.100) is

$$\psi^+(i\hbar(\gamma^\mu)^+\overleftarrow{\partial_\mu} + mc) = 0. \tag{2.90}$$

In other words

$$\psi^+(i\hbar\gamma^0\gamma^\mu\gamma^0\overleftarrow{\partial_\mu} + mc) = 0. \tag{2.91}$$

The Hermitian conjugate of a Dirac spinor is not ψ^+ but it is defined by

$$\bar{\psi} = \psi^+\gamma^0. \tag{2.92}$$

Thus the Hermitian conjugate of the Dirac equation is

$$\bar{\psi}(i\hbar\gamma^\mu\overleftarrow{\partial_\mu} + mc) = 0. \tag{2.93}$$

Equivalently

$$\bar{\psi}(i\hbar\overleftarrow{\partial\!\!\!/} + mc) = 0. \tag{2.94}$$

Putting (2.88) and (2.94) together we obtain

$$\bar{\psi}(i\hbar\overleftarrow{\partial\!\!\!/} + i\hbar\overrightarrow{\partial\!\!\!/})\psi = 0. \tag{2.95}$$

We obtain the continuity equation

$$\partial_\mu J^\mu = 0 \ , \ J^\mu = \bar{\psi}\gamma^\mu\psi. \tag{2.96}$$

Explicitly we have

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\vec{J}=0. \tag{2.97}$$

$$\rho=\frac{J^0}{c}=\frac{1}{c}\bar{\psi}\gamma^0\psi=\frac{1}{c}\psi^+\psi. \tag{2.98}$$

$$\vec{J}=\bar{\psi}\vec{\gamma}\psi=\psi^+\vec{\alpha}\psi. \tag{2.99}$$

The probability density ρ is positive definite as desired.

2.4 Free Solutions of The Dirac Equation

We seek solutions of the Dirac equation

$$(i\hbar\gamma^\mu\partial_\mu-mc)\psi=0. \tag{2.100}$$

The plane-wave solutions are of the form

$$\psi(x)=a\ e^{-\frac{i}{\hbar}px}u(p). \tag{2.101}$$

Explicitly

$$\psi(t,\vec{x})=a\ e^{-\frac{i}{\hbar}(Et-\vec{p}\vec{x})}u(E,\vec{p}). \tag{2.102}$$

The spinor $u(p)$ must satisfy

$$(\gamma^\mu p_\mu-mc)u=0. \tag{2.103}$$

We write

$$u=\begin{pmatrix} u_A \\ u_B \end{pmatrix}. \tag{2.104}$$

We compute

$$\gamma^\mu p_\mu-mc=\begin{pmatrix} -mc & \frac{E}{c}-\vec{\sigma}\vec{p} \\ \frac{E}{c}+\vec{\sigma}\vec{p} & -mc \end{pmatrix}. \tag{2.105}$$

We get immediately

$$u_A=\frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}u_B. \tag{2.106}$$

$$u_B=\frac{\frac{E}{c}+\vec{\sigma}\vec{p}}{mc}u_A. \tag{2.107}$$

A consistency condition is

$$u_A=\frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}\frac{\frac{E}{c}+\vec{\sigma}\vec{p}}{mc}u_A=\frac{\frac{E^2}{c^2}-(\vec{\sigma}\vec{p})^2}{m^2c^2}u_A. \tag{2.108}$$

Thus one must have

$$\frac{E^2}{c^2} - (\vec{\sigma}\vec{p})^2 = m^2c^2 \Leftrightarrow E^2 = \vec{p}^2c^2 + m^2c^4. \tag{2.109}$$

Thus we have a single condition

$$u_B = \frac{\frac{E}{c} + \vec{\sigma}\vec{p}}{mc} u_A. \tag{2.110}$$

There are four possible solutions. These are

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow u^{(1)} = N^{(1)} \begin{pmatrix} 1 \\ 0 \\ \frac{\frac{E}{c}+p^3}{mc} \\ \frac{p^1+ip^2}{mc} \end{pmatrix}. \tag{2.111}$$

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow u^{(4)} = N^{(4)} \begin{pmatrix} 0 \\ 1 \\ \frac{p^1-ip^2}{mc} \\ \frac{\frac{E}{c}-p^3}{mc} \end{pmatrix}. \tag{2.112}$$

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow u^{(3)} = N^{(3)} \begin{pmatrix} \frac{\frac{E}{c}-p^3}{mc} \\ -\frac{p^1+ip^2}{mc} \\ 1 \\ 0 \end{pmatrix}. \tag{2.113}$$

$$u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow u^{(2)} = N^{(2)} \begin{pmatrix} -\frac{p^1-ip^2}{mc} \\ \frac{\frac{E}{c}+p^3}{mc} \\ 0 \\ 1 \end{pmatrix}. \tag{2.114}$$

The first and the fourth solutions will be normalized such that

$$\bar{u}u = u^+\gamma^0 u = u_A^+ u_B + u_B^+ u_A = 2mc. \tag{2.115}$$

We obtain

$$N^{(1)} = N^{(2)} = \sqrt{\frac{m^2c^2}{\frac{E}{c} + p^3}}. \tag{2.116}$$

Clearly one must have $E \geq 0$ otherwise the square root will not be well defined. In other words $u^{(1)}$ and $u^{(2)}$ correspond to positive-energy solutions associated with particles. The spinors $u^{(i)}(p)$ can be rewritten as

$$u^{(i)} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\xi^i \\ \sqrt{\bar{\sigma}_\mu p^\mu}\xi^i \end{pmatrix}. \tag{2.117}$$

The 2−dimensional spinors ξ^i satisfy

$$(\xi^r)^+\xi^s = \delta^{rs}. \tag{2.118}$$

The remaining spinors $u^{(3)}$ and $u^{(4)}$ must correspond to negative-energy solutions which must be reinterpreted as positive-energy antiparticles. Thus we flip the signs of the energy and the momentum such that the wave function (2.102) becomes

$$\psi(t,\vec{x}) \quad = \quad a\ e^{\frac{i}{\hbar}(Et-\vec{p}\vec{x})}u(-E,-\vec{p}). \tag{2.119}$$

The solutions u^3 and u^4 become

$$v^{(1)}(E,\vec{p}) = u^{(3)}(-E,-\vec{p}) = N^{(3)}\begin{pmatrix} -\frac{\frac{E}{c}-p^3}{mc} \\ \frac{p^1+ip^2}{mc} \\ 1 \\ 0 \end{pmatrix}\ ,\ v^{(2)}(E,\vec{p}) = u^{(4)}(-E,-\vec{p}) = N^{(4)}\begin{pmatrix} 0 \\ 1 \\ -\frac{p^1-ip^2}{mc} \\ -\frac{\frac{E}{c}-p^3}{mc} \end{pmatrix}. \tag{2.120}$$

We impose the normalization condition

$$\bar{v}v = v^+\gamma^0 v = v_A^+ v_B + v_B^+ v_A = -2mc. \tag{2.121}$$

We obtain

$$N^{(3)} = N^{(4)} = \sqrt{\frac{m^2c^2}{\frac{E}{c}-p^3}}. \tag{2.122}$$

The spinors $v^{(i)}(p)$ can be rewritten as

$$v^{(i)} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\eta^i \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\eta^i \end{pmatrix}. \tag{2.123}$$

Again the 2−dimensional spinors η^i satisfy

$$(\eta^r)^+\eta^s = \delta^{rs}. \tag{2.124}$$

2.5 Lorentz Covariance

In this section we will refer to the Klein-Gordon wave function ϕ as a scalar field and to the Dirac wave function ψ as a Dirac spinor field although we are still thinking of them as quantum wave functions and not classical fields.

Scalar Fields: Let us recall that the set of all Lorentz transformations form a group called the Lorentz group. An arbitrary Lorentz transformation acts as

$$x^\mu \longrightarrow x^{'\mu} = \Lambda^\mu{}_\nu x^\nu. \tag{2.125}$$

In the inertial reference frame O the Klein-Gordon wave function is $\phi = \phi(x)$. It is a scalar field. Thus in the transformed reference frame $O^{'}$ the wave function must be $\phi^{'} = \phi^{'}(x^{'})$ where

$$\phi^{'}(x^{'}) = \phi(x). \tag{2.126}$$

For a one-component field this is the only possible linear transformation law. The Klein-Gordon equation in the reference frame $O^{'}$ if it holds is of the form

$$\left(\partial^{'}_{\mu}\partial^{'\mu}+\frac{m^2c^2}{\hbar^2}\right)\phi^{'}(x^{'})=0. \quad (2.127)$$

It is not difficult to show that

$$\partial^{'}_{\mu}\partial^{'\mu}=\partial_{\mu}\partial^{\mu} \quad (2.128)$$

The Klein-Gordon (2.127) becomes

$$\left(\partial_{\mu}\partial^{\mu}+\frac{m^2c^2}{\hbar^2}\right)\phi(x)=0. \quad (2.129)$$

Vector Fields: Let $V^{\mu}=V^{\mu}(x)$ be an arbitrary vector field (for example $\partial^{\mu}\phi$ and the electromagnetic vector potential A^{μ}). Under Lorentz transformations it must transform as a $4-$vector, i.e. as in (2.125) and hence

$$V^{'\mu}(x^{'})=\Lambda^{\mu}_{\nu}V^{\nu}(x). \quad (2.130)$$

This should be contrasted with the transformation law of an ordinary vector field $V^{i}(x)$ under rotations in three dimensional space given by

$$V^{'i}(x^{'})=R^{ij}V^{j}(x). \quad (2.131)$$

The group of rotations in three dimensional space is a continuous group. The set of infinitesimal transformations (the transformations near the identity) form a vector space which we call the Lie algebra of the group. The basis vectors of this vector space are called the generators of the Lie algebra and they are given by the angular momentum operators J^{i} which satisfy the commutation relations

$$[J^{i},J^{j}]=i\hbar\epsilon^{ijk}J^{k}. \quad (2.132)$$

A rotation with an angle $|\theta|$ about the axis $\hat{\theta}$ is obtained by exponentiation, viz

$$R=e^{-i\theta^{i}J^{i}}. \quad (2.133)$$

The matrices R form an $n-$dimensional representation with $n=2j+1$ where j is the spin quantum number. The angular momentum operators J^{i} are given by

$$J^{i}=-i\hbar\epsilon^{ijk}x^{j}\partial^{k}. \quad (2.134)$$

This is equivalent to

$$\begin{aligned}J^{ij}&=\epsilon^{ijk}J^{k}\\&=-i\hbar(x^{i}\partial^{j}-x^{j}\partial^{i}).\end{aligned} \quad (2.135)$$

Generalization of this result to $4-$dimensional Minkowski space yields the six generators of the Lorentz group given by

$$J^{\mu\nu}=-i\hbar(x^{\mu}\partial^{\nu}-x^{\nu}\partial^{\mu}). \quad (2.136)$$

We compute the commutation relations

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\hbar\Big(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}\Big). \tag{2.137}$$

A solution of (2.137) is given by the 4×4 matrices

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\hbar(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha). \tag{2.138}$$

Equivalently we can write this solution as

$$(\mathcal{J}^{\mu\nu})^\alpha{}_\beta = i\hbar(\eta^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta\eta^{\nu\alpha}). \tag{2.139}$$

This representation is the $4-$dimensional vector representation of the Lorentz group which is denoted by $(1/2, 1/2)$. It is an irreducible representation of the Lorentz group. A scalar field transforms in the trivial representation of the Lorentz group denoted by $(0, 0)$. It remains to determine the transformation properties of spinor fields.

Spinor Fields We go back to the Dirac equation in the form

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0. \tag{2.140}$$

This equation is assumed to be covariant under Lorentz transformations and hence one must have the transformed equation

$$(i\hbar\gamma'^\mu\partial'_\mu - mc)\psi' = 0. \tag{2.141}$$

The Dirac γ matrices are assumed to be invariant under Lorentz transformations and thus

$$\gamma'_\mu = \gamma_\mu. \tag{2.142}$$

The spinor ψ will be assumed to transform under Lorentz transformations linearly, namely

$$\psi(x) \longrightarrow \psi'(x') = S(\Lambda)\psi(x). \tag{2.143}$$

Furthermore we have

$$\partial'_\nu = (\Lambda^{-1})^\mu{}_\nu\partial_\mu. \tag{2.144}$$

Thus equation (2.141) is of the form

$$(i\hbar(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\gamma'^\mu S(\Lambda)\partial_\nu - mc)\psi = 0. \tag{2.145}$$

We can get immediately

$$(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\gamma'^\mu S(\Lambda) = \gamma^\nu. \tag{2.146}$$

Equivalently

$$(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \gamma^\nu. \tag{2.147}$$

This is the transformation law of the γ matrices under Lorentz transformations. Thus the γ matrices are invariant under the simultaneous rotations of the vector and spinor indices under

Lorentz transformations. This is analogous to the fact that Pauli matrices σ^i are invariant under the simultaneous rotations of the vector and spinor indices under spatial rotations.

The matrix $S(\Lambda)$ form a 4−dimensional representation of the Lorentz group which is called the spinor representation. This representation is reducible and it is denoted by $(1/2,0)\oplus(0,1/2)$. It remains to find the matrix $S(\Lambda)$. We consider an infinitesimal Lorentz transformation

$$\Lambda = 1 - \frac{i}{2\hbar}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta} \ , \ \Lambda^{-1} = 1 + \frac{i}{2\hbar}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}. \tag{2.148}$$

We can write $S(\Lambda)$ as

$$S(\Lambda) = 1 - \frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta} \ , \ S^{-1}(\Lambda) = 1 + \frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta}. \tag{2.149}$$

The infinitesimal form of (2.147) is

$$-(\mathcal{J}^{\alpha\beta})^{\mu}{}_{\nu}\gamma_{\mu} = [\gamma_{\nu}, \Gamma^{\alpha\beta}]. \tag{2.150}$$

The fact that the index μ is rotated with $\mathcal{J}^{\alpha\beta}$ means that it is a vector index. The spinor indices are the matrix components of the γ matrices which are rotated with the generators $\Gamma^{\alpha\beta}$. A solution is given by

$$\Gamma^{\mu\nu} = \frac{i\hbar}{4}[\gamma^{\mu}, \gamma^{\nu}]. \tag{2.151}$$

Explicitly

$$\begin{aligned}
\Gamma^{0i} &= \frac{i\hbar}{4}[\gamma^0, \gamma^i] = -\frac{i\hbar}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \\
\Gamma^{ij} &= \frac{i\hbar}{4}[\gamma^i, \gamma^j] = -\frac{i\hbar}{4}\begin{pmatrix} [\sigma^i, \sigma^j] & 0 \\ 0 & [\sigma^i, \sigma^j] \end{pmatrix} = \frac{\hbar}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}.
\end{aligned} \tag{2.152}$$

Clearly Γ^{ij} are the generators of rotations. They are the direct sum of two copies of the generators of rotation in three dimensional space. Thus immediately we conclude that Γ^{0i} are the generators of boosts.

2.6 Exercises and Problems

Scalar Product Show explicitly that the scalar product of two 4−vectors in spacetime is invariant under boosts. Show that the scalar product is then invariant under all Lorentz transformations.

Relativistic Mechanics

- Show that the proper time of a point particle -the proper time is the time measured by an inertial observer flying with the particle- is invariant under Lorentz transformations. We assume that the particle is moving with a velocity $\vec{u}$ with respect to an inertial observer O.
- Define the 4−vector velocity of the particle in spacetime. What is the spatial component.
- Define the energy-momentum 4−vector in spacetime and deduce the relativistic energy.
- Express the energy in terms of the momentum.
- Define the 4−vector force.

Einstein's Velocity Addition Rule Derive the velocity addition rule in special relativity.

Weyl Representation

- Show that the Weyl representation of Dirac matrices given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix} \ , \ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

  solves Dirac-Clifford algebra.

- Show that

$$(\gamma^\mu)^+ = \gamma^0 \gamma^\mu \gamma^0.$$

- Show that the Dirac equation can be put in the form of a schrödinger equation

$$i\hbar \frac{\partial}{\partial t}\psi = H\psi,$$

  with some Hamiltonian H.

Lorentz Invariance of the D'Alembertian Show that

$$\eta = \Lambda^T \eta \Lambda.$$

$$\Lambda_\rho{}^\mu = (\Lambda^{-1})^\mu{}_\rho.$$

$$\partial_\nu^{'} = (\Lambda^{-1})^\mu{}_\nu \partial_\mu.$$

$$\partial_\mu^{'} \partial^{'\mu} = \partial_\mu \partial^\mu.$$

Covariance of the Klein-Gordon equation Show that the Klein-Gordon equation is covariant under Lorentz transformations.

Vector Representations

- Write down the transformation property under ordinary rotations of a vector in three dimensions. What are the generators J^i. What is the dimensions of the irreducible representations and the corresponding quantum numbers.

- The generators of rotation can be alternatively given by

$$J^{ij} = \epsilon^{ijk} J^k.$$

  Calculate the commutators $[J^{ij}, J^{kl}]$.

- Write down the generators of the Lorentz group $J^{\mu\nu}$ by simply generalizing J^{ij} and show that

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\hbar\bigg(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}\bigg).$$

- Verify that

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\hbar(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha),$$

  is a solution. This is called the vector representation of the Lorentz group.

- Write down a finite Lorentz transformation matrix in the vector representation. Write down an infinitesimal rotation in the $xy-$plane and an infinitesimal boost along the $x-$axis.

Dirac Spinors

- Introduce $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$. Show that

$$(\sigma_\mu p^\mu)(\bar{\sigma}_\mu p^\mu) = m^2c^2.$$

- Show that the normalization condition $\bar{u}u = 2mc$ for $u^{(1)}$ and $u^{(2)}$ yields

$$N^{(1)} = N^{(2)} = \sqrt{\frac{m^2c^2}{\frac{E}{c} + p^3}}.$$

- Show that the normalization condition $\bar{v}v = -2mc$ for $v^{(1)}(p) = u^{(3)}(-p)$ and $v^{(2)}(p) = u^{(4)}(-p)$ yields

$$N^{(3)} = N^{(4)} = \sqrt{\frac{m^2c^2}{\frac{E}{c} - p^3}}.$$

- Show that we can rewrite the spinors u and v as

$$u^{(i)} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\xi^i \\ \sqrt{\bar{\sigma}_\mu p^\mu}\xi^i \end{pmatrix}.$$

$$v^{(i)} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\eta^i \\ -\sqrt{\bar{\sigma}_\mu p^\mu}\eta^i \end{pmatrix}.$$

  Determine ξ^i and η^i.

Spin Sums Let $u^{(r)}(p)$ and $v^{(r)}(p)$ be the positive-energy and negative-energy solutions of the free Dirac equation. Show that

- $$\bar{u}^{(r)}u^{(s)} = 2mc\delta^{rs}\ ,\ \bar{v}^{(r)}v^{(s)} = -2mc\delta^{rs}\ ,\ \bar{u}^{(r)}v^{(s)} = 0\ ,\ \bar{v}^{(r)}u^{(s)} = 0.$$

- $$u^{(r)+}u^{(s)} = \frac{2E}{c}\delta^{rs}\ ,\ v^{(r)+}v^{(s)} = \frac{2E}{c}\delta^{rs}.$$

$$u^{(r)+}(E, \vec{p})v^{(s)}(E, -\vec{p}) = 0\ ,\ v^{(r)+}(E, -\vec{p})u^{(s)}(E, \vec{p}) = 0.$$

- $$\sum_{s=1}^{2} u^{(s)}(E, \vec{p})\bar{u}^{(s)}(E, \vec{p}) = \gamma^\mu p_\mu + mc\ ,\ \sum_{s=1}^{2} v^{(s)}(E, \vec{p})\bar{v}^{(s)}(E, \vec{p}) = \gamma^\mu p_\mu - mc.$$

Covariance of the Dirac Equation Determine the transformation property of the spinor ψ under Lorentz transformations in order that the Dirac equation is covariant.

Spinor Bilinears Determine the transformation rule under Lorentz transformations of $\bar{\psi}$, $\bar{\psi}\psi$, $\bar{\psi}\gamma^5\psi$, $\bar{\psi}\gamma^\mu\psi$, $\bar{\psi}\gamma^\mu\gamma^5\psi$ and $\bar{\psi}\Gamma^{\mu\nu}\psi$.

Clifford Algebra

- Write down the solution of the Clifford algebra in three Euclidean dimensions. Construct a basis for 2×2 matrices in terms of Pauli matrices.
- Construct a basis for 4×4 matrices in terms of Dirac matrices.
  Hint: Show that there are 16 antisymmetric combinations of the Dirac gamma matrices in $1+3$ dimensions.

Chirality Operator and Weyl Fermions

- We define the gamma five matrix (chirality operator) by
  $$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3.$$
  Show that
  $$\gamma^5 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma.$$
  $$(\gamma^5)^2 = 1.$$
  $$(\gamma^5)^+ = \gamma^5.$$
  $$\{\gamma^5,\gamma^\mu\} = 0.$$
  $$[\gamma^5,\Gamma^{\mu\nu}] = 0.$$
- We write the Dirac spinor as
  $$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}.$$
  By working in the Weyl representation show that Dirac representation is reducible.
  Hint: Compute the eigenvalues of γ^5 and show that they do not mix under Lorentz transformations.
- Rewrite Dirac equation in terms of ψ_L and ψ_R. What is their physical interpretation.

3

Canonical Quantization of Free Fields

3.1 Classical Mechanics

3.1.1 D'Alembert Principle

We consider a system of many particles and let $\vec{r}_i$ and m_i be the radius vector and the mass respectively of the ith particle. Newton's second law of motion for the ith particle reads

$$\vec{F}_i = \vec{F}_i^{(e)} + \sum_j \vec{F}_{ji} = \frac{d\vec{p}_i}{dt}. \tag{3.1}$$

The external force acting on the ith particle is $\vec{F}_i^{(e)}$ whereas $\vec{F}_{ji}$ is the internal force on the ith particle due to the jth particle ($\vec{F}_{ii} = 0$ and $\vec{F}_{ij} = -\vec{F}_{ji}$). The momentum vector of the ith particle is $\vec{p}_i = m_i\vec{v}_i = m_i\frac{d\vec{r}_i}{dt}$. Thus we have

$$\vec{F}_i = \vec{F}_i^{(e)} + \sum_j \vec{F}_{ji} = m_i\frac{d^2\vec{r}_i}{dt^2}. \tag{3.2}$$

By summing over all particles we get

$$0\sum_i \vec{F}_i = \sum_i \vec{F}_i^{(e)} = \sum_i m_i\frac{d^2\vec{r}_i}{dt^2} = M\frac{d^2\vec{R}}{dt^2}. \tag{3.3}$$

The total mass M is $M = \sum_i m_i$ and the average radius vector $\vec{R}$ is $\vec{R} = \sum_i m_i\vec{r}_i/M$. This is the radius vector of the center of mass of the system. Thus the internal forces if they obey Newton's third law of motion will have no effect on the motion of the center of mass.

The goal of mechanics is to solve the set of second order differential equations (3.2) for $\vec{r}_i$ given the forces $\vec{F}_i^{(e)}$ and $\vec{F}_{ji}$. This task is in general very difficult and it is made even more complicated by the possible presence of constraints which limit the motion of the system. As an example we take the class of systems known as rigid bodies in which the motion of the particles is constrained in such a way that the distances between the particles are kept fixed and do not change in time. It is clear that constraints correspond to forces which can not be specified directly but are only known via their effect on the motion of the system. We will only consider holonomic constraints which can be expressed by equations of the form

$$f(\vec{r}_1,\vec{r}_2,\vec{r}_3,...,t)=0. \tag{3.4}$$

The constraints which can not be expressed in this way are called nonholonomic. In the example of rigid bodies the constraints are holonomic since they can be expressed as

$$(\vec{r}_i-\vec{r}_j)^2-c_{ij}^2=0. \tag{3.5}$$

The presence of constraints means that not all the vectors $\vec{r}_i$ are independent, i.e not all the differential equations (3.2) are independent. We assume that the system contains N particles and that we have k holonomic constraints. Then there must exist $3N-k$ independent degrees of freedom q_i which are called generalized coordinates. We can therefore express the vectors $\vec{r}_i$ as functions of the independent generalized coordinates q_i as

$$\begin{aligned}
&\vec{r}_1=\vec{r}_1(q_1,q_2,....,q_{3N-k},t)\\
&.\\
&.\\
&.\\
&\vec{r}_N=\vec{r}_N(q_1,q_2,....,q_{3N-k},t).
\end{aligned} \tag{3.6}$$

Let us compute the work done by the forces $\vec{F}_i^{(e)}$ and $\vec{F}_{ji}$ in moving the system from an initial configuration 1 to a final configuration 2. We have

$$W_{12}=\sum_i\int_1^2\vec{F}_i d\vec{s}_i=\sum_i\int_1^2\vec{F}_i^{(e)}d\vec{s}_i+\sum_{i,j}\int_1^2\vec{F}_{ji}d\vec{s}_i. \tag{3.7}$$

We have from one hand

$$\begin{aligned}
W_{12}=\sum_i\int_1^2\vec{F}_i d\vec{s}_i &= \sum_i\int_1^2 m_i\frac{d\vec{v}_i}{dt}\vec{v}_i dt\\
&= \sum_i\int_1^2 d(\frac{1}{2}m_i v_i^2)\\
&= T_2-T_1.
\end{aligned} \tag{3.8}$$

The total kinetic energy is defined by

$$T=\sum_i\frac{1}{2}m_i v_i^2. \tag{3.9}$$

We assume that the external forces $\vec{F}_i^{(e)}$ are conservative, i.e they are derived from potentials V_i such that

$$\vec{F}_i^{(e)}=-\vec{\nabla}_i V_i. \tag{3.10}$$

Then we compute

$$\sum_i \int_1^2 \vec{F}_i^{(e)} d\vec{s}_i = -\sum_i \int_1^2 \vec{\nabla}_i V_i d\vec{s}_i = -\sum_i V_i|_1^2. \tag{3.11}$$

We also assume that the internal forces $\vec{F}_{ji}$ are derived from potentials V_{ij} such that

$$\vec{F}_{ji} = -\vec{\nabla}_i V_{ij}. \tag{3.12}$$

Since we must have $\vec{F}_{ij} = -\vec{F}_{ji}$ we must take V_{ij} as a function of the distance $|\vec{r}_i - \vec{r}_j|$ only, i.e $V_{ij} = V_{ji}$. We can also check that the force $\vec{F}_{ij}$ lies along the line joining the particles i and j.

We define the difference vector by $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$. We have then $\vec{\nabla}_i V_{ij} = -\vec{\nabla}_j V_{ij} = \vec{\nabla}_{ij} V_{ij}$. We then compute

$$\begin{aligned}
\sum_{i,j} \int_1^2 \vec{F}_{ji} d\vec{s}_i &= -\frac{1}{2} \sum_{i,j} \int_1^2 (\vec{\nabla}_i V_{ij} d\vec{s}_i + \vec{\nabla}_j V_{ij} d\vec{s}_j) \\
&= -\frac{1}{2} \sum_{i,j} \int_1^2 \vec{\nabla}_{ij} V_{ij} (d\vec{s}_i - d\vec{s}_j) \\
&= -\frac{1}{2} \sum_{i,j} \int_1^2 \vec{\nabla}_{ij} V_{ij} d\vec{r}_{ij} \\
&= -\frac{1}{2} \sum_{i \neq j} V_{ij}|_1^2.
\end{aligned} \tag{3.13}$$

Thus the work done is found to be given by

$$W_{12} = -V_2 + V_1. \tag{3.14}$$

The total potential is given by

$$V = \sum_i V_i + \frac{1}{2} \sum_{i \neq j} V_{ij}. \tag{3.15}$$

From the results $W_{12} = T_2 - T_1$ and $W_{12} = -V_2 + V_1$ we conclude that the total energy $T + V$ is conserved. The term $\frac{1}{2}\sum_{i \neq j} V_{ij}$ in V is called the internal potential energy of the system.

For rigid bodies the internal energy is constant since the distances $|\vec{r}_i - \vec{r}_j|$ are fixed. Indeed in rigid bodies the vectors $d\vec{r}_{ij}$ can only be perpendicular to $\vec{r}_{ij}$ and therefore perpendicular to $\vec{F}_{ij}$ and as a consequence the internal forces do no work and the internal energy remains constant. In this case the forces $\vec{F}_{ij}$ are precisely the forces of constraints, i.e. the forces of constraint do no work.

We consider virtual infinitesimal displacements $\delta\vec{r}_i$ which are consistent with the forces and constraints imposed on the system at time t. A virtual displacement $\delta\vec{r}_i$ is to be compared with a real displacement $d\vec{r}_i$ which occurs during a time interval dt. Thus during a real displacement the forces and constraints imposed on the system may change. To be more precise an actual displacement is given in general by the equation

$$d\vec{r}_i = \frac{\partial \vec{r}_i}{\partial t} dt + \sum_{j=1}^{3N-k} \frac{\partial \vec{r}_i}{\partial q_j} dq_j. \tag{3.16}$$

A virtual displacement is given on the other hand by an equation of the form

$$\delta\vec{r}_i=\sum_{j=1}^{3N-k}\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j. \tag{3.17}$$

The effective force on each particle is zero, i.e $\vec{F}_{i\ \mathrm{eff}}=\vec{F}_i-\frac{d\vec{p}_i}{dt}=0$. The virtual work of this effective force in the displacement $\delta\vec{r}_i$ is therefore trivially zero. Summed over all particles we get

$$\sum_i(\vec{F}_i-\frac{d\vec{p}_i}{dt})\delta\vec{r}_i=0. \tag{3.18}$$

We decompose the force $\vec{F}_i$ into the applied force $\vec{F}_i^{(a)}$ and the force of constraint $\vec{f}_i$, viz $\vec{F}_i=\vec{F}_i^{(a)}+\vec{f}_i$. Thus we have

$$\sum_i(\vec{F}_i^{(a)}-\frac{d\vec{p}_i}{dt})\delta\vec{r}_i+\sum_i\vec{f}_i\delta\vec{r}_i=0. \tag{3.19}$$

We restrict ourselves to those systems for which the net virtual work of the forces of constraints is zero. In fact virtual displacements which are consistent with the constraints imposed on the system are precisely those displacements which are prependicular to the forces of constraints in such a way that the net virtual work of the forces of constraints is zero. We get then

$$\sum_i(\vec{F}_i^{(a)}-\frac{d\vec{p}_i}{dt})\delta\vec{r}_i=0. \tag{3.20}$$

This is the principle of virtual work of D'Alembert. The forces of constraints which as we have said are generally unknown but only their effect on the motion is known do not appear explicitly in D'Alembert principle which is our goal. Their only effect in the equation is to make the virtual displacements $\delta\vec{r}_i$ not all independent.

3.1.2 Lagrange's Equations

We compute

$$\begin{aligned}\sum_i\vec{F}_i^{(a)}\delta\vec{r}_i &= \sum_{i,j}\vec{F}_i^{(a)}\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j\\ &= \sum_j Q_j\delta q_j.\end{aligned} \tag{3.21}$$

The Q_j are the components of the generalized force. They are defined by

$$Q_j=\sum_i\vec{F}_i^{(a)}\frac{\partial\vec{r}_i}{\partial q_j}. \tag{3.22}$$

Let us note that since the generalized coordinates q_i need not have the dimensions of lenght the components Q_i of the generalized force need not have the dimensions of force.

We also compute

$$\begin{aligned}\sum_i \frac{d\vec{p}_i}{dt}\delta\vec{r}_i &= \sum_{i,j} m_i \frac{d^2\vec{r}_i}{dt^2}\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j \\ &= \sum_{i,j} m_i \bigg[\frac{d}{dt}\bigg(\frac{d\vec{r}_i}{dt}\frac{\partial\vec{r}_i}{\partial q_j}\bigg) - \frac{d\vec{r}_i}{dt}\frac{d}{dt}\bigg(\frac{\partial\vec{r}_i}{\partial q_j}\bigg)\bigg]\delta q_j \\ &= \sum_{i,j} m_i \bigg[\frac{d}{dt}\bigg(\vec{v}_i\frac{\partial\vec{r}_i}{\partial q_j}\bigg) - \vec{v}_i\frac{\partial\vec{v}_i}{\partial q_j}\bigg]\delta q_j.\end{aligned} \tag{3.23}$$

By using the result $\frac{\partial\vec{v}_i}{\partial\dot{q}_j} = \frac{\partial\vec{r}_i}{\partial q_j}$ we obtain

$$\begin{aligned}\sum_i \frac{d\vec{p}_i}{dt}\delta\vec{r}_i &= \sum_{i,j} m_i \bigg[\frac{d}{dt}\bigg(\vec{v}_i\frac{\partial\vec{v}_i}{\partial\dot{q}_j}\bigg) - \vec{v}_i\frac{\partial\vec{v}_i}{\partial q_j}\bigg]\delta q_j \\ &= \sum_j \bigg[\frac{d}{dt}\bigg(\frac{\partial T}{\partial\dot{q}_j}\bigg) - \frac{\partial T}{\partial q_j}\bigg]\delta q_j.\end{aligned} \tag{3.24}$$

The total kinetic term is $T = \sum_i \frac{1}{2}m_i v_i^2$. Hence D'Alembert's principle becomes

$$\sum_i (\vec{F}_i^{(a)} - \frac{d\vec{p}_i}{dt})\delta\vec{r}_i = -\sum_j \bigg[Q_j - \frac{d}{dt}\bigg(\frac{\partial T}{\partial\dot{q}_j}\bigg) + \frac{\partial T}{\partial q_j}\bigg]\delta q_j = 0. \tag{3.25}$$

Since the generalized coordinates q_i for holonomic constraints can be chosen such that they are all independent we get the equations of motion

$$-Q_j + \frac{d}{dt}\bigg(\frac{\partial T}{\partial\dot{q}_j}\bigg) - \frac{\partial T}{\partial q_j} = 0. \tag{3.26}$$

In above $j = 1, ..., n$ where $n = 3N - k$ is the number of independent generalized coordinates. For conservative forces we have $\vec{F}_i^{(a)} = -\vec{\nabla}_i V$, i.e

$$Q_j = -\frac{\partial V}{\partial q_j}. \tag{3.27}$$

Hence we get the equations of motion

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial\dot{q}_j}\bigg) - \frac{\partial L}{\partial q_j} = 0. \tag{3.28}$$

These are Lagrange's equations of motion where the Lagrangian L is defined by

$$L = T - V. \tag{3.29}$$

3.1.3 Hamilton's Principle: The Principle of Least Action

In the previous section we have derived Lagrange's equations from considerations involving virtual displacements around the instantaneous state of the system using the differential principle of D'Alembert. In this section we will rederive Lagrange's equations from considerations involving virtual variations of the entire motion between times t_1 and t_2 around the actual entire motion between t_1 and t_2 using the integral principle of Hamilton.

The instantaneous state or configuration of the system at time t is described by the n generalized coordinates q_1, q_2,...,q_n. This is a point in the n-dimensional configuration space with axes given by the generalized coordinates q_i. As time evolves the system changes and the point $(q_1, q_2, ..., q_n)$ moves in configuration space tracing out a curve called the path of motion of the system.

Hamilton's principle is less general than D'Alembert's principle in that it describes only systems in which all forces (except the forces of constraints) are derived from generalized scalar potentials U. The generalized potentials are velocity-dependent potentials which may also depend on time, i.e $U = U(q_i, \dot{q}_i, t)$. The generalized forces are obtained form U as

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\bigg(\frac{\partial U}{\partial \dot{q}_j}\bigg). \tag{3.30}$$

Such systems are called monogenic where Lagrange's equations of motion will still hold with Lagrangians given by $L = T - U$. The systems become conservative if the potentials depend only on coordinates. We define the action between times t_1 and t_2 by the line integral

$$I[q] = \int_{t_1}^{t_2} Ldt\ ,\ L = T - V. \tag{3.31}$$

The Lagrangian is a function of the generalized coordinates and velocities q_i and $\dot{q}_i$ and of time t, i.e $L = L(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t)$. The action I is a functional.

Hamilton's principle can be states as follows. The line integral I has a stationary value, i.e it is an extremum for the actual path of the motion. Therefore any first order variation of the actual path results in a second order change in I so that all neighboring paths which differ from the actual path by infintesimal displacements have the same action. This is a variational problem for the action functional which is based on one single function which is the Lagrangian. Clearly I is invariant to the system of generalized coordinates used to express L and as a consequence the equations of motion which will be derived from I will be covariant. We write Hamilton's principle as follows

$$\frac{\delta}{\delta q_i} I[q] = \frac{\delta}{\delta q_i}\int_{t_1}^{t_2} L(q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n, t)dt. \tag{3.32}$$

For systems with holonomic constraints it can be shown that Hamilton's principle is a necessary and sufficient condition for Lagrange's equations. Thus we can take Hamilton's principle as the basic postulate of mechanics rather than Newton's laws when all forces (except the forces of constraints) are derived from potentials which can depend on the coordinates, velocities and time.

Let us denote the soultions of the extremum problem by $q_i(t, 0)$. We write any other path around the correct path $q_i(t, 0)$ as $q_i(t, \alpha) = q_i(t, 0) + \alpha\eta_i(t)$ where the η_i are arbitrary functions of t which must vanish at the end points t_1 and t_2 and are continuous through the second derivative and α is an infinitesimal parameter which labels the set of neighboring paths which have the same action as the correct path. For this parametric family of curves the action becomes an ordinary function of α given by

$$I(\alpha) = \int_{t_1}^{t_2} L(q_i(t, \alpha), \dot{q}_i(t, \alpha), t)dt. \tag{3.33}$$

We define the virtual displacements δq_i by

$$\delta q_i = \bigg(\frac{\partial q_i}{\partial \alpha}\bigg)|_{\alpha=0}d\alpha = \eta_i d\alpha. \tag{3.34}$$

Similarly the infinitesimal variation of I is defined by

$$\delta I = \left(\frac{dI}{d\alpha}\right)|_{\alpha=0} d\alpha. \tag{3.35}$$

We compute

$$\begin{aligned}
\frac{dI}{d\alpha} &= \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha}+\frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial \alpha}\right)dt\\
&= \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha}+\frac{\partial L}{\partial \dot{q}_i}\frac{\partial}{\partial t}\frac{\partial q_i}{\partial \alpha}\right)dt\\
&= \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha}+\frac{\partial L}{\partial \dot{q}_i}\frac{d}{dt}\frac{\partial q_i}{\partial \alpha}\right)dt\\
&= \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial \alpha}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial \alpha}\right)dt+\left(\frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial \alpha}\right)_{t_1}^{t_2}.
\end{aligned} \tag{3.36}$$

The last term vanishes since all varied paths pass through the points $(t_1, y_i(t_1,0)$ and $(t_2, y_i(t_2,0))$. Thus we get

$$\delta I = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)\delta q_i dt. \tag{3.37}$$

Hamilton's principle reads

$$\frac{\delta I}{d\alpha} = \left(\frac{dI}{d\alpha}\right)|_{\alpha=0} = 0. \tag{3.38}$$

This leads to the equations of motion

$$\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\right)\eta_i dt = 0. \tag{3.39}$$

This should hold for any set of functions η_i. Thus by the fundamental lemma of the calculus of variations we must have

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0. \tag{3.40}$$

Formaly we write Hamilton's principle as

$$\frac{\delta I}{\delta q_i} = \frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0. \tag{3.41}$$

These are Lagrange's equations.

3.1.4 The Hamilton Equations of Motion

Again we will assume that the constraints are holonomic and the forces are monogenic, i.e they are derived from generalized scalar potentials as in (3.30). For a system with n degrees of freedom we have n Lagrange's equations of motion. Since Lagrange's equations are second order differential equations the motion of the system can be completely determined only after we also supply $2n$ initial conditions. As an example of initial conditions we can provide the n q_is and the n $\dot{q}_i$'s at an initial time t_0.

In the Hamiltonian formulation we want to describe the motion of the system in terms of first order differential equations. Since the number of initial conditions must remain $2n$ the number of first order differential equation which are needed to describe the system must be equal $2n$, i.e we must have $2n$ independent variables. It is only natural to choose the first half of the $2n$ independent variables to be the n generalized coordinates q_i. The second half will be chosen to be the n generalized momenta p_i defined by

$$p_i = \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_i}. \tag{3.42}$$

The pairs (q_i, p_i) are known as canonical variables. The generalized momenta p_i are also known as canonical or conjugate momenta.

In the Hamiltonian formulation the state or configuration of the system is described by the point $(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n)$ in the $2n$-dimensional space known as the phase space of the system with axes given by the generalized coordinates and momenta q_i and p_i. The $2n$ first order differential equations will describe how the point $(q_1, q_2, ..., q_n, p_1, p_2, ..., p_n)$ moves inside the phase space as the configuration of the system evolves in time.

The transition from the Lagrangian formulation to the Hamiltonian formulation corresponds to the change of variables $(q_i, \dot{q}_i, t) \longrightarrow (q_i, p_i, t)$ which is an example of a Legendre transformation. Instead of the Lagrangian which is a function of q_i,$\dot{q}_i$ and t, viz $L = L(q_i, \dot{q}_i, t)$ we will work in the Hamiltonian formulation with the Hamiltonian H which is a function of q_i, p_i and t defined by

$$H(q_i, p_i, t) = \sum_i \dot{q}_i p_i - L(q_i, \dot{q}_i, t). \tag{3.43}$$

We compute from one hand

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt. \tag{3.44}$$

From the other hand we compute

$$\begin{aligned} dH &= \dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \\ &= \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \\ &= \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt. \end{aligned} \tag{3.45}$$

By comparison we get the canonical equations of motion of Hamilton

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \ , \ -\dot{p}_i = \frac{\partial H}{\partial q_i}. \tag{3.46}$$

We also get

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}. \tag{3.47}$$

For a large class of systems and sets of generalized coordinates the Lagrangian can be decomposed as $L(q_i, \dot{q}_i, t) = L_0(q_i, t) + L_1(q_i, \dot{q}_i, t) + L_2(q_i, \dot{q}_i, t)$ where L_2 is a homogeneous function of degree 2 in $\dot{q}_i$ whereas L_1 is a homogeneous function of degree 1 in $\dot{q}_i$. In this case we compute

$$\dot{q}_i p_i = \dot{q}_i \frac{\partial L_1}{\partial \dot{q}_i} + \dot{q}_i \frac{\partial L_2}{\partial \dot{q}_i} = L_1 + 2L_2. \tag{3.48}$$

Hence

$$H = L_2 - L_0. \tag{3.49}$$

If the transformation equations which define the generalized coordinates do not depend on time explicitly, i.e $\vec{r}_i = \vec{r}_i(q_1, q_2, ..., q_n)$ then $\vec{v}_i = \sum_j \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j$ and as a consequence $T = T_2$ where T_2 is a function of q_i and $\dot{q}_i$ which is quadratic in the $\dot{q}_i$'s. In general the kinetic term will be of the form $T = T_2(q_i, \dot{q}_i, t) + T_1(q_i, \dot{q}_i, t) + T_0(q_i, t)$. Further if the potential does not depend on the generalized velocities $\dot{q}_i$ then $L_2 = T$, $L_1 = 0$ and $L_0 = -V$. Hence we get

$$H = T + V. \tag{3.50}$$

This is the total energy of the system. It is not difficult to show using Hamilton's equations that $\frac{dH}{dt} = \frac{\partial H}{\partial t}$. Thus if V does not depend on time explicitly then L will not depend on time explicitly and as a consequence H will be conserved.

3.2 Classical Free Field Theories

3.2.1 The Klein-Gordon Lagrangian Density

The Klein-Gordon wave equation is given by

$$\left(\partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2}\right)\phi(x) = 0. \tag{3.51}$$

We will consider a complex field ϕ so that we have also the independent equation

$$\left(\partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2}\right)\phi^*(x) = 0. \tag{3.52}$$

From now on we will reinterpret the wave functions ϕ and ϕ^* as fields and the corresponding Klein-Gordon wave equations as field equations.

A field is a dynamical system with an infinite number of degrees of freedom. Here the degrees of freedom $q_{\vec{x}}(t)$ and $\bar{q}_{\vec{x}}(t)$ are the values of the fields ϕ and ϕ^* at the points $\vec{x}$, viz

$$\begin{aligned} q_{\vec{x}}(t) &= \phi(x^0, \vec{x}) \\ \bar{q}_{\vec{x}}(t) &= \phi^*(x^0, \vec{x}). \end{aligned} \tag{3.53}$$

Remark that

$$\begin{aligned} \dot{q}_{\vec{x}} &= \frac{dq_{\vec{x}}}{dt} = c\partial_0 \phi + \frac{dx^i}{dt}\partial_i \phi \\ \dot{\bar{q}}_{\vec{x}} &= \frac{d\bar{q}_{\vec{x}}}{dt} = c\partial_0 \phi^* + \frac{dx^i}{dt}\partial_i \phi^*. \end{aligned} \tag{3.54}$$

Thus the role of $\dot{q}_{\vec{x}}$ and $\dot{\bar{q}}_{\vec{x}}$ will be played by the values of the derivatives of the fields $\partial_\mu\phi$ and $\partial_\mu\phi^*$ at the points $\vec{x}$.

The field equations (3.51) and (3.52) should be thought of as the equations of motion of the degrees of freedom $q_{\vec{x}}$ and $\bar{q}_{\vec{x}}$ respectively. These equations of motion should be derived from a Lagrangian density $\mathcal{L}$ which must depend only on the fields and their first derivatives at the point $\vec{x}$. In other words $\mathcal{L}$ must be local. This is also the reason why $\mathcal{L}$ is a Lagrangian density and not a Lagrangian. We have then

$$\mathcal{L}=\mathcal{L}(\phi,\phi^*,\partial_\mu\phi,\partial_\mu\phi^*)=\mathcal{L}(x^0,\vec{x}). \tag{3.55}$$

The Lagrangian is the integral over $\vec{x}$ of the Lagrangian density, viz

$$L=\int d\vec{x}\mathcal{L}(x^0,\vec{x}). \tag{3.56}$$

The action is the integral over time of L, namely

$$S=\int dtL=\int d^4x\mathcal{L}. \tag{3.57}$$

The Lagrangian density $\mathcal{L}$ is thus a Lorentz scalar. In other words it is a scalar under Lorentz transformations since the volume form d^4x is a scalar under Lorentz transformations. We compute

$$\begin{aligned}
\delta S &= \int d^4x\delta\mathcal{L}\\
&= \int d^4x\bigg[\delta\phi\frac{\delta\mathcal{L}}{\delta\phi}+\delta\partial_\mu\phi\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}+\text{h.c}\bigg]\\
&= \int d^4x\bigg[\delta\phi\frac{\delta\mathcal{L}}{\delta\phi}+\partial_\mu\delta\phi\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}+\text{h.c}\bigg]\\
&= \int d^4x\bigg[\delta\phi\frac{\delta\mathcal{L}}{\delta\phi}-\delta\phi\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}+\partial_\mu\bigg(\delta\phi\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\bigg)+\text{h.c}\bigg].
\end{aligned} \tag{3.58}$$

The surface term is zero because the field ϕ at infinity is assumed to be zero and hence

$$\delta\phi=0\ ,\ x^\mu\longrightarrow\pm\infty. \tag{3.59}$$

We get

$$\delta S = \int d^4x\bigg[\delta\phi\bigg(\frac{\delta\mathcal{L}}{\delta\phi}-\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\bigg)+\text{h.c}\bigg]. \tag{3.60}$$

The principle of least action states that

$$\delta S=0. \tag{3.61}$$

We obtain the Euler-Lagrange equations

$$\frac{\delta\mathcal{L}}{\delta\phi}-\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}=0. \tag{3.62}$$

$$\frac{\delta\mathcal{L}}{\delta\phi^*}-\partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}=0. \tag{3.63}$$

These must be the equations of motion (3.52) and (3.51) respectively. A solution is given by

$$\mathcal{L}_{\rm KG} = \frac{\hbar^2}{2}\bigg(\partial_\mu\phi^*\partial^\mu\phi - \frac{m^2c^2}{\hbar^2}\phi^*\phi\bigg). \tag{3.64}$$

The factor $\hbar^2$ is included so that the quantity $\int d^3x\mathcal{L}_{\rm KG}$ has dimension of energy. The coefficient 1/2 is the canonical convention.

The conjugate momenta $\pi(x)$ and $\pi^*(x)$ associated with the fields $\phi(x)$ and $\phi^*(x)$ are defined by

$$\pi(x) = \frac{\delta\mathcal{L}_{KG}}{\delta\partial_t\phi} \ , \ \pi^*(x) = \frac{\delta\mathcal{L}_{KG}}{\delta\partial_t\phi^*}. \tag{3.65}$$

We compute

$$\pi(x) = \frac{\hbar^2}{2c^2}\partial_t\phi^* \ , \ \pi^*(x) = \frac{\hbar^2}{2c^2}\partial_t\phi. \tag{3.66}$$

The Hamiltonian density $\mathcal{H}_{KG}$ is the Legendre transform of $\mathcal{L}_{KG}$ defined by

$$\begin{aligned}\mathcal{H}_{\rm KG} &= \pi(x)\partial_t\phi(x) + \pi^*(x)\partial_t\phi^*(x) - \mathcal{L}_{\rm KG} \\ &= \frac{\hbar^2}{2}\bigg(\partial_0\phi^*\partial_0\phi + \vec{\nabla}\phi^*\vec{\nabla}\phi + \frac{m^2c^2}{\hbar^2}\phi^*\phi\bigg).\end{aligned} \tag{3.67}$$

The Hamiltonian is given by

$$H_{\rm KG} = \int d^3x\mathcal{H}_{\rm KG}. \tag{3.68}$$

3.2.2 The Dirac Lagrangian Density

The Dirac equation and its Hermitian conjugate are given by

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0. \tag{3.69}$$

$$\bar{\psi}(i\hbar\gamma^\mu\overleftarrow{\partial_\mu} + mc) = 0. \tag{3.70}$$

The spinors ψ and $\bar{\psi}$ will now be interpreted as fields. In other words at each point $\vec{x}$ the dynamical variables are $\psi(x^0,\vec{x})$ and $\bar{\psi}(x^0,\vec{x})$. The two field equations (3.69) and (3.70) will be viewed as the equations of motion of the dynamical variables $\psi(x^0,\vec{x})$ and $\bar{\psi}(x^0,\vec{x})$. The local Lagrangian density will be of the form

$$\mathcal{L} = \mathcal{L}(\psi,\bar{\psi},\partial_\mu\psi,\partial_\mu\bar{\psi}) = \mathcal{L}(x^0,\vec{x}). \tag{3.71}$$

The Euler-Lagrange equations are

$$\frac{\delta\mathcal{L}}{\delta\psi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\psi} = 0. \tag{3.72}$$

$$\frac{\delta\mathcal{L}}{\delta\bar{\psi}} - \partial_\mu\frac{\delta\mathcal{L}}{\delta\partial_\mu\bar{\psi}} = 0. \tag{3.73}$$

A solution is given by

$$\mathcal{L}_{\rm Dirac} = \bar{\psi}(i\hbar c\gamma^\mu \partial_\mu - mc^2)\psi. \tag{3.74}$$

The conjugate momenta $\bar{\Pi}(x)$ and $\Pi(x)$ associated with the fields $\psi(x)$ and $\bar{\psi}(x)$ are defined by

$$\Pi(x) = \frac{\delta \mathcal{L}_{\rm Dirac}}{\delta \partial_t \psi}\ ,\ \bar{\Pi}(x) = \frac{\delta \mathcal{L}_{\rm Dirac}}{\delta \partial_t \bar{\psi}}. \tag{3.75}$$

We compute

$$\Pi(x) = \bar{\psi} i\hbar\gamma^0\ ,\ \bar{\Pi}(x) = 0. \tag{3.76}$$

The Hamiltonian density $\mathcal{H}_{\rm Dirac}$ is the Legendre transform of $\mathcal{L}_{\rm Dirac}$ defined by

$$\begin{aligned}
\mathcal{H}_{\rm Dirac} &= \Pi(x)\partial_t\psi(x) + \partial_t\bar{\psi}(x)\bar{\Pi}(x) - \mathcal{L}_{\rm Dirac} \\
&= \bar{\psi}(-i\hbar c\gamma^i\partial_i + mc^2)\psi \\
&= \psi^+(-i\hbar c\vec{\alpha}\vec{\nabla} + mc^2\beta)\psi.
\end{aligned} \tag{3.77}$$

3.3 Canonical Quantization of a Real Scalar Field

We will assume here that the scalar field ϕ is real. Thus $\phi^* = \phi$. This is a classical field theory governed by the Lagrangian density and the Lagrangian

$$\mathcal{L}_{\rm KG} = \frac{\hbar^2}{2}\bigg(\partial_\mu\phi\partial^\mu\phi - \frac{m^2c^2}{\hbar^2}\phi^2\bigg). \tag{3.78}$$

$$L_{\rm KG} = \int d^3x \mathcal{L}_{\rm KG}. \tag{3.79}$$

The conjugate momentum is

$$\pi = \frac{\delta \mathcal{L}_{\rm KG}}{\delta\partial_t\phi} = \frac{\hbar^2}{c^2}\partial_t\phi. \tag{3.80}$$

We expand the classical field ϕ as

$$\phi(x^0,\vec{x}) = \frac{c}{\hbar}\int \frac{d^3p}{(2\pi\hbar)^3} Q(x^0,\vec{p}) e^{\frac{i}{\hbar}\vec{p}\vec{x}}. \tag{3.81}$$

In other words $Q(x^0,\vec{p})$ is the Fourier transform of $\phi(x^0,\vec{x})$ which is given by

$$\frac{c}{\hbar}Q(x^0,\vec{p}) = \int d^3x \phi(x^0,\vec{x}) e^{-\frac{i}{\hbar}\vec{p}\vec{x}}. \tag{3.82}$$

Since $\phi^* = \phi$ we have $Q(x^0,-\vec{p}) = Q^*(x^0,\vec{p})$. We compute

$$\begin{aligned}
L_{\rm KG} &= \frac{1}{2}\int \frac{d^3p}{(2\pi\hbar)^3}\bigg[\partial_t Q^*(x^0,\vec{p})\partial_t Q(x^0,\vec{p}) - \omega(\vec{p})^2 Q^*(x^0,\vec{p})Q(x^0,\vec{p})\bigg] \\
&= \int_+ \frac{d^3p}{(2\pi\hbar)^3}\bigg[\partial_t Q^*(x^0,\vec{p})\partial_t Q(x^0,\vec{p}) - \omega(\vec{p})^2 Q^*(x^0,\vec{p})Q(x^0,\vec{p})\bigg].
\end{aligned} \tag{3.83}$$

$$\omega^2(\vec{p}) = \frac{1}{\hbar^2}(\vec{p}^2c^2 + m^2c^4). \quad (3.84)$$

The sign $\int_+$ stands for the integration over positive values of p^1, p^2 and p^3. The equation of motion obeyed by Q derived from the Lagrangian $L_{\rm KG}$ is

$$(\partial_t^2 + \omega(\vec{p}))Q(x^0, \vec{p}) = 0. \quad (3.85)$$

The general solution is of the form

$$Q(x^0, \vec{p}) = \frac{1}{\sqrt{2\omega(\vec{p})}}\bigg[a(\vec{p})\ e^{-i\omega(\vec{p})t} + a(-\vec{p})^*\ e^{i\omega(\vec{p})t}\bigg]. \quad (3.86)$$

This satisfies $Q(x^0, -\vec{p}) = Q^*(x^0, \vec{p})$. The conjugate momentum is

$$\pi(x^0, \vec{x}) = \frac{\hbar}{c}\int \frac{d^3p}{(2\pi\hbar)^3}P(x^0, \vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}}\ ,\ P(x^0, \vec{p}) = \partial_t Q(x^0, \vec{p}). \quad (3.87)$$

$$\frac{\hbar}{c}P(x^0, \vec{p}) = \int d^3x \pi(x^0, \vec{x})e^{-\frac{i}{\hbar}\vec{p}\vec{x}}. \quad (3.88)$$

Since $\pi^* = \pi$ we have $P(x^0, -\vec{p}) = P^*(x^0, \vec{p})$. We observe that

$$P(x^0, \vec{p}) = \frac{\delta L_{\rm KG}}{\delta\partial_t Q^*(x^0, \vec{p})}. \quad (3.89)$$

The Hamiltonian is

$$H_{\rm KG} \quad = \quad \int_+ \frac{d^3p}{(2\pi\hbar)^3}\bigg[P^*(x^0, \vec{p})P(x^0, \vec{p}) + \omega^2(\vec{p})Q^*(x^0, \vec{p})Q(x^0, \vec{p})\bigg]. \quad (3.90)$$

The real scalar field is therefore equivalent to an infinite collection of independent harmonic oscillators with frequencies $\omega(\vec{p})$ which depend on the momenta $\vec{p}$ of the Fourier modes.

Quantization of this dynamical system means replacing the scalar field ϕ and the conjugate momentum field π by operators $\hat{\phi}$ and $\hat{\pi}$ respectively which are acting in some Hilbert space. This means that the coefficients a and a^* become operators $\hat{a}$ and $\hat{a}^+$ and hence Q and P become operators $\hat{Q}$ and $\hat{P}$. The operators $\hat{\phi}$ and $\hat{\pi}$ will obey the equal-time canonical commutation relations due to Dirac, viz

$$[\hat{\phi}(x^0, \vec{x}), \hat{\pi}(x^0, \vec{y})] = i\hbar\delta^3(\vec{x} - \vec{y}). \quad (3.91)$$

$$[\hat{\phi}(x^0, \vec{x}), \hat{\phi}(x^0, \vec{y})] = [\hat{\pi}(x^0, \vec{x}), \hat{\pi}(x^0, \vec{y})] = 0. \quad (3.92)$$

These commutation relations should be compared with

$$[q_i, p_j] = i\hbar\delta_{ij}. \quad (3.93)$$

$$[q_i, q_j] = [p_i, p_j] = 0. \quad (3.94)$$

The field operator $\hat{\phi}$ and the conjugate momentum operator $\hat{\pi}$ are given by

$$\frac{\hbar}{c}\hat{\phi}(x^0, \vec{x}) = \int \frac{d^3p}{(2\pi\hbar)^3}\hat{Q}(x^0, \vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}} = \int_+ \frac{d^3p}{(2\pi\hbar)^3}\hat{Q}(x^0, \vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}} + \int_+ \frac{d^3p}{(2\pi\hbar)^3}\hat{Q}^+(x^0, \vec{p})e^{-\frac{i}{\hbar}\vec{p}\vec{x}} \quad (3.95)$$

$$\frac{c}{\hbar}\hat{\pi}(x^0,\vec{x})=\int\frac{d^3p}{(2\pi\hbar)^3}\hat{P}(x^0,\vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}}=\int_+\frac{d^3p}{(2\pi\hbar)^3}\hat{P}(x^0,\vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}}+\int_+\frac{d^3p}{(2\pi\hbar)^3}\hat{P}^+(x^0,\vec{p})e^{-\frac{i}{\hbar}\vec{p}\vec{x}}. \quad (3.96)$$

It is then not difficult to see that the commutation relations (3.91) and (3.92) are equivalent to the equal-time commutation rules

$$[\hat{Q}(x^0,\vec{p}),\hat{P}^+(x^0,\vec{q})]=i\hbar(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}). \quad (3.97)$$

$$[\hat{Q}(x^0,\vec{p}),\hat{P}(x^0,\vec{q})]=0. \quad (3.98)$$

$$[\hat{Q}(x^0,\vec{p}),\hat{Q}(x^0,\vec{q})]=[\hat{P}(x^0,\vec{p}),\hat{P}(x^0,\vec{q})]=0. \quad (3.99)$$

We have

$$\hat{Q}(x^0,\vec{p})=\frac{1}{\sqrt{2\omega(\vec{p})}}\bigg[\hat{a}(\vec{p})\ e^{-i\omega(\vec{p})t}+\hat{a}(-\vec{p})^+\ e^{i\omega(\vec{p})t}\bigg]. \quad (3.100)$$

$$\hat{P}(x^0,\vec{p})=-i\sqrt{\frac{\omega(\vec{p})}{2}}\bigg[\hat{a}(\vec{p})\ e^{-i\omega(\vec{p})t}-\hat{a}(-\vec{p})^+\ e^{i\omega(\vec{p})t}\bigg]. \quad (3.101)$$

Since $\hat{Q}(x^0,\vec{p})$ and $\hat{P}(x^0,\vec{p})$ satisfy (3.97), (3.98) and (3.99) the annihilation and creation operators $a(\vec{p})$ and $a(\vec{p})^+$ must satisfy

$$[\hat{a}(\vec{p}),\hat{a}(\vec{q})^+]=\hbar(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}). \quad (3.102)$$

The Hamiltonian operator is

$$\begin{aligned}\hat{H}_{\rm KG} &= \int_+\frac{d^3p}{(2\pi\hbar)^3}\bigg[\hat{P}^+(x^0,\vec{p})\hat{P}(x^0,\vec{p})+\omega^2(\vec{p})\hat{Q}^+(x^0,\vec{p})\hat{Q}(x^0,\vec{p})\bigg]\\ &= \int_+\frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\bigg[\hat{a}(\vec{p})^+\hat{a}(\vec{p})+\hat{a}(\vec{p})\hat{a}(\vec{p})^+\bigg]\\ &= 2\int_+\frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\bigg[\hat{a}(\vec{p})^+\hat{a}(\vec{p})+\frac{\hbar}{2}(2\pi\hbar)^3\delta^3(0)\bigg]\\ &= \int\frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\bigg[\hat{a}(\vec{p})^+\hat{a}(\vec{p})+\frac{\hbar}{2}(2\pi\hbar)^3\delta^3(0)\bigg]. \quad (3.103)\end{aligned}$$

Let us define the vacuum (ground) state $|0>$ by

$$\hat{a}(\vec{p})|0>=0. \quad (3.104)$$

The energy of the vacuum is therefore infinite since

$$\hat{H}_{\rm KG}|0> \ = \ \int\frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\bigg[\frac{\hbar}{2}(2\pi\hbar)^3\delta^3(0)\bigg]|0>. \quad (3.105)$$

This is a bit disturbing. But since all we can measure experimentally are energy differences from the ground state this infinite energy is unobservable. We can ignore this infinite energy by the so-called normal (Wick's) ordering procedure defined by

$$:\hat{a}(\vec{p})\hat{a}(\vec{p})^+:=\hat{a}(\vec{p})^+\hat{a}(\vec{p})\ ,\ \ :\hat{a}(\vec{p})^+\hat{a}(\vec{p}):=\hat{a}(\vec{p})^+\hat{a}(\vec{p}). \quad (3.106)$$

We then get

$$: \hat{H}_{\rm KG} : \ = \ \int \frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\hat{a}(\vec{p})^+\hat{a}(\vec{p}). \quad (3.107)$$

Clearly

$$: \hat{H}_{\rm KG} : |0> \ = \ 0. \quad (3.108)$$

It is easy to calculate

$$[\hat{H}_{\rm KG}, \hat{a}(\vec{p})^+] = \hbar\omega(\vec{p})\hat{a}(\vec{p})^+ \ , \ [\hat{H}, \hat{a}(\vec{p})] = -\hbar\omega(\vec{p})\hat{a}(\vec{p}). \quad (3.109)$$

This establishes that $\hat{a}(\vec{p})^+$ and $\hat{a}(\vec{p})$ are raising and lowering operators. The one-particle states are states of the form

$$|\vec{p}>= \frac{1}{c}\sqrt{2\omega(\vec{p})}\hat{a}(\vec{p})^+|0> . \quad (3.110)$$

Indeed we compute

$$\hat{H}_{\rm KG}|\vec{p}>= \hbar\omega(\vec{p})|\vec{p}>= E(\vec{p})|\vec{p}> \ , \ E(\vec{p}) = \sqrt{\vec{p}^2c^2 + m^2c^4}. \quad (3.111)$$

The energy $E(\vec{p})$ is precisely the energy of a relativistic particle of mass m and momentum $\vec{p}$. This is the underlying reason for the interpretation of $|\vec{p}>$ as a state of a free quantum particle carrying momentum $\vec{p}$ and energy $E(\vec{p})$. The normalization of the one-particle state $|\vec{p}>$ is chosen such that

$$< \vec{p}|\vec{q}>= \frac{2}{c^2}(2\pi\hbar)^3E(\vec{p})\delta^3(\vec{p}-\vec{q}). \quad (3.112)$$

We have assumed that $< 0|0 >= 1$. The factor $\sqrt{2\omega(\vec{p})}$ in (3.110) is chosen so that the normalization (3.112) is Lorentz invariant.

The two-particle states are states of the form (not bothering about normalization)

$$|\vec{p}, \vec{q}>= \hat{a}(\vec{p})^+\hat{a}(\vec{q})^+|0> . \quad (3.113)$$

We compute in this case

$$\hat{H}_{\rm KG}|\vec{p}, \vec{q}>= \hbar(\omega(\vec{p}) + \omega(\vec{q}))|\vec{p}> . \quad (3.114)$$

Since the creation operators for different momenta commute the state $|\vec{p}, \vec{q}>$ is the same as the state $|\vec{q}, \vec{p}>$ and as a consequence our particles obey the Bose-Einstein statistics. In general multiple-particle states will be of the form $\hat{a}(\vec{p})^+\hat{a}(\vec{q})^+...\hat{a}(\vec{k})^+|0>$ with energy equal to $\hbar(\omega(\vec{p}) + \omega(\vec{q}) + ... + \omega(\vec{k}))$.

Let us compute (with $px = cp^0t - \vec{p}\vec{x}$)

$$\begin{aligned}\frac{\hbar}{c}\hat{\phi}(x) &= \int \frac{d^3p}{(2\pi\hbar)^3}\hat{Q}(x^0, \vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}}\\ &= \int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{\sqrt{2\omega(\vec{p})}}\bigg(\hat{a}(\vec{p})e^{-\frac{i}{\hbar}px} + \hat{a}(\vec{p})^+e^{\frac{i}{\hbar}px}\bigg)_{p^0=E(\vec{p})/c}. \quad (3.115)\end{aligned}$$

Finally we remark that the unit of $\hbar$ is $[\hbar] = ML^2/T$, the unit of ϕ is $[\phi] = 1/(L^{3/2}M^{1/2})$, the unit of π is $[\pi] = (M^{3/2}L^{1/2})/T$, the unit of Q is $[Q] = M^{1/2}L^{5/2}$, the unit of P is $[P] = (M^{1/2}L^{5/2})/T$, the unit of a is $[a] = (M^{1/2}L^{5/2})/T^{1/2}$, the unit of H is $[H] = (ML^2)/T^2$ and the unit of momentum p is $[p] = (ML)/T$.

3.4 Canonical Quantization of Free Spinor Field

We expand the spinor field as

$$\psi(x^0,\vec{x})=\frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\chi(x^0,\vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}}. \tag{3.116}$$

The Lagrangian in terms of χ and χ^+ is given by

$$\begin{aligned}L_{\rm Dirac} &= \int d^3x\mathcal{L}_{\rm Dirac}\\ &= \int d^3x\bar{\psi}(i\hbar c\gamma^\mu\partial_\mu-mc^2)\psi\\ &= \frac{c}{\hbar^2}\int\frac{d^3p}{(2\pi\hbar)^3}\bar{\chi}(x^0,\vec{p})(i\hbar\gamma^0\partial_0-\gamma^ip^i-mc)\chi(x^0,\vec{p}).\end{aligned} \tag{3.117}$$

The classical equation of motion obeyed by the field $\chi(x^0,\vec{p})$ is

$$(i\hbar\gamma^0\partial_0-\gamma^ip^i-mc)\chi(x^0,\vec{p})=0. \tag{3.118}$$

This can be solved by plane-waves of the form

$$\chi(x^0,\vec{p})=e^{-\frac{i}{\hbar}Et}\chi(\vec{p}), \tag{3.119}$$

with

$$(\gamma^\mu p_\mu-mc)\chi(\vec{p})=0. \tag{3.120}$$

We know how to solve this equation. The positive-energy solutions are given by

$$\chi_+(\vec{p})=u^{(i)}(E,\vec{p}). \tag{3.121}$$

The corresponding plane-waves are

$$\chi_+(x^0,\vec{p})=e^{-i\omega(\vec{p})t}u^{(i)}(E(\vec{p}),\vec{p})=e^{-i\omega(\vec{p})t}u^{(i)}(\vec{p}). \tag{3.122}$$

$$\omega(\vec{p})=\frac{E}{\hbar}=\frac{\sqrt{\vec{p}^2c^2+m^2c^4}}{\hbar}. \tag{3.123}$$

The negative-energy solutions are given by

$$\chi_-(\vec{p})=v^{(i)}(-E,-\vec{p}). \tag{3.124}$$

The corresponding plane-waves are

$$\chi_+(x^0,\vec{p})=e^{i\omega(\vec{p})t}v^{(i)}(E(\vec{p}),-\vec{p})=e^{i\omega(\vec{p})t}v^{(i)}(-\vec{p}). \tag{3.125}$$

In the above equations

$$E(\vec{p})=E=\hbar\omega(\vec{p}). \tag{3.126}$$

Thus the general solution is a linear combination of the form

$$\chi(x^0,\vec{p})=\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i\bigg(e^{-i\omega(\vec{p})t}u^{(i)}(\vec{p})b(\vec{p},i)+e^{i\omega(\vec{p})t}v^{(i)}(-\vec{p})d(-\vec{p},i)^*\bigg). \tag{3.127}$$

The spinor field becomes

$$\psi(x^0,\vec{x}) = \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{p})}}e^{\frac{i}{\hbar}\vec{p}\vec{x}}\sum_i\bigg(e^{-i\omega(\vec{p})t}u^{(i)}(\vec{p})b(\vec{p},i)+e^{i\omega(\vec{p})t}v^{(i)}(-\vec{p})d(-\vec{p},i)^*\bigg). \tag{3.128}$$

The conjugate momentum field is

$$\begin{aligned}\Pi(x^0,\vec{x}) &= i\hbar\psi^+\\ &= i\int\frac{d^3p}{(2\pi\hbar)^3}\chi^+(x^0,\vec{p})e^{-\frac{i}{\hbar}\vec{p}\vec{x}}.\end{aligned} \tag{3.129}$$

After quantization the coefficients $b(\vec{p},i)$ and $d(-\vec{p},i)^*$ and a s a consequence the spinors $\chi(x^0,\vec{p})$ and $\chi^+(x^0,\vec{p})$ become operators $\hat{b}(\vec{p},i)$, $\hat{d}(-\vec{p},i)^+$, $\hat{\chi}(x^0,\vec{p})$ and $\hat{\chi}^+(x^0,\vec{p})$ respectively. As we will see shortly the quantized Poisson brackets for a spinor field are given by anticommutation relations and not commutation relations. In other words we must impose anticommutation relations between the spinor field operator $\hat{\psi}$ and the conjugate momentum field operator $\hat{\Pi}$. In the following we will consider both possibilities for the sake of completeness. We set then

$$[\hat{\psi}_\alpha(x^0,\vec{x}),\hat{\Pi}_\beta(x^0,\vec{y})]_\pm = i\hbar\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y}). \tag{3.130}$$

The plus sign corresponds to anticommutator whereas the minus sign corresponds to commutator. We can immediately compute

$$[\hat{\chi}_\alpha(x^0,\vec{p}),\hat{\chi}^+_\beta(x^0,\vec{q})]_\pm = \hbar^2\delta_{\alpha\beta}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}). \tag{3.131}$$

This is equivalent to

$$[\hat{b}(\vec{p},i),\hat{b}(\vec{q},j)^+]_\pm = \hbar\delta_{ij}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}), \tag{3.132}$$

$$[\hat{d}(\vec{p},i)^+,\hat{d}(\vec{q},j)]_\pm = \hbar\delta_{ij}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}), \tag{3.133}$$

and

$$[\hat{b}(\vec{p},i),\hat{d}(\vec{q},j)]_\pm = [\hat{d}(\vec{q},j)^+,\hat{b}(\vec{p},i)]_\pm = 0. \tag{3.134}$$

We go back to the classical theory for a moment. The Hamiltonian in terms of χ and χ^+ is given by

$$\begin{aligned}H_{\rm Dirac} &= \int d^3x\mathcal{H}_{\rm Dirac}\\ &= \int d^3x\bar{\psi}(-i\hbar c\gamma^i\partial_i+mc^2)\psi\\ &= \frac{c}{\hbar^2}\int\frac{d^3p}{(2\pi\hbar)^3}\bar{\chi}(x^0,\vec{p})(\gamma^ip^i+mc)\chi(x^0,\vec{p})\\ &= \frac{c}{\hbar^2}\int\frac{d^3p}{(2\pi\hbar)^3}\chi^+(x^0,\vec{p})\gamma^0(\gamma^ip^i+mc)\chi(x^0,\vec{p}).\end{aligned} \tag{3.135}$$

The eigenvalue equation (3.120) can be put in the form

$$\gamma^0(\gamma^i p^i + mc)\chi(x^0,\vec{p}) = \frac{E}{c}\chi(x^0,\vec{p}). \tag{3.136}$$

On the positive-energy solution we have

$$\gamma^0(\gamma^i p^i + mc)\chi_+(x^0,\vec{p}) = \frac{\hbar\omega(\vec{p})}{c}\chi_+(x^0,\vec{p}). \tag{3.137}$$

On the negative-energy solution we have

$$\gamma^0(\gamma^i p^i + mc)\chi_-(x^0,\vec{p}) = -\frac{\hbar\omega(\vec{p})}{c}\chi_-(x^0,\vec{p}). \tag{3.138}$$

Hence we have explicitly

$$c\gamma^0(\gamma^i p^i + mc)\chi(x^0,\vec{p}) = \frac{\hbar\omega(\vec{p})}{\sqrt{2\omega(\vec{p})}}\sum_i\bigg(e^{-i\omega(\vec{p})t}u^{(i)}(\vec{p})b(\vec{p},i) - e^{i\omega(\vec{p})t}v^{(i)}(-\vec{p})d(-\vec{p},i)^*\bigg). \tag{3.139}$$

The Hamiltonian becomes

$$\begin{aligned} H_{\rm Dirac} &= \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}E(\vec{p})\sum_i\bigg(b(\vec{p},i)^*b(\vec{p},i) - d(-\vec{p},i)d(-\vec{p},i)^*\bigg) \\ &= \int\frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\sum_i\bigg(b(\vec{p},i)^*b(\vec{p},i) - d(\vec{p},i)d(\vec{p},i)^*\bigg). \end{aligned} \tag{3.140}$$

After quantization the Hamiltonian becomes an operator given by

$$\hat{H}_{\rm Dirac} = \int\frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\sum_i\bigg(\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},i) - \hat{d}(\vec{p},i)\hat{d}(\vec{p},i)^+\bigg). \tag{3.141}$$

At this stage we will decide once and for all whether the creation and annihilation operators of the theory obey commutation relations or anticommutation relations. In the case of commutation relations we see from the commutation relations (3.133) that $\hat{d}$ is the creation operator and $\hat{d}^+$ is the annihilation operator. Thus the second term in the above Hamiltonian operator is already normal ordered. However we observe that the contribution of the $d-$particles to the energy is negative and thus by creating more and more d particles the energy can be lowered without limit. The theory does not admit a stable ground state.

In the case of anticommutation relations the above Hamiltonian operator becomes

$$\hat{H}_{\rm Dirac} = \int\frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\sum_i\bigg(\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},i) + \hat{d}(\vec{p},i)^+\hat{d}(\vec{p},i)\bigg). \tag{3.142}$$

This expression is correct modulo an infinite constant which can be removed by normal ordering as in the scalar field theory. The vacuum state is defined by

$$\hat{b}(\vec{p},i)|0>=\hat{d}(\vec{p},i)|0>=0. \tag{3.143}$$

Clearly

$$\hat{H}_{\rm Dirac}|0> = 0. \tag{3.144}$$

We calculate

$$[\hat{H}_{\rm Dirac}, \hat{b}(\vec{p},i)^+] = \hbar\omega(\vec{p})\hat{b}(\vec{p},i)^+ \ , \ [\hat{H}_{\rm Dirac}, \hat{b}(\vec{p},i)] = -\hbar\omega(\vec{p})\hat{b}(\vec{p},i). \tag{3.145}$$

$$[\hat{H}_{\rm Dirac}, \hat{d}(\vec{p},i)^+] = \hbar\omega(\vec{p})\hat{d}(\vec{p},i)^+ \ , \ [\hat{H}_{\rm Dirac}, \hat{d}(\vec{p},i)] = -\hbar\omega(\vec{p})\hat{d}(\vec{p},i). \tag{3.146}$$

Excited particle states are obtained by acting with $\hat{b}(\vec{p},i)^+$ on $|0>$ and excited antiparticle states are obtained by acting with $\hat{d}(\vec{p},i)^+$ on $|0>$. The normalization of one-particle excited states can be fixed in the same way as in the scalar field theory, viz

$$|\vec{p},ib>= \sqrt{2\omega(\vec{p})}\hat{b}(\vec{p},i)^+|0> \ , \ |\vec{p},id>= \sqrt{2\omega(\vec{p})}\hat{d}(\vec{p},i)^+|0> . \tag{3.147}$$

Indeed we compute

$$\hat{H}_{\rm Dirac}|\vec{p},ib>= E(\vec{p})|\vec{p},ib> \ , \ \hat{H}_{\rm Dirac}|\vec{p},id>= E(\vec{p})|\vec{p},id> . \tag{3.148}$$

$$< \vec{p},ib|\vec{q},jb>=< \vec{p},id|\vec{q},jd>= 2E(\vec{p})\delta_{ij}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}). \tag{3.149}$$

Furthermore we compute

$$< 0|\hat{\psi}(x)|\vec{p},ib>= u^{(i)}(\vec{p})e^{-\frac{i}{\hbar}px}. \tag{3.150}$$

$$< 0|\hat{\bar{\psi}}(x)|\vec{p},id>= \bar{v}^{(i)}(\vec{p})e^{-\frac{i}{\hbar}px}. \tag{3.151}$$

The field operator $\hat{\bar{\psi}}(x)$ acting on the vacuum $|0>$ creates a particle at $\vec{x}$ at time $t = x^0/c$ whereas $\hat{\psi}(x)$ acting on $|0>$ creates an antiparticle at $\vec{x}$ at time $t = x^0/c$.

General multiparticle states are obtained by acting with $\hat{b}(\vec{p},i)^+$ and $\hat{d}(\vec{p},i)^+$ on $|0>$. Since the creation operators anticommute our particles will obey the Fermi-Dirac statistics. For example particles can not occupy the same state, i.e. $\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},i)^+|0>= 0$.

The spinor field operator can be put in the form

$$\hat{\psi}(x) = \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i\bigg(e^{-\frac{i}{\hbar}px}u^{(i)}(\vec{p})\hat{b}(\vec{p},i) + e^{\frac{i}{\hbar}px}v^{(i)}(\vec{p})\hat{d}(\vec{p},i)^+\bigg). \tag{3.152}$$

3.5 Propagators

3.5.1 Scalar Propagator

The probability amplitude for a scalar particle to propagate from the spacetime point y to the spacetime x is

$$D(x-y) =< 0|\hat{\phi}(x)\hat{\phi}(y)|0> . \tag{3.153}$$

We compute

$$\begin{aligned} D(x-y) &= \frac{c^2}{\hbar^2}\int\frac{d^3p}{(2\pi\hbar)^3}\int\frac{d^3q}{(2\pi\hbar)^3}\frac{e^{-\frac{i}{\hbar}px}}{\sqrt{2\omega(\vec{p})}}\frac{e^{\frac{i}{\hbar}qy}}{\sqrt{2\omega(\vec{q})}} < 0|\hat{a}(\vec{p})\hat{a}(\vec{q})^+|0> \\ &= c^2\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}p(x-y)}. \end{aligned} \tag{3.154}$$

This is Lorentz invariant since $d^3p/E(\vec{p})$ is Lorentz invariant. Now we will relate this probability amplitude with the commutator $[\hat{\phi}(x),\hat{\phi}(y)]$. We compute

$$\begin{aligned}
[\hat{\phi}(x),\hat{\phi}(y)] &= \frac{c^2}{\hbar^2}\int\frac{d^3p}{(2\pi\hbar)^3}\int\frac{d^3q}{(2\pi\hbar)^3}\frac{1}{\sqrt{2\omega(\vec{p})}}\frac{1}{\sqrt{2\omega(\vec{q})}}\\
&\times \left(e^{-\frac{i}{\hbar}px}e^{\frac{i}{\hbar}qy}[\hat{a}(\vec{p}),\hat{a}(\vec{q})^+]-e^{\frac{i}{\hbar}px}e^{-\frac{i}{\hbar}qy}[\hat{a}(\vec{q}),\hat{a}(\vec{p})^+]\right)\\
&= D(x-y)-D(y-x).
\end{aligned}\tag{3.155}$$

In the case of a spacelike interval, i.e. $(x-y)^2=(x^0-y^0)^2-(\vec{x}-\vec{y})^2<0$ the amplitudes $D(x-y)$ and $D(y-x)$ are equal and thus the commutator vanishes. To see this more clearly we place the event x at the origin of spacetime. The event y if it is spacelike it will lie outside the light-cone. In this case there is an inertial reference frame in which the two events occur at the same time, viz $y^0=x^0$. In this reference frame the amplitude takes the form

$$D(x-y) = c^2\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{\frac{i}{\hbar}\vec{p}(\vec{x}-\vec{y})}.\tag{3.156}$$

It is clear that $D(x-y)=D(y-x)$ and hence

$$[\hat{\phi}(x),\hat{\phi}(y)] = 0\ ,\ \text{iff}\ (x-y)^2<0.\tag{3.157}$$

In conclusion any two measurements in the Klein-Gordon theory with one measurement lying outside the light-cone of the other measurement will not affect each other. In other words measurements attached to events separated by spacelike intervals will commute.

In the case of a timelike interval, i.e. $(x-y)^2>0$ the event y will lie inside the light-cone of the event x. Furthermore there is an inertial reference frame in which the two events occur at the same point, viz $\vec{y}=\vec{x}$. In this reference frame the amplitude is

$$D(x-y) = c^2\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}p^0(x^0-y^0)}.\tag{3.158}$$

Thus in this case the amplitudes $D(x-y)$ and $D(y-x)$ are not equal. As a consequence the commutator $[\hat{\phi}(x),\hat{\phi}(y)]$ does not vanish and hence measurements attached to events separated by timelike intervals can affect each.

Let us rewrite the commutator as

$$\begin{aligned}
<0|[\hat{\phi}(x),\hat{\phi}(y)]|0> &= [\hat{\phi}(x),\hat{\phi}(y)]\\
&= c^2\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}\left(e^{-\frac{i}{\hbar}p(x-y)}-e^{\frac{i}{\hbar}p(x-y)}\right)\\
&= c^2\int\frac{d^3p}{(2\pi\hbar)^3}\left(\frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}\left(\frac{E(\vec{p})}{c}(x^0-y^0)-\vec{p}(\vec{x}-\vec{y})\right)}\right.\\
&+ \left.\frac{1}{-2E(\vec{p})}e^{-\frac{i}{\hbar}\left(-\frac{E(\vec{p})}{c}(x^0-y^0)-\vec{p}(\vec{x}-\vec{y})\right)}\right).
\end{aligned}\tag{3.159}$$

Let us calculate from the other hand

$$\begin{aligned}
\frac{1}{c}\int\frac{dp^0}{2\pi}\frac{i}{p^2-m^2c^2}e^{-\frac{i}{\hbar}p(x-y)} &= \frac{1}{c}\int\frac{dp^0}{2\pi}\frac{i}{(p^0)^2-\frac{E(\vec{p})^2}{c^2}}e^{-\frac{i}{\hbar}p(x-y)}\\
&= \frac{1}{c}\int\frac{dp^0}{2\pi}\frac{i}{(p^0)^2-\frac{E(\vec{p})^2}{c^2}}e^{-\frac{i}{\hbar}\left(p^0(x^0-y^0)-\vec{p}(\vec{x}-\vec{y})\right)}.
\end{aligned}\tag{3.160}$$

There are two poles on the real axis at $p^0 = \pm E(\vec{p})/c$. In order to use the residue theorem we must close the contour of integration. In this case we close the contour such that both poles are included and assuming that $x^0 - y^0 > 0$ the contour must be closed below. Clearly for $x^0 - y^0 < 0$ we must close the contour above which then yields zero. We get then

$$\begin{aligned}
\frac{1}{c}\int \frac{dp^0}{2\pi}\frac{i}{p^2 - m^2c^2}e^{-\frac{i}{\hbar}p(x-y)} &= \frac{i}{2\pi c}(-2\pi i)\Bigg[\Bigg(\frac{p^0 - \frac{E(\vec{p})}{c}}{(p^0)^2 - \frac{E(\vec{p})^2}{c^2}}e^{-\frac{i}{\hbar}\left(p^0(x^0-y^0) - \vec{p}(\vec{x}-\vec{y})\right)}\Bigg)_{p^0 = E(\vec{p})/c} \\
&+ \Bigg(\frac{p^0 + \frac{E(\vec{p})}{c}}{(p^0)^2 - \frac{E(\vec{p})^2}{c^2}}e^{-\frac{i}{\hbar}\left(p^0(x^0-y^0) - \vec{p}(\vec{x}-\vec{y})\right)}\Bigg)_{p^0 = -E(\vec{p})/c}\Bigg] \\
&= \frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}\left(\frac{E(\vec{p})}{c}(x^0-y^0) - \vec{p}(\vec{x}-\vec{y})\right)} + \frac{1}{-2E(\vec{p})}e^{-\frac{i}{\hbar}\left(-\frac{E(\vec{p})}{c}(x^0-y^0) - \vec{p}(\vec{x}-\vec{y})\right)}.
\end{aligned} \tag{3.161}$$

Thus we get

$$\begin{aligned}
D_R(x-y) &= \theta(x^0 - y^0) < 0|[\hat{\phi}(x), \hat{\phi}(y)]|0 > \\
&= c\hbar \int \frac{d^4p}{(2\pi\hbar)^4}\frac{i}{p^2 - m^2c^2}e^{-\frac{i}{\hbar}p(x-y)}.
\end{aligned} \tag{3.162}$$

Clearly this function satisfies

$$(\partial_\mu \partial^\mu + \frac{m^2c^2}{\hbar^2})D_R(x-y) = -i\frac{c}{\hbar}\delta^4(x-y). \tag{3.163}$$

This is a retarded (since it vanishes for $x^0 < y^0$) Green's function of the Klein-Gordon equation.

In the above analysis the contour used is only one possibility among four possible contours. It yielded the retarded Green's function which is non-zero only for $x^0 > y^0$. The second contour is the contour which gives the advanced Green's function which is non-zero only for $x^0 < y^0$. The third contour corresponds to the so-called Feynman prescription given by

$$D_F(x-y) = c\hbar \int \frac{d^4p}{(2\pi\hbar)^4}\frac{i}{p^2 - m^2c^2 + i\epsilon}e^{-\frac{i}{\hbar}p(x-y)}. \tag{3.164}$$

The convention is to take $\epsilon > 0$. The fourth contour corresponds to $\epsilon < 0$.

In the case of the Feynman prescription we close for $x^0 > y^0$ the contour below so only the pole $p^0 = E(\vec{p})/c - i\epsilon^{'}$ will be included. The integral reduces to $D(x-y)$. For $x^0 < y^0$ we close the contour above so only the pole $p^0 = -E(\vec{p})/c + i\epsilon^{'}$ will be included. The integral reduces to $D(y-x)$. In summary we have

$$\begin{aligned}
D_F(x-y) &= \theta(x^0 - y^0)D(x-y) + \theta(y^0 - x^0)D(y-x) \\
&= < 0|T\hat{\phi}(x)\hat{\phi}(y)|0 > .
\end{aligned} \tag{3.165}$$

The time-ordering operator is defined by

$$\begin{aligned}
&T\hat{\phi}(x)\hat{\phi}(y) = \hat{\phi}(x)\hat{\phi}(y)\ ,\ x^0 > y^0 \\
&T\hat{\phi}(x)\hat{\phi}(y) = \hat{\phi}(y)\hat{\phi}(x)\ , x^0 < y^0.
\end{aligned} \tag{3.166}$$

By construction $D_F(x-y)$ must satisfy the Green's function equation (3.163). The Green's function $D_F(x-y)$ is called the Feynman propagator for a real scalar field.

3.5.2 Dirac Propagator

The probability amplitudes for a Dirac particle to propagate from the spacetime point y to the spacetime x is

$$S_{ab}(x-y) =< 0|\hat{\psi}_a(x)\bar{\hat{\psi}}_b(y)|0> . \quad (3.167)$$

The probability amplitudes for a Dirac antiparticle to propagate from the spacetime point x to the spacetime y is

$$\bar{S}_{ba}(y-x) =< 0|\bar{\hat{\psi}}_b(y)\hat{\psi}_a(x)|0> . \quad (3.168)$$

We compute

$$S_{ab}(x-y) = \frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D(x-y). \quad (3.169)$$

$$\bar{S}_{ba}(y-x) = -\frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D(y-x). \quad (3.170)$$

The retarded Green's function of the Dirac equation can be defined by

$$(S_R)_{ab}(x-y) = \frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D_R(x-y). \quad (3.171)$$

It is not difficult to convince ourselves that

$$(S_R)_{ab}(x-y) = \theta(x^0-y^0) < 0|\{\hat{\psi}_a(x), \bar{\hat{\psi}}_b(y)\}_+|0> . \quad (3.172)$$

This satisfies the equation

$$(i\hbar\gamma^\mu\partial_\mu^x - mc)_{ca}(S_R)_{ab}(x-y) = i\hbar\delta^4(x-y)\delta_{cb}. \quad (3.173)$$

Another solution of this equation is the so-called Feynman propagator for a Dirac spinor field given by

$$(S_F)_{ab}(x-y) = \frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D_F(x-y). \quad (3.174)$$

We compute

$$(S_F)_{ab}(x-y) = < 0|T\hat{\psi}_a(x)\bar{\hat{\psi}}_b(y)|0> . \quad (3.175)$$

The time-ordering operator is defined by

$$T\hat{\psi}(x)\hat{\psi}(y) = \hat{\psi}(x)\hat{\psi}(y) \ , \ x^0 > y^0$$
$$T\hat{\psi}(x)\hat{\psi}(y) = -\hat{\psi}(y)\hat{\psi}(x) \ , x^0 < y^0. \quad (3.176)$$

By construction $S_F(x-y)$ must satisfy the Green's function equation (3.173). This can also be checked directly from the Fourier expansion of $S_F(x-y)$ given by

$$(S_F)_{ab}(x-y) = \hbar\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i(\gamma^\mu p_\mu + mc)_{ab}}{p^2-m^2c^2+i\epsilon}e^{-\frac{i}{\hbar}p(x-y)}. \quad (3.177)$$

3.6 Discrete Symmetries

In the quantum theory corresponding to each continuous Lorentz transformation Λ there is a unitary transformation $U(\Lambda)$ acting in the Hilbert space of state vectors. Indeed all state vectors $|\alpha>$ will transform under Lorentz transformations as $|\alpha>\longrightarrow U(\Lambda)|\alpha>$. In order that the general matrix elements $<\beta|\mathcal{O}(\hat{\psi},\bar{\hat{\psi}})|\alpha>$ be Lorentz invariant the field operator $\hat{\psi}(x)$ must transform as

$$\hat{\psi}(x)\longrightarrow\hat{\psi}^{'}(x)=U(\Lambda)^{+}\hat{\psi}(x)U(\Lambda). \tag{3.178}$$

Hence we must have

$$S(\Lambda)\hat{\psi}(\Lambda^{-1}x)=U(\Lambda)^{+}\hat{\psi}(x)U(\Lambda). \tag{3.179}$$

In the case of a scalar field $\hat{\phi}(x)$ we must have instead

$$\hat{\phi}(\Lambda^{-1}x)=U(\Lambda)^{+}\hat{\phi}(x)U(\Lambda). \tag{3.180}$$

There are two discrete spacetime symmetries of great importance to particle physics. The first discrete transformation is parity defined by

$$(t,\vec{x})\longrightarrow P(t,\vec{x})=(t,-\vec{x}). \tag{3.181}$$

The second discrete transformation is time reversal defined by

$$(t,\vec{x})\longrightarrow T(t,\vec{x})=(-t,\vec{x}). \tag{3.182}$$

The Lorentz group consists of four disconnected subroups. The subgroup of continuous Lorentz transformations consists of all Lorentz transfomrations which can be obtained from the identity transformation. This is called the proper orthochronous Lorentz group. The improper orthochronous Lorentz group is obtained by the action of parity on the proper orthochronous Lorentz group. The proper nonorthochronous Lorentz group is obtained by the action of time reversal on the proper orthochronous Lorentz group. The improper nonorthochronous Lorentz group is obtained by the action of parity and then time reversal or by the action of time reversal and then parity on the proper orthochronous Lorentz group.

A third discrete symmetry of fundamental importance to particle physics is charge conjugation operation C. This is not a spacetime symmetry. This is a symmetry under which particles become their antiparticles. It is well known that parity P, time reversal T and charge conjugation C are symmetries of gravitational, electromagnetic and strong interactions. The weak interactions violate P and C and to a lesser extent T and CP but it is observed that all fundamental forces conserve CPT.

3.6.1 Parity

The action of parity on the spinor field operator is

$$\begin{aligned}
U(P)^{+}\hat{\psi}(x)U(P) &= \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i\bigg(e^{-\frac{i}{\hbar}px}u^{(i)}(\vec{p})U(P)^{+}\hat{b}(\vec{p},i)U(P)\\
&+ e^{\frac{i}{\hbar}px}v^{(i)}(\vec{p})U(P)^{+}\hat{d}(\vec{p},i)^{+}U(P)\bigg)\\
&= S(P)\hat{\psi}(P^{-1}x).
\end{aligned} \tag{3.183}$$

We need to rewrite this operator in terms of $\tilde{x} = P^{-1}x = (x^0, -\vec{x})$. Thus $px = \tilde{p}\tilde{x}$ where $\tilde{p} = P^{-1}x = (p^0, -\vec{p})$. We have also $\sigma p = \bar{\sigma}\tilde{p}$ and $\bar{\sigma} p = \sigma\tilde{p}$. As a consequence we have

$$u^{(i)}(\vec{p}) = \gamma^0 u^{(i)}(\vec{\tilde{p}})\ ,\ v^{(i)}(\vec{p}) = -\gamma^0 v^{(i)}(\vec{\tilde{p}}). \tag{3.184}$$

Hence

$$\begin{aligned} U(P)^+\hat{\psi}(x)U(P) &= \gamma^0\frac{1}{\hbar}\int\frac{d^3\tilde{p}}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{\tilde{p}})}}\sum_i\bigg(e^{-\frac{i}{\hbar}\tilde{p}\tilde{x}}u^{(i)}(\vec{\tilde{p}})U(P)^+\hat{b}(\vec{p},i)U(P)\\ &- e^{\frac{i}{\hbar}\tilde{p}\tilde{x}}v^{(i)}(\vec{\tilde{p}})U(P)^+\hat{d}(\vec{p},i)^+U(P)\bigg). \end{aligned} \tag{3.185}$$

The parity operation flips the direction of the momentum but not the direction of the spin. Thus we expect that

$$U(P)^+\hat{b}(\vec{p},i)U(P) = \eta_b\hat{b}(-\vec{p},i)\ ,\ U(P)^+\hat{d}(\vec{p},i)U(P) = \eta_d\hat{d}(-\vec{p},i). \tag{3.186}$$

The phases η_b and η_a must clearly satisfy

$$\eta_b^2 = 1\ ,\ \eta_d^2 = 1. \tag{3.187}$$

Hence we obtain

$$U(P)^+\hat{\psi}(x)U(P) = \gamma^0\frac{1}{\hbar}\int\frac{d^3\tilde{p}}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{\tilde{p}})}}\sum_i\bigg(\eta_b e^{-\frac{i}{\hbar}\tilde{p}\tilde{x}}u^{(i)}(\vec{\tilde{p}})\hat{b}(\vec{\tilde{p}},i) - \eta_d^* e^{\frac{i}{\hbar}\tilde{p}\tilde{x}}v^{(i)}(\vec{\tilde{p}})\hat{d}(\vec{\tilde{p}},i)^+\bigg). \tag{3.188}$$

This should equal $S(P)\hat{\psi}(\tilde{x})$. Immediately we conclude that we must have

$$\eta_d^* = -\eta_b. \tag{3.189}$$

Hence

$$U(P)^+\hat{\psi}(x)U(P) = \eta_b\gamma^0\hat{\psi}(\tilde{x}). \tag{3.190}$$

3.6.2 Time Reversal

The action of time reversal on the spinor field operator is

$$\begin{aligned} U(T)^+\hat{\psi}(x)U(T) &= \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i\bigg(U(T)^+e^{-\frac{i}{\hbar}px}u^{(i)}(\vec{p})\hat{b}(\vec{p},i)U(T)\\ &+ U(T)^+e^{\frac{i}{\hbar}px}v^{(i)}(\vec{p})\hat{d}(\vec{p},i)^+U(T)\bigg)\\ &= S(T)\hat{\psi}(T^{-1}x). \end{aligned} \tag{3.191}$$

This needs to be rewritten in terms of $\tilde{x} = T^{-1}x = (-x^0, \vec{x})$. Time reversal reverses the direction of the momentum in the sense that $px = -\tilde{p}\tilde{x}$ where $\tilde{p} = (p^0, -\vec{p})$. Clearly if $U(T)$ is an ordinary unitary operator the phases $e^{\mp\frac{i}{\hbar}px}$ will go to their complex conjugates $e^{\pm\frac{i}{\hbar}px}$ under time reversal. In other words if $U(T)$ is an ordinary unitary operator the field operator $U(T)^+\hat{\psi}(x)U(T)$ can not be written as a constant matrix times $\hat{\psi}(\tilde{x})$. The solution is to choose $U(T)$ to be an antilinear operator defined by

$$U(T)^+c = c^*U(T)^+. \tag{3.192}$$

Hence we get

$$\begin{aligned} U(T)^+\hat{\psi}(x)U(T) &= \frac{1}{\hbar}\int\frac{d^3\tilde{p}}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{\tilde{p}})}}\sum_i\bigg(e^{-\frac{i}{\hbar}\tilde{p}\tilde{x}}u^{(i)*}(\vec{p})U(T)^+\hat{b}(\vec{p},i)U(T) \\ &+ e^{\frac{i}{\hbar}\tilde{p}\tilde{x}}v^{(i)*}(\vec{p})U(T)^+\hat{d}(\vec{p},i)^+U(T)\bigg). \end{aligned} \tag{3.193}$$

We recall that

$$u^{(1)}(\vec{p}) = N^{(1)}\begin{pmatrix} \xi_0^1 \\ \frac{\frac{E}{c}+\vec{\sigma}\vec{p}}{mc}\xi_0^1 \end{pmatrix}\ ,\ v^{(1)} = N^{(3)}\begin{pmatrix} -\frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}\eta_0^1 \\ \eta_0^1 \end{pmatrix}. \tag{3.194}$$

Hence (by using $\sigma^{i*}=-\sigma^2\sigma^i\sigma^2$) we obtain

$$u^{(1)*}(\vec{p}) = N^{(1)}\begin{pmatrix} \xi_0^{1*} \\ \sigma^2\frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}\sigma^2\xi_0^{1*} \end{pmatrix} = N^{(1)}\gamma^1\gamma^3\begin{pmatrix} -i\sigma^2\xi_0^{1*} \\ \frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}(-i\sigma^2\xi_0^{1*}) \end{pmatrix}. \tag{3.195}$$

$$v^{(1)*}(\vec{p}) = N^{(3)}\begin{pmatrix} \sigma^2\frac{-\frac{E}{c}+\vec{\sigma}\vec{p}}{mc}\sigma^2\eta_0^{1*} \\ \eta_0^{1*} \end{pmatrix} = N^{(3)}\gamma^1\gamma^3\begin{pmatrix} \frac{-\frac{E}{c}+\vec{\sigma}\vec{p}}{mc}(-i\sigma^2\eta_0^{1*}) \\ -i\sigma^2\eta_0^{1*} \end{pmatrix}. \tag{3.196}$$

We define

$$\xi_0^{-s} = -i\sigma^2\xi_0^{s*}\ ,\ \eta_0^{-s} = i\sigma^2\eta_0^{s*}. \tag{3.197}$$

Note that we can take ξ_0^{-s} proportional to η_0^s. We obtain then

$$u^{(1)*}(\vec{p}) = N^{(1)}\gamma^1\gamma^3\begin{pmatrix} \xi_0^{-1} \\ \frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}\xi_0^{-1} \end{pmatrix} = \gamma^1\gamma^3\begin{pmatrix} \sqrt{\sigma_\mu\tilde{p}^\mu}\xi^{-1} \\ \sqrt{\bar{\sigma}_\mu\tilde{p}^\mu}\xi^{-1} \end{pmatrix} = \gamma^1\gamma^3u^{(-1)}(\vec{\tilde{p}}). \tag{3.198}$$

$$v^{(1)*}(\vec{p}) = -N^{(3)}\gamma^1\gamma^3\begin{pmatrix} \frac{-\frac{E}{c}+\vec{\sigma}\vec{p}}{mc}\eta_0^{-1} \\ \eta_0^{-1} \end{pmatrix} = -\gamma^1\gamma^3\begin{pmatrix} \sqrt{\sigma_\mu\tilde{p}^\mu}\eta^{-1} \\ -\sqrt{\bar{\sigma}_\mu\tilde{p}^\mu}\eta^{-1} \end{pmatrix} = -\gamma^1\gamma^3v^{(-1)}(\vec{\tilde{p}}). \tag{3.199}$$

Similarly we can show that

$$u^{(2)*}(\vec{p}) = \gamma^1\gamma^3u^{(-2)}(\vec{\tilde{p}})\ ,\ v^{(2)*}(\vec{p}) = -\gamma^1\gamma^3v^{(-2)}(\vec{\tilde{p}}). \tag{3.200}$$

In the above equations

$$\xi^{-s} = N^{(1)}(-\tilde{p}^3)\frac{1}{\sqrt{\sigma_\mu\tilde{p}^\mu}}\xi_0^{-s}\ ,\ \eta^{-s} = -N^{(3)}(-\tilde{p}^3)\frac{1}{\sqrt{\bar{\sigma}_\mu\tilde{p}^\mu}}\eta_0^s. \tag{3.201}$$

Let us remark that if ξ_0^i is an eigenvector of $\vec{\sigma}\hat{n}$ with spin s then ξ_0^{-i} is an eigenvector of $\vec{\sigma}\hat{n}$ with spin $-s$, viz

$$\vec{\sigma}\hat{n}\xi_0^i = s\xi_0^i \ \Leftrightarrow \vec{\sigma}\hat{n}\xi_0^{-i} = -s\xi_0^{-i}. \tag{3.202}$$

Now going back to equation (3.193) we get

$$\begin{aligned} U(T)^+\hat{\psi}(x)U(T) &= \frac{1}{\hbar}\gamma^1\gamma^3\int\frac{d^3\tilde{p}}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{\tilde{p}})}}\sum_i\bigg(e^{-\frac{i}{\hbar}\tilde{p}\tilde{x}}u^{(-i)}(\vec{\tilde{p}})U(T)^+\hat{b}(\vec{p},i)U(T) \\ &- e^{\frac{i}{\hbar}\tilde{p}\tilde{x}}v^{(-i)}(\vec{\tilde{p}})U(T)^+\hat{d}(\vec{p},i)^+U(T)\bigg). \end{aligned} \tag{3.203}$$

Time reversal reverses the direction of the momentum and of the spin. Thus we write

$$U(T)^+\hat{b}(\vec{p},i)U(T) = \eta_b\hat{b}(-\vec{p},-i)\ ,\ U(T)^+\hat{d}(\vec{p},i)U(T) = \eta_d\hat{d}(-\vec{p},-i). \tag{3.204}$$

We get then

$$\begin{aligned} U(T)^+\hat{\psi}(x)U(T) &= \frac{1}{\hbar}\gamma^1\gamma^3\int\frac{d^3\tilde{p}}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{\tilde{p}})}}\sum_i\bigg(\eta_b e^{-\frac{i}{\hbar}\tilde{p}\tilde{x}}u^{(-i)}(\vec{\tilde{p}})\hat{b}(\vec{\tilde{p}},-i) \\ &- \eta_d^* e^{\frac{i}{\hbar}\tilde{p}\tilde{x}}v^{(-i)}(\vec{\tilde{p}})\hat{d}(\vec{\tilde{p}},-i)^+\bigg). \end{aligned} \tag{3.205}$$

By analogy with $\xi_0^{-s} = -i\sigma^2\xi_0^{s*}$ we define

$$\hat{b}(\vec{\tilde{p}},-i) = -(-i\sigma^2)_{ij}\hat{b}(\vec{\tilde{p}},j)\ ,\ \hat{d}(\vec{\tilde{p}},-i) = -(-i\sigma^2)_{ij}\hat{d}(\vec{\tilde{p}},j). \tag{3.206}$$

Also we choose

$$\eta_d^* = -\eta_b. \tag{3.207}$$

Hence

$$\begin{aligned} U(T)^+\hat{\psi}(x)U(T) &= \frac{\eta_b}{\hbar}\gamma^1\gamma^3\int\frac{d^3\tilde{p}}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{\tilde{p}})}}\sum_i\bigg(e^{-\frac{i}{\hbar}\tilde{p}\tilde{x}}u^{(-i)}(\vec{\tilde{p}})\hat{b}(\vec{\tilde{p}},-i) + e^{\frac{i}{\hbar}\tilde{p}\tilde{x}}v^{(-i)}(\vec{\tilde{p}})\hat{d}(\vec{\tilde{p}},-i)^+\bigg) \\ &= \eta_b\gamma^1\gamma^3\hat{\psi}(-x^0,\vec{x}). \end{aligned} \tag{3.208}$$

3.6.3 Charge Conjugation

This is defined simply by (with $C^+ = C^{-1} = C$)

$$C\hat{b}(\vec{p},i)C = \hat{d}(\vec{p},i)\ ,\ C\hat{d}(\vec{p},i)C = \hat{b}(\vec{p},i) \tag{3.209}$$

Hence

$$C\hat{\psi}(x)C = \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i\bigg(e^{-\frac{i}{\hbar}px}u^{(i)}(\vec{p})\hat{d}(\vec{p},i) + e^{\frac{i}{\hbar}px}v^{(i)}(\vec{p})\hat{b}(\vec{p},i)^+\bigg). \tag{3.210}$$

Let us remark that (by choosing $N^{(1)}\xi_0^{-i}=-N^{(3)}\eta_0^i$ or equivalently $\xi^{-i}=\eta^i$)

$$u^{(1)*}(\vec{p}) \;=\; iN^{(1)}\gamma^2\begin{pmatrix} -\frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}\xi_0^{-1} \\ \xi_0^{-1}\end{pmatrix}=-iN^{(3)}\gamma^2\begin{pmatrix} -\frac{\frac{E}{c}-\vec{\sigma}\vec{p}}{mc}\eta_0^{1} \\ \eta_0^{1}\end{pmatrix}=-i\gamma^2 v^{(1)}(\vec{p}). \tag{3.211}$$

In other words

$$u^{(1)}(\vec{p})=-i\gamma^2 v^{(1)*}(\vec{p})\ ,\ v^{(1)}(\vec{p})=-i\gamma^2u^{(1)*}(\vec{p}). \tag{3.212}$$

Similarly we find

$$u^{(2)}(\vec{p})=-i\gamma^2 v^{(2)*}(\vec{p})\ ,\ v^{(2)}(\vec{p})=-i\gamma^2u^{(2)*}(\vec{p}). \tag{3.213}$$

Thus we have

$$\begin{aligned}
C\hat{\psi}(x)C &= \frac{1}{\hbar}(-i\gamma^2)\int\frac{d^3p}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i\Big(e^{-\frac{i}{\hbar}px}v^{(i)*}(\vec{p})\hat{d}(\vec{p},i)+e^{\frac{i}{\hbar}px}u^{(i)*}(\vec{p})\hat{b}(\vec{p},i)^+\Big)\\
&= \frac{1}{\hbar}(-i\gamma^2)\int\frac{d^3p}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i\Big(e^{\frac{i}{\hbar}px}v^{(i)}(\vec{p})\hat{d}(\vec{p},i)^++e^{-\frac{i}{\hbar}px}u^{(i)}(\vec{p})\hat{b}(\vec{p},i)\Big)^*\\
&= -i\gamma^2\psi^*(x).
\end{aligned} \tag{3.214}$$

3.7 Exercises and Problems

Scalars Commutation Relations Show that

- $$\hat{Q}(x^0,-\vec{p})=\hat{Q}^+(x^0,\vec{p}).$$
- $$[\hat{Q}(x^0,\vec{p}),\hat{P}^+(x^0,\vec{q})]=i\hbar(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}).$$
- $$[\hat{a}(\vec{p}),\hat{a}(\vec{q})^+]=\hbar(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}).$$

The One-Particle States For a real scalar field theory the one-particle states are defined by

$$|\vec{p}>=\frac{1}{c}\sqrt{2\omega(\vec{p})}\hat{a}(\vec{p})^+|0>.$$

- Compute the energy of this state.
- Compute the scalar product $<\vec{p}|\vec{q}>$ and show that it is Lorentz invariant.
- Show that $\hat{\phi}(x)|0>$ can be interpreted as the eigenstate $|\vec{x}>$ of the position operator at time x^0.

Momentum Operator

1) Compute the total momentum operator of a quantum real scalar field in terms of the creation and annihilation operators $\hat{a}(\vec{p})^+$ and $\hat{a}(\vec{p})$.

2) What is the total momentum operator for a Dirac field.

Fermions Anticommutation Relations Show that

- $$[\hat{\chi}_\alpha(x^0,\vec{p}),\hat{\chi}^+_\beta(x^0,\vec{q})]_+ = \hbar^2\delta_{\alpha\beta}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}).$$

- $$[\hat{b}(\vec{p},i),\hat{b}(\vec{q},j)^+]_+ = [\hat{d}(\vec{p},i)^+,\hat{d}(\vec{q},j)]_+ = \hbar\delta_{ij}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}).$$

$$[\hat{b}(\vec{p},i),\hat{d}(\vec{q},j)]_+ = [\hat{d}(\vec{q},j)^+,\hat{b}(\vec{p},i)]_\pm = 0.$$

Retarded Propagator The retarded propagator is

$$D_R(x-y) = c\hbar\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i}{p^2-m^2c^2}e^{-\frac{i}{\hbar}p(x-y)}.$$

Show that the Klein-Gordon equation with contact term, viz

$$(\partial_\mu\partial^\mu+\frac{m^2c^2}{\hbar^2})D_R(x-y) = -i\frac{c}{\hbar}\delta^4(x-y).$$

Feynman Propagator We give the scalar Feynman propagator by the equation

$$D_F(x-y) = c\hbar\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i}{p^2-m^2c^2+i\epsilon}e^{-\frac{i}{\hbar}p(x-y)}.$$

- Perform the integral over p_0 and show that

$$D_F(x-y) = \theta(x^0-y^0)D(x-y)+\theta(y^0-x^0)D(y-x).$$

- Show that

$$D_F(x-y) = <0|T\hat{\phi}(x)\hat{\phi}(y)|0>,$$

where T is the time-ordering operator.

The Dirac Propagator The probability amplitudes for a Dirac particle (antiparticle) to propagate from the spacetime point y (x) to the spacetime x (y) are

$$S_{ab}(x-y) = <0|\hat{\psi}_a(x)\bar{\hat{\psi}}_b(y)|0> .$$

$$\bar{S}_{ba}(y-x) = <0|\bar{\hat{\psi}}_b(y)\hat{\psi}_a(x)|0> .$$

1) Compute S and $\bar{S}$ in terms of the Klein-Gordon propagator $D(x-y)$ given by

$$D(x-y) = \int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}p(x-y)}.$$

2) Show that the retarded Green's function of the Dirac equation is given by

$$(S_R)_{ab}(x-y) = <0|\{\hat{\psi}_a(x),\bar{\hat{\psi}}_b(y)\}|0> .$$

3) Verify that S_R satisfies the Dirac equation

$$(i\hbar\gamma^\mu\partial_\mu^x - mc)_{ca}(S_R)_{ab}(x-y) = i\frac{\hbar}{c}\delta^4(x-y)\delta_{cb}.$$

4) Derive an expression of the Feynman propagator in terms of the Dirac fields $\hat{\psi}$ and $\bar{\hat{\psi}}$ and then write down its Fourier Expansion.

Dirac Hamiltonian Show that the Dirac Hamiltonian

$$\hat{H}_{\rm Dirac} = \int \frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\sum_i \Big(\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},i) + \hat{d}(\vec{p},i)^+\hat{d}(\vec{p},i)\Big),$$

satisfies

$$[\hat{H}_{\rm Dirac},\hat{b}(\vec{p},i)^+] = \hbar\omega(\vec{p})\hat{b}(\vec{p},i)^+ \ , \ [\hat{H}_{\rm Dirac},\hat{d}(\vec{p},i)^+] = \hbar\omega(\vec{p})\hat{d}(\vec{p},i)^+.$$

Energy-Momentum Tensor Noether's theorem states that each continuous symmetry transformation which leaves the action invariant corresponds to a conservation law and as a consequence leads to a constant of the motion.

We consider a single real scalar field ϕ with a Lagrangian density $\mathcal{L}(\phi,\partial_\mu\phi)$. Prove Noether's theorem for spacetime translations given by

$$x^\mu \longrightarrow x^{'\mu} = x^\mu + a^\mu.$$

In particular determine the four conserved currents and the four conserved charges (constants of the motion) in terms of the field ϕ.

Electric Charge

1) The continuity equation for a Dirac wave function is

$$\partial_\mu J^\mu = 0 \ , \ J^\mu = \bar{\psi}\gamma^\mu\psi.$$

The current J^μ is conserved. According to Noether's theorem this conserved current (when we go to the field theory) must correspond to the invariance of the action under a symmetry principle. Determine the symmetry transformations in this case.

2) The associated conserved charge is

$$Q=\int d^3xJ^0.$$

Compute Q for a quantized Dirac field. What is the physical interpretation of Q.

Chiral Invariance

1) Rewrite the Dirac Lagrangian in terms of ψ_L and ψ_R.

2) The Dirac Lagrangian is invariant under the vector transformations

$$\psi\longrightarrow e^{i\alpha}\psi.$$

Derive the conserved current j^μ.

3) The Dirac Lagrangian is almost invariant under the axial vector transformations

$$\psi\longrightarrow e^{i\gamma^5\alpha}\psi.$$

Derive the would-be current $j^{\mu 5}$ in this case. Determine the condition under which this becomes a conserved current.

4) Show that in the massless limit

$$j^\mu=j^\mu_L+j^\mu_R\ ,\ j^{\mu 5}=-j^\mu_L+j^\mu_R.$$

$$j^\mu_L=\bar{\Psi}_L\gamma^\mu\Psi_L\ ,\ j^\mu_R=\bar{\Psi}_R\gamma^\mu\Psi_R.$$

Parity and Time Reversal Determine the transformation rule under parity and time reversal transformations of $\bar{\psi}$, $\bar{\psi}\psi$, $i\bar{\psi}\gamma^5\psi$, $\bar{\psi}\gamma^\mu\psi$ and $\bar{\psi}\gamma^\mu\gamma^5\psi$.

Angular Momentum of Dirac Field

- Write down the infinitesimal Lorentz transformation corresponding to an infintesimal rotation around the z axis with an angle θ.
- From the effect of a Lorentz transformation on a Dirac spinor calculate the variation in the field at a fixed point, viz

$$\delta\psi(x)=\psi^{'}(x)-\psi(x).$$

- Using Noether's theorem compute the conserved current j^μ associated with the invariance of the Lagrangian under the above rotation. The charge J^3 is defined by

$$J^3\ =\ \int d^3xj^0.$$

Show that J^3 is conserved and derive an expression for it in terms of the Dirac field. What is the physical interpretation of J^3. What is the charge in the case of a general rotation.

- In the quantum theory J^3 becomes an operator. What is the angular momentum of the vacuum.

- What is the angular momentum of a one-particle zero-momentum state defined by

$$|\vec{0}, sb >= \sqrt{\frac{2mc^2}{\hbar}} \hat{b}(\vec{0}, s)^+ |0 > .$$

  Hint: In order to answer this question we need to compute the commutator $[\hat{J}^3, \hat{b}(\vec{0}, s)^+]$.

- By analogy what is the angular momentum of a one-antiparticle zero-momentum state defined by

$$|\vec{0}, sd >= \sqrt{\frac{2mc^2}{\hbar}} \hat{d}(\vec{0}, s)^+ |0 > .$$

4

The $S-$Matrix and Feynman Diagrams For Phi-Four Theory

4.1 Forced Scalar Field

4.1.1 Asymptotic Solutions

We have learned that a free neutral particle of spin 0 can be described by a real scalar field with a Lagrangian density given by (with $\hbar = c = 1$)

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2. \tag{4.1}$$

The free field operator can be expanded as (with $p^0 = E(\vec{p}) = E_{\vec{p}}$)

$$\begin{aligned}
\hat{\phi}(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{p})}}\bigg(\hat{a}(\vec{p})e^{-ipx} + \hat{a}(\vec{p})^+e^{ipx}\bigg) \\
&= \int \frac{d^3p}{(2\pi)^3}\hat{Q}(t,\vec{p})e^{i\vec{p}\vec{x}}.
\end{aligned} \tag{4.2}$$

$$\hat{Q}(t,\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}\bigg(\hat{a}(\vec{p})e^{-iE_{\vec{p}}t} + \hat{a}(-\vec{p})^+e^{iE_{\vec{p}}t}\bigg). \tag{4.3}$$

The simplest interaction we can envisage is the action of an arbitrary external force $J(x)$ on the real scalar field $\phi(x)$. This can be described by adding a term of the form $J\phi$ to the Lagrangian density $\mathcal{L}_0$. We get then the Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 + J\phi. \tag{4.4}$$

The equations of motion become

$$(\partial_\mu\partial^\mu + m^2)\phi = J. \tag{4.5}$$

We expand the source in Fourier modes as

$$J(x) = \int \frac{d^3p}{(2\pi)^3} j(t,\vec{p}) e^{i\vec{p}\vec{x}}. \tag{4.6}$$

We get then the equations of motion in momentum space

$$(\partial_t^2 + E_{\vec{p}}^2) Q(t,\vec{p}) = j(t,\vec{p}). \tag{4.7}$$

By assuming that $j(t,\vec{p})$ vanishes outside a finite time interval we conclude that for early and late times where $j(t,\vec{p})$ is zero the field is effectively free. Thus for early times we have

$$\hat{Q}(t,\vec{p}) = \hat{Q}_{\rm in}(t,\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}\bigg(\hat{a}_{\rm in}(\vec{p}) e^{-iE_{\vec{p}}t} + \hat{a}_{\rm in}(-\vec{p})^+ e^{iE_{\vec{p}}t}\bigg)\ ,\ t \longrightarrow -\infty. \tag{4.8}$$

For late times we have

$$\hat{Q}(t,\vec{p}) = \hat{Q}_{\rm out}(t,\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}\bigg(\hat{a}_{\rm out}(\vec{p}) e^{-iE_{\vec{p}}t} + \hat{a}_{\rm out}(-\vec{p})^+ e^{iE_{\vec{p}}t}\bigg)\ ,\ t \longrightarrow +\infty. \tag{4.9}$$

The general solution is of the form

$$\hat{Q}(t,\vec{p}) = \hat{Q}_{\rm in}(t,\vec{p}) + \frac{1}{E_{\vec{p}}}\int_{-\infty}^{t} dt^{'} \sin E_{\vec{p}}(t-t^{'}) j(t^{'},\vec{p}). \tag{4.10}$$

Clearly for early times $t \longrightarrow -\infty$ we get $\hat{Q} \longrightarrow \hat{Q}_{\rm in}$. On the other hand since for late times $t \longrightarrow +\infty$ we have $\hat{Q} \longrightarrow \hat{Q}_{\rm out}$ we must have

$$\hat{Q}_{\rm out}(t,\vec{p}) = \hat{Q}_{\rm in}(t,\vec{p}) + \frac{1}{E_{\vec{p}}}\int_{-\infty}^{+\infty} dt^{'} \sin E_{\vec{p}}(t-t^{'}) j(t^{'},\vec{p}). \tag{4.11}$$

We define the positive-energy and the negative-energy parts of $\hat{Q}$ by

$$\hat{Q}^+(t,\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}\hat{a}(\vec{p}) e^{-iE_{\vec{p}}t}\ ,\ \hat{Q}^-(t,\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}\hat{a}(-\vec{p})^+ e^{iE_{\vec{p}}t}. \tag{4.12}$$

Equation (4.10) is equivalent to the two equations

$$\hat{Q}^{\pm}(t,\vec{p}) = \hat{Q}^{\pm}_{\rm in}(t,\vec{p}) \pm \frac{i}{2E_{\vec{p}}}\int_{-\infty}^{t} dt^{'}\, e^{\mp iE_{\vec{p}}(t-t^{'})} j(t^{'},\vec{p}). \tag{4.13}$$

The Feynman propagator in one-dimension is given by

$$G_{\vec{p}}(t-t^{'}) = \frac{e^{-iE_{\vec{p}}|t-t^{'}|}}{2E_{\vec{p}}} = \int \frac{dE}{2\pi}\frac{i}{E^2 - E_{\vec{p}}^2 + i\epsilon} e^{-iE(t-t^{'})}. \tag{4.14}$$

Note that in our case $t - t^{'} > 0$. Hence

$$\hat{Q}^+(t,\vec{p}) = \hat{Q}^+_{\rm in}(t,\vec{p}) + i\int_{-\infty}^{t} dt^{'}\, G_{\vec{p}}(t-t^{'}) j(t^{'},\vec{p}). \tag{4.15}$$

$$\hat{Q}^-(t,\vec{p}) = \hat{Q}^-_{\rm in}(t,\vec{p}) - i\int_{-\infty}^{t} dt^{'}\, G_{\vec{p}}(t^{'}-t) j(t^{'},\vec{p}). \tag{4.16}$$

For late times we get

$$\hat{Q}^{+}_{\rm out}(t,\vec{p})=\hat{Q}^{+}_{\rm in}(t,\vec{p})+i\int_{-\infty}^{+\infty}dt^{'}G_{\vec{p}}(t-t^{'})j(t^{'},\vec{p}). \tag{4.17}$$

$$\hat{Q}^{-}_{\rm out}(t,\vec{p})=\hat{Q}^{-}_{\rm in}(t,\vec{p})-i\int_{-\infty}^{+\infty}dt^{'}G_{\vec{p}}(t^{'}-t)j(t^{'},\vec{p}). \tag{4.18}$$

These two equations are clearly equivalent to equation (4.11).

The above two equations can be rewritten as

$$\hat{Q}^{\pm}_{\rm out}(t,\vec{p})=\hat{Q}^{\pm}_{\rm in}(t,\vec{p})\pm\frac{i}{2E_{\vec{p}}}\int_{-\infty}^{+\infty}dt^{'}e^{\mp iE_{\vec{p}}(t-t^{'})}j(t^{'},\vec{p}). \tag{4.19}$$

In terms of the creation and annihilation operators this becomes

$$\hat{a}_{\rm out}(\vec{p})=\hat{a}_{\rm in}(\vec{p})+\frac{i}{\sqrt{2E_{\vec{p}}}}j(p)~,~\hat{a}_{\rm out}(\vec{p})^{+}=\hat{a}_{\rm in}(\vec{p})^{+}-\frac{i}{\sqrt{2E_{\vec{p}}}}j(-p). \tag{4.20}$$

$$j(p)\equiv j(E_{\vec{p}},\vec{p})=\int dte^{iE_{\vec{p}}t}j(t,\vec{p}). \tag{4.21}$$

We observe that the "in" operators and the "out" operators are different. Hence there exists two different Hilbert spaces and as a consequence two different vacua $|0~{\rm in}>$ and $|0~{\rm out}>$ defined by

$$\hat{a}_{\rm out}(\vec{p})|0~{\rm out}>=0~,~\hat{a}_{\rm in}(\vec{p})|0~{\rm in}>=0~\forall\vec{p}. \tag{4.22}$$

4.1.2 The Schrodinger, Heisenberg and Dirac Pictures

The Lagrangian from which the equation of motion (4.7) is derived is

$$\int_{+}\frac{d^3p}{(2\pi)^3}\bigg(\partial_tQ(t,\vec{p})^{*}\partial_tQ(t,\vec{p})-E_{\vec{p}}^2Q(t,\vec{p})^{*}Q(t,\vec{p})+j(t,\vec{p})^{*}Q(t,\vec{p})+j(t,\vec{p})Q(t,\vec{p})^{*}\bigg). \tag{4.23}$$

The corresponding Hamiltonian is (with $P(t,\vec{p})=\partial_tQ(t,\vec{p})$)

$$\int_{+}\frac{d^3p}{(2\pi)^3}\bigg(P(t,\vec{p})^{*}P(t,\vec{p})+E_{\vec{p}}^2Q(t,\vec{p})^{*}Q(t,\vec{p})-j(t,\vec{p})^{*}Q(t,\vec{p})-j(t,\vec{p})Q(t,\vec{p})^{*}\bigg). \tag{4.24}$$

The operators $\hat{P}(t,\vec{p})$ and $\hat{Q}(t,\vec{p})$ are the time-dependent Heisenberg operators. The time-independent Schrodinger operators will be denoted by $\hat{P}(\vec{p})$ and $\hat{Q}(\vec{p})$. In the Schrodinger picture the Hamiltonian is given by

$$\int_{+}\frac{d^3p}{(2\pi)^3}\bigg(P(\vec{p})^{*}P(\vec{p})+E_{\vec{p}}^2Q(\vec{p})^{*}Q(\vec{p})-j(t,\vec{p})^{*}Q(\vec{p})-j(t,\vec{p})Q(\vec{p})^{*}\bigg). \tag{4.25}$$

This Hamiltonian depends on time only through the time-dependence of the source. Using box normalization the momenta become discrete and the measure $\int d^3p/(2\pi)^3$ becomes the sum $\sum_{\vec{p}}/V$. Thus the Hamiltonian becomes

$$\sum_{p^1>0}\sum_{p^2>0}\sum_{p^3>0}\mathcal{H}_{\vec{p}}(t). \tag{4.26}$$

We recall the equal time commutation relations $[\hat{Q}(t,\vec{p}),\hat{P}(t,\vec{p})^+]=i(2\pi)^3\delta^3(\vec{p}-\vec{q})$ and $[\hat{Q}(t,\vec{p}),\hat{P}(t,\vec{p})]=[\hat{Q}(t,\vec{p}),\hat{Q}(t,\vec{p})]=[\hat{P}(t,\vec{p}),\hat{P}(t,\vec{p})]=0$. Using box normalization the equal time commutation relations take the form

$$\begin{aligned}&[\hat{Q}(t,\vec{p}),\hat{P}(t,\vec{p})^+]=iV\delta_{\vec{p},\vec{q}}\\&[\hat{Q}(t,\vec{p}),\hat{P}(t,\vec{p})]=[\hat{Q}(t,\vec{p}),\hat{Q}(t,\vec{p})]=[\hat{P}(t,\vec{p}),\hat{P}(t,\vec{p})]=0.\end{aligned} \tag{4.27}$$

The Hamiltonian of a single forced oscillator which has a momentum $\vec{p}$ is

$$\mathcal{H}_{\vec{p}}(t)=\frac{1}{V}\bigg(P(\vec{p})^*P(\vec{p})+E_{\vec{p}}^2Q(\vec{p})^*Q(\vec{p})\bigg)+V(t,\vec{p}). \tag{4.28}$$

The potential is defined by

$$V(t,\vec{p}) \;=\; -\frac{1}{V}\bigg(j(t,\vec{p})^*Q(\vec{p})+j(t,\vec{p})Q(\vec{p})^*\bigg). \tag{4.29}$$

The unitary time evolution operator must solve the Schrodinger equation

$$i\partial_tU(t)=\hat{\mathcal{H}}_{\vec{p}}(t)U(t). \tag{4.30}$$

The Heisenberg and Schrodinger operators are related by

$$\hat{Q}(t,\vec{p}) \;=\; U(t)^{-1}\hat{Q}(\vec{p})U(t). \tag{4.31}$$

We introduce the interaction picture through the unitary operator Ω defined by

$$U(t)=e^{-it\hat{\mathcal{H}}_{\vec{p}}}\Omega(t). \tag{4.32}$$

In the above equation $\mathcal{H}_{\vec{p}}$ is the free Hamiltonian density, viz

$$\mathcal{H}_{\vec{p}} \;=\; \frac{1}{V}\bigg(P(\vec{p})^*P(\vec{p})+E_{\vec{p}}^2Q(\vec{p})^*Q(\vec{p})\bigg). \tag{4.33}$$

The operator Ω satisfies the Schrodinger equation

$$i\partial_t\Omega(t)=\hat{V}_I(t,\vec{p})\Omega(t). \tag{4.34}$$

$$\begin{aligned}\hat{V}_I(t,\vec{p}) &= e^{it\hat{\mathcal{H}}_{\vec{p}}}\hat{V}(t,\vec{p})e^{-it\hat{\mathcal{H}}_{\vec{p}}}\\&= -\frac{1}{V}(j(t,\vec{p})^*\hat{Q}_I(t,\vec{p})+j(t,\vec{p})\hat{Q}_I(t,\vec{p})^+).\end{aligned} \tag{4.35}$$

The interaction, Schrodinger and Heisenberg operators are related by

$$\begin{aligned}\hat{Q}_I(t,\vec{p}) &= e^{it\hat{\mathcal{H}}_{\vec{p}}}\hat{Q}(\vec{p})e^{-it\hat{\mathcal{H}}_{\vec{p}}}\\&= \Omega(t)U(t)^{-1}\hat{Q}(\vec{p})U(t)\Omega(t)^{-1}\\&= \Omega(t)\hat{Q}(t,\vec{p})\Omega(t)^{-1}.\end{aligned} \tag{4.36}$$

We write this as

$$\hat{Q}(t,\vec{p}) \quad = \quad \Omega(t)^{-1}\hat{Q}_I(t,\vec{p})\Omega(t). \tag{4.37}$$

It is not difficult to show that the operators $\hat{Q}_I(t,\vec{p})$ and $\hat{P}_I(t,\vec{p})$ describe free oscillators, viz

$$(\partial_t^2+E_{\vec{p}}^2)\hat{Q}_I(t,\vec{p})=0 \ , \ (\partial_t^2+E_{\vec{p}}^2)\hat{P}_I(t,\vec{p})=0. \tag{4.38}$$

4.1.3 The $S-$Matrix

Single Oscillator: The probability amplitude that the oscillator remains in the ground state is <0 out$|0$ in $>$. In general the matrix of transition amplitudes is

$$S_{mn}=<m \text{ out}|n \text{ in}> . \tag{4.39}$$

We define the $S-$matrix S by

$$S_{mn}=<m \text{ in}|S|n \text{ in}> . \tag{4.40}$$

In other words

$$<m \text{ out}|=<m \text{ in}|S. \tag{4.41}$$

It is not difficult to see that S is a unitary matrix since the states $|m$ in $>$ and $|m$ in $>$ are normalized and complete. Equation (4.41) is equivalent to

$$\begin{aligned}<0 \text{ out}|(\hat{a}_{\text{out}}(\vec{p}))^m &= <0 \text{ in}|(\hat{a}_{\text{in}}(\vec{p}))^m S\\ &= <0 \text{ out}|S^{-1}(\hat{a}_{\text{in}}(\vec{p}))^m S\\ &= <0 \text{ out}|(S^{-1}\hat{a}_{\text{in}}(\vec{p})S)^m.\end{aligned} \tag{4.42}$$

Thus

$$\hat{a}_{\text{out}}(\vec{p})=S^{-1}\hat{a}_{\text{in}}(\vec{p})S. \tag{4.43}$$

This can also be written as

$$\hat{Q}_{\text{out}}(t,\vec{p}) \quad = \quad S^{-1}\hat{Q}_{\text{in}}(t,\vec{p})S. \tag{4.44}$$

From the other hand, the solution of the differential equation (4.34) can be obtained by iteration as follows. We write

$$\Omega(t)=1+\Omega_1(t)+\Omega_2(t)+\Omega_3(t)+... \tag{4.45}$$

The operator $\Omega_n(t)$ is proportional to the nth power of the interaction $\hat{V}_I(t)$. By substitution we get the differential equations

$$i\partial_t\Omega_1(t)=\hat{V}_I(t,\vec{p})\Leftrightarrow\Omega_1(t)=-i\int_{-\infty}^t dt_1\hat{V}_I(t_1,\vec{p}). \tag{4.46}$$

$$i\partial_t\Omega_n(t)=\hat{V}_I(t,\vec{p})\Omega_{n-1}(t)\Leftrightarrow\Omega_n(t)=-i\int_{-\infty}^t dt_1\hat{V}_I(t,\vec{p})\Omega_{n-1}(t_1) \ , \ n\geq 2. \tag{4.47}$$

Thus we get the solution

$$\begin{aligned}\Omega(t) &= 1-i\int_{-\infty}^{t}dt_1\hat{V}_I(t_1,\vec{p})+(-i)^2\int_{-\infty}^{t}dt_1\hat{V}_I(t_1,\vec{p})\int_{-\infty}^{t_1}dt_2\hat{V}_I(t_2,\vec{p})\\ &+ (-i)^3\int_{-\infty}^{t}dt_1\hat{V}_I(t_1,\vec{p})\int_{-\infty}^{t_1}dt_2\hat{V}_I(t_2,\vec{p})\int_{-\infty}^{t_2}dt_3\hat{V}_I(t_3,\vec{p})+...\\ &= \sum_{n=0}^{\infty}(-i)^n\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2...\int_{-\infty}^{t_{n-1}}dt_n\hat{V}_I(t_1,\vec{p})...\hat{V}_I(t_n,\vec{p}).\end{aligned} \tag{4.48}$$

This expression can be simplified by using the time-ordering operator T. Let us first recall that

$$\begin{aligned}T(\hat{V}_I(t_1)\hat{V}_I(t_2)) &= \hat{V}_I(t_1)\hat{V}_I(t_2)\ ,\ \text{if}\ t_1>t_2\\ T(\hat{V}_I(t_1)\hat{V}_I(t_2)) &= \hat{V}_I(t_2)\hat{V}_I(t_1)\ ,\ \text{if}\ t_2>t_1.\end{aligned} \tag{4.49}$$

For ease of notation we have suppressed momentarily the momentum-dependence of $\hat{V}_I$. Clearly $T(\hat{V}_I(t_1)\hat{V}_I(t_2))$ is a function of t_1 and t_2 which is symmetric about the axis $t_1=t_2$. Hence

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2T(\hat{V}_I(t_1)\hat{V}_I(t_2)) &= \frac{1}{2}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\hat{V}_I(t_1)\hat{V}_I(t_2)+\frac{1}{2}\int_{-\infty}^{t}dt_2\int_{-\infty}^{t_2}dt_1\hat{V}_I(t_2)\hat{V}_I(t_1)\\ &= \int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\hat{V}_I(t_1)\hat{V}_I(t_2).\end{aligned} \tag{4.50}$$

The generalized result we will use is therefore given by

$$\frac{1}{n!}\int_{-\infty}^{t}dt_1...\int_{-\infty}^{t}dt_nT(\hat{V}_I(t_1)...\hat{V}_I(t_n)) = \int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2...\int_{-\infty}^{t_{n-1}}dt_n\hat{V}_I(t_1)\hat{V}_I(t_2)...\hat{V}_I(t_n). \tag{4.51}$$

By substituting this identity in (4.48) we obtain

$$\begin{aligned}\Omega(t) &= \sum_{n=0}^{\infty}(-i)^n\frac{1}{n!}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2...\int_{-\infty}^{t}dt_nT(\hat{V}_I(t_1,\vec{p})\hat{V}_I(t_2,\vec{p})...\hat{V}_I(t_n,\vec{p}))\\ &= T\bigg(e^{-i\int_{-\infty}^{t}ds\hat{V}_I(s,\vec{p})}\bigg).\end{aligned} \tag{4.52}$$

It is clear that

$$\Omega(-\infty) = 1. \tag{4.53}$$

This can only be consistent with the assumption that $j(t,\vec{p})\longrightarrow 0$ as $t\longrightarrow -\infty$. As we will see shortly we need actually to assume the stronger requirement that the source $j(t,\vec{p})$ vanishes outside a finite time interval. Hence for early times $t\longrightarrow -\infty$ we have $\Omega(t)\longrightarrow 1$ and as a consequence we get $\hat{Q}(t,\vec{p})\longrightarrow \hat{Q}_I(t,\vec{p})$ from (4.37). However we know that $\hat{Q}(t,\vec{p})\longrightarrow \hat{Q}_{\rm in}(t,\vec{p})$ as $t\longrightarrow -\infty$. Since $\hat{Q}_I(t,\vec{p})$ and $\hat{Q}_{\rm in}(t,\vec{p})$ are both free fields, i.e. they solve the same differential equation we conclude that they must be the same field for all times, viz

$$\hat{Q}_I(t,\vec{p}) = \hat{Q}_{\rm in}(t,\vec{p})\ ,\ \forall t. \tag{4.54}$$

Equation (4.37) becomes

$$\hat{Q}(t,\vec{p}) = \Omega(t)^{-1}\hat{Q}_{\rm in}(t,\vec{p})\Omega(t). \tag{4.55}$$

For late times $t \longrightarrow \infty$ we know that $\hat{Q}(t,\vec{p}) \longrightarrow \hat{Q}_{\rm out}(t,\vec{p})$. Thus from the above equation we obtain

$$\hat{Q}_{\rm out}(t,\vec{p}) = \Omega(+\infty)^{-1}\hat{Q}_{\rm in}(t,\vec{p})\Omega(+\infty). \quad (4.56)$$

Comparing this equation with (4.44) we conclude that the $S-$matrix is given by

$$S = \Omega(+\infty) = T\bigg(e^{-i\int_{-\infty}^{+\infty} ds \hat{V}_I(s,\vec{p})}\bigg). \quad (4.57)$$

Scalar Field: Generalization of (4.57) is straightforward. The full $S-$matrix of the forced scalar field is the tensor product of the individual $S-$matrices of the forced harmonic oscillators one for each momentum $\vec{p}$. Since $\hat{Q}(t,-\vec{p}) = \hat{Q}(t,\vec{p})^+$ we only consider momenta $\vec{p}$ with positive components. In the tensor product all factors commute because they involve momenta which are different. We obtain then the evolution operator and the $S-$matrix

$$\begin{aligned}
\Omega(t) &= T\bigg(e^{-i\int_{-\infty}^{t} ds \sum_{p^1>0}\sum_{p^2>0}\sum_{p^3>0}\hat{V}_I(s,\vec{p})}\bigg) \\
&= T\bigg(e^{\frac{i}{2}\int_{-\infty}^{t} ds \int \frac{d^3p}{(2\pi)^3}\big(j(s,\vec{p})^*\hat{Q}_I(s,\vec{p})+j(s,\vec{p})\hat{Q}_I(s,\vec{p})^+\big)}\bigg) \\
&= T\bigg(e^{i\int_{-\infty}^{t} ds \int d^3x J(x)\hat{\phi}_I(x)}\bigg) \\
&= T\bigg(e^{i\int_{-\infty}^{t} ds \int d^3x \mathcal{L}_{\rm int}(x)}\bigg).
\end{aligned} \quad (4.58)$$

$$S = \Omega(+\infty) = T\bigg(e^{i\int d^4x \mathcal{L}_{\rm int}(x)}\bigg). \quad (4.59)$$

The interaction Lagrangian density depends on the interaction field operator $\hat{\phi}_I = \hat{\phi}_{\rm in}$, viz

$$\begin{aligned}
\mathcal{L}_{\rm int}(x) &= \mathcal{L}_{\rm int}(\hat{\phi}_{\rm in}) \\
&= J(x)\hat{\phi}_{\rm in}(x).
\end{aligned} \quad (4.60)$$

4.1.4 Wick's Theorem For Forced Scalar Field

Let us recall the Fourier expansion of the field $\hat{\phi}_{\rm in}$ given by

$$\hat{\phi}_{\rm in}(x) = \int \frac{d^3p}{(2\pi)^3}\hat{Q}_{\rm in}(t,\vec{p})e^{i\vec{p}\vec{x}}. \quad (4.61)$$

We compute immediately

$$\begin{aligned}
\int d^3x \mathcal{L}_{\rm int}(x) &= \frac{1}{V}\sum_{\vec{p}} j(t,\vec{p})^*\hat{Q}_{\rm in}(t,\vec{p}) \\
&= \frac{1}{V}\sum_{\vec{p}}\frac{j(t,\vec{p})^*}{\sqrt{2E_{\vec{p}}}}\bigg(\hat{a}_{\rm in}(\vec{p})e^{-iE_{\vec{p}}t} + \hat{a}_{\rm in}(-\vec{p})^+e^{iE_{\vec{p}}t}\bigg).
\end{aligned} \quad (4.62)$$

Also we compute

$$\begin{aligned}\Omega(t) &= T\bigg(e^{\sum_{\vec{p}}\left(\alpha_{\vec{p}}(t)\hat{a}_{\rm in}(\vec{p})^+ - \alpha_{\vec{p}}(t)^*\hat{a}_{\rm in}(\vec{p})\right)}\bigg)\\ &= T\prod_{\vec{p}}\bigg(e^{\alpha_{\vec{p}}(t)\hat{a}_{\rm in}(\vec{p})^+ - \alpha_{\vec{p}}(t)^*\hat{a}_{\rm in}(\vec{p})}\bigg). \qquad (4.63)\end{aligned}$$

$$\alpha_{\vec{p}}(t) = \frac{i}{V}\frac{1}{\sqrt{2E_{\vec{p}}}}\int_{-\infty}^{t} ds j(s,\vec{p})e^{iE_{\vec{p}}s}. \qquad (4.64)$$

It is clear that the solution $\Omega(t)$ is of the form (including also an arbitrary phase $\beta_{\vec{p}}(t)$)

$$\Omega(t) = \prod_{\vec{p}}\bigg(e^{\alpha_{\vec{p}}(t)\hat{a}_{\rm in}(\vec{p})^+ - \alpha_{\vec{p}}(t)^*\hat{a}_{\rm in}(\vec{p}) + i\beta_{\vec{p}}(t)}\bigg). \qquad (4.65)$$

We use the Campbell-Baker-Hausdorff formula

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]} \ , \text{ if } [A,[A,B]] = [B,[A,B]] = 0. \qquad (4.66)$$

We also use the commutation relations

$$[\hat{a}_{\rm in}(\vec{p}), \hat{a}_{\rm in}(\vec{q})^+] = V\delta_{\vec{p},\vec{q}}. \qquad (4.67)$$

$$\begin{aligned}\Omega(t) &= \prod_{\vec{p}}\bigg(e^{\alpha_{\vec{p}}(t)\hat{a}_{\rm in}(\vec{p})^+}e^{-\alpha_{\vec{p}}(t)^*\hat{a}_{\rm in}(\vec{p})}e^{-\frac{1}{2}V|\alpha_{\vec{p}}(t)|^2 + i\beta_{\vec{p}}(t)}\bigg)\\ &= \prod_{\vec{p}}\Omega_{\vec{p}}(t). \qquad (4.68)\end{aligned}$$

In the limit $t \longrightarrow \infty$ we compute

$$-\frac{1}{2}V\sum_{\vec{p}}|\alpha_{\vec{p}}(+\infty)|^2 = -\frac{1}{2}\int d^4x\int d^4x^{'} J(x)J(x^{'})\frac{1}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{ip(x-x^{'})}. \qquad (4.69)$$

We also need to compute the limit of $i\beta_{\vec{p}}(t)$ when $t \longrightarrow +\infty$. After some calculation, we obtain

$$i\sum_{\vec{p}}\beta_{\vec{p}}(+\infty) = \frac{1}{2}\int d^4x\int d^4x^{'} J(x)J(x^{'})\bigg(\frac{\theta(t-t^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{ip(x-x^{'})} - \frac{\theta(t-t^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{-ip(x-x^{'})}\bigg). \qquad (4.70)$$

Putting (4.69) and (4.70) together we get finally

$$\begin{aligned}-\frac{1}{2}V\sum_{\vec{p}}|\alpha_{\vec{p}}(+\infty)|^2 + i\sum_{\vec{p}}\beta_{\vec{p}}(+\infty) &= -\frac{1}{2}\int d^4x\int d^4x^{'} J(x)J(x^{'})\bigg(\frac{\theta(t^{'}-t)}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{ip(x-x^{'})}\\ &+ \frac{\theta(t-t^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{-ip(x-x^{'})}\bigg)\\ &= -\frac{1}{2}\int d^4x\int d^4x^{'} J(x)J(x^{'})D_F(x-x^{'}). \qquad (4.71)\end{aligned}$$

From this last equation and from equation (4.68) we obtain the $S-$matrix in its pre-final form given by

$$S=\Omega(+\infty) \;=\; \prod_{\vec{p}}\bigg(e^{\alpha_{\vec{p}}(+\infty)\hat{a}_{\rm in}(\vec{p})^+}e^{-\alpha_{\vec{p}}(+\infty)^*\hat{a}_{\rm in}(\vec{p})}\bigg)e^{-\frac{1}{2}\int d^4x\int d^4x^{'}J(x)J(x^{'})D_F(x-x^{'})}. \tag{4.72}$$

This expression is already normal-ordered since

$$:\bigg(e^{\sum_{\vec{p}}\big(\alpha_{\vec{p}}(+\infty)\hat{a}_{\rm in}(\vec{p})^+-\alpha_{\vec{p}}(+\infty)^*\hat{a}_{\rm in}(\vec{p})\big)}\bigg): \;=\; \prod_{\vec{p}}\bigg(e^{\alpha_{\vec{p}}(+\infty)\hat{a}_{\rm in}(\vec{p})^+}e^{-\alpha_{\vec{p}}(+\infty)^*\hat{a}_{\rm in}(\vec{p})}\bigg). \tag{4.73}$$

In summary we have

$$\begin{aligned} S=\Omega(+\infty) \;&=\; T\bigg(e^{\sum_{\vec{p}}\big(\alpha_{\vec{p}}(+\infty)\hat{a}_{\rm in}(\vec{p})^+-\alpha_{\vec{p}}(+\infty)^*\hat{a}_{\rm in}(\vec{p})\big)}\bigg)\\ &=\; :\bigg(e^{\sum_{\vec{p}}\big(\alpha_{\vec{p}}(+\infty)\hat{a}_{\rm in}(\vec{p})^+-\alpha_{\vec{p}}(+\infty)^*\hat{a}_{\rm in}(\vec{p})\big)}\bigg):e^{-\frac{1}{2}\int d^4x\int d^4x^{'}J(x)J(x^{'})D_F(x-x^{'})}. \end{aligned} \tag{4.74}$$

More explicitly we write

$$S=T\bigg(e^{i\int d^4xJ(x)\hat{\phi}_{\rm in}(x)}\bigg)=:e^{i\int d^4xJ(x)\hat{\phi}_{\rm in}(x)}:e^{-\frac{1}{2}\int d^4x\int d^4x^{'}J(x)J(x^{'})D_F(x-x^{'})}. \tag{4.75}$$

This is Wick's theorem.

4.2 The $\Phi-$Four Theory

4.2.1 The Lagrangian Density

In this section we consider more general interacting scalar field theories. In principle we can add any interaction Lagrangian density $\mathcal{L}_{\rm int}$ to the free Lagrangian density $\mathcal{L}_0$ given by equation (10.478) in order to obtain an interacting scalar field theory. This interaction Lagrangian density can be for example any polynomial in the field ϕ. However there exists only one single interacting scalar field theory of physical interest which is also renormalizable known as the $\phi-$four theory. This is obtained by adding to (10.478) a quartic interaction Lagrangian density of the form

$$\mathcal{L}_{\rm int}=-\frac{\lambda}{4!}\phi^4. \tag{4.76}$$

The equation of motion becomes

$$\begin{aligned} (\partial_{\mu}\partial^{\mu}+m^2)\phi \;&=\; \frac{\delta\mathcal{L}_{\rm int}}{\delta\phi}\\ &=\; -\frac{\lambda}{6}\phi^3. \end{aligned} \tag{4.77}$$

Equivalently

$$(\partial_t^2+E_{\vec{p}}^2)Q(t,\vec{p}) \;=\; \int d^3x\frac{\delta\mathcal{L}_{\rm int}}{\delta\phi}e^{-i\vec{p}\vec{x}}. \tag{4.78}$$

We will suppose that the right-hand side of the above equation goes to zero as $t \longrightarrow \pm\infty$. In other words we must require that $\delta\mathcal{L}_{\rm int}/\delta\phi \longrightarrow 0$ as $t \longrightarrow \pm\infty$. If this is not true (which is generically the case) then we will assume implicitly an adiabatic switching off process for the interaction in the limits $t \longrightarrow \pm\infty$ given by the replacement

$$\mathcal{L}_{\rm int} \longrightarrow e^{-\epsilon|t|}\mathcal{L}_{\rm int}. \tag{4.79}$$

With this assumption the solutions of the equation of motion in the limits $t \longrightarrow -\infty$ and $t \longrightarrow +\infty$ are given respectively by

$$\hat{Q}_{\rm in}(t,\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}\bigg(\hat{a}_{\rm in}(\vec{p})e^{-iE_{\vec{p}}t} + \hat{a}_{\rm in}(-\vec{p})^+e^{iE_{\vec{p}}t}\bigg)\ ,\ t \longrightarrow -\infty. \tag{4.80}$$

$$\hat{Q}_{\rm out}(t,\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}\bigg(\hat{a}_{\rm out}(\vec{p})e^{-iE_{\vec{p}}t} + \hat{a}_{\rm out}(-\vec{p})^+e^{iE_{\vec{p}}t}\bigg)\ ,\ t \longrightarrow +\infty. \tag{4.81}$$

4.2.2 The $S-$Matrix

The Hamiltonian operator in the Schrodinger picture is time-independent of the form

$$\hat{H} = \hat{H}_0(\hat{Q},\hat{Q}^+,\hat{P},\hat{P}^+) + \hat{V}(\hat{Q},\hat{Q}^+). \tag{4.82}$$

$$\begin{aligned}\hat{H}_0(\hat{Q},\hat{Q}^+,\hat{P},\hat{P}^+) &= \int_+ \frac{d^3p}{(2\pi)^3}\bigg[\hat{P}^+(\vec{p})\hat{P}(\vec{p}) + E_{\vec{p}}^2\hat{Q}^+(\vec{p})\hat{Q}(\vec{p})\bigg]\\ &= \frac{1}{2}\sum_{\vec{p}}\hat{\mathcal{H}}_{\vec{p}}.\end{aligned} \tag{4.83}$$

$$\begin{aligned}\hat{V}(\hat{Q},\hat{Q}^+) &= (+\frac{\lambda}{4!})\frac{1}{V^3}\sum_{\vec{p}_1,\vec{p}_2,\vec{p}_3}\hat{Q}(\vec{p}_1)\hat{Q}(\vec{p}_2)\hat{Q}(\vec{p}_3)\hat{Q}(\vec{p}_1+\vec{p}_2+\vec{p}_3)^+\\ &= -\int d^3x\mathcal{L}_{\rm int}.\end{aligned} \tag{4.84}$$

The scalar field operator and the conjugate momentum field operator in the Schrodinger picture are given by

$$\hat{\phi}(\vec{x}) = \frac{1}{V}\sum_{\vec{p}}\hat{Q}(\vec{p})e^{i\vec{p}\vec{x}}. \tag{4.85}$$

$$\hat{\pi}(\vec{x}) = \frac{1}{V}\sum_{\vec{p}}\hat{P}(\vec{p})e^{i\vec{p}\vec{x}}. \tag{4.86}$$

The unitary time evolution operator of the scalar field must solve the Schrodinger equation

$$i\partial_t U(t) = \hat{H}U(t). \tag{4.87}$$

The Heisenberg and Schrodinger operators are related by

$$\hat{\phi}(t,\vec{x}) = U(t)^{-1}\hat{\phi}(\vec{x})U(t). \tag{4.88}$$

We introduce the interaction picture through the unitary operator Ω defined by

$$U(t) = e^{-it\hat{H}_0}\Omega(t). \tag{4.89}$$

The operator Ω satisfies the Schrodinger equation

$$i\partial_t\Omega(t) = \hat{V}_I(t)\Omega(t). \tag{4.90}$$

$$\hat{V}_I(t) \equiv \hat{V}_I(\hat{Q}, \hat{Q}^+, t) \quad = \quad e^{it\hat{H}_0}\hat{V}(\hat{Q}, \hat{Q}^+)e^{-it\hat{H}_0}. \tag{4.91}$$

The interaction, Schrodinger and Heisenberg operators are related by

$$\begin{aligned}\hat{\phi}_I(t, \vec{x}) &= e^{it\hat{H}_0}\hat{\phi}(\vec{x})e^{-it\hat{H}_0} \\ &= \Omega(t)U(t)^{-1}\hat{\phi}(\vec{x})U(t)\Omega(t)^{-1} \\ &= \Omega(t)\hat{\phi}(t, \vec{x})\Omega(t)^{-1}.\end{aligned} \tag{4.92}$$

We write this as

$$\hat{\phi}(x) \quad = \quad \Omega(t)^{-1}\hat{\phi}_I(x)\Omega(t). \tag{4.93}$$

Similarly we should have for the conjugate momentum field $\hat{\pi}(x) = \partial_t\hat{\phi}(x)$ the result

$$\hat{\pi}_I(x) \quad = \quad e^{it\hat{H}_0}\hat{\pi}(\vec{x})e^{-it\hat{H}_0}. \tag{4.94}$$

$$\hat{\pi}(x) \quad = \quad \Omega(t)^{-1}\hat{\pi}_I(x)\Omega(t). \tag{4.95}$$

It is not difficult to show that the interaction fields $\hat{\phi}_I$ and $\hat{\pi}_I$ are free fields. Indeed we can show for example that $\hat{\phi}_I$ obeys the equation of motion

$$(\partial_t^2 - \vec{\nabla}^2 + m^2)\hat{\phi}_I(t, \vec{x}) = 0. \tag{4.96}$$

Thus all information about interaction is encoded in the evolution operator $\Omega(t)$ which in turn is obtained from the solution of the Schrodinger equation (4.90). From our previous experience this task is trivial. In direct analogy with the solution given by the formula (4.52) of the differential equation (4.34) the solution of (4.90) must be of the form

$$\begin{aligned}\Omega(t) &= \sum_{n=0}^{\infty}(-i)^n\frac{1}{n!}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t}dt_2...\int_{-\infty}^{t}dt_n T(\hat{V}_I(t_1)\hat{V}_I(t_2)...\hat{V}_I(t_n)) \\ &= T\bigg(e^{-i\int_{-\infty}^{t}ds\hat{V}_I(s)}\bigg) \\ &= T\bigg(e^{i\int_{-\infty}^{t}ds\int d^3x\mathcal{L}_{\rm int}(\hat{\phi}_I(s,\vec{x}))}\bigg).\end{aligned} \tag{4.97}$$

Clearly this satisfies the boundary condition

$$\Omega(-\infty) = 1. \tag{4.98}$$

As before this boundary condition can only be consistent with the assumption that $V_I(t) \longrightarrow 0$ as $t \longrightarrow -\infty$. This requirement is contained in the condition (4.79).

The $S-$matrix is defined by

$$S = \Omega(+\infty) = T\bigg(e^{-i\int_{-\infty}^{+\infty} ds\hat{V}_I(s)}\bigg)$$
$$= T\bigg(e^{i\int d^4x\mathcal{L}_{\rm int}(\hat{\phi}_I(x))}\bigg). \quad (4.99)$$

Taking the limit $t \longrightarrow -\infty$ in equation (4.93) we see that we have $\hat{\phi}(x) \longrightarrow \phi_I(x)$. But we already know that $\hat{\phi}(x) \longrightarrow \hat{\phi}_{\rm in}(x)$ when $t \longrightarrow -\infty$. Since the fields $\hat{\phi}_I(x)$ and $\hat{\phi}_{\rm in}(x)$ are free fields and satisfy the same differential equation we conclude that the two fields are identical at all times, viz

$$\hat{\phi}_I(x) = \hat{\phi}_{\rm in}(x)\ ,\ \forall t. \quad (4.100)$$

The $S-$matrix relates the "in" vacuum $|0\ {\rm in}>$ to the "out" vacuum $|0\ {\rm out}>$ as follows

$$<0\ {\rm out}| =<0\ {\rm in}|S. \quad (4.101)$$

For the $\phi-$four theory (as opposed to the forced scalar field) the vacuum is stable. In other words the "in" vacuum is identical to the "out" vacuum, viz

$$|0\ {\rm out}>=|0\ {\rm in}>=|0>. \quad (4.102)$$

Hence

$$<0| =<0|S. \quad (4.103)$$

The consistency of the supposition that the "in" vacuum is identical to the "out" vacuum will be verified order by order in perturbation theory. In fact we will also verify that the same holds also true for the one-particle states, viz

$$|\vec{p}\ {\rm out}>=|\vec{p}\ {\rm in}>. \quad (4.104)$$

4.2.3 The Gell-Mann Low Formula

We go back to equation

$$\hat{\phi}(x) = \Omega(t)^+\hat{\phi}_I(x)\Omega(t). \quad (4.105)$$

We compute

$$\begin{aligned}
\hat{\phi}(x) &= \Omega(t)^{+}\hat{\phi}_I(x)\Omega(t)\\
&= S^{-1}T\bigg(e^{-i\int_t^{+\infty}ds\hat{V}_{\rm in}(s)}\bigg)\hat{\phi}_{\rm in}(x)T\bigg(e^{-i\int_{-\infty}^{t}ds\hat{V}_{\rm in}(s)}\bigg)\\
&= S^{-1}\bigg(1-i\int_t^{+\infty}dt_1\hat{V}_{\rm in}(t_1)+(-i)^2\int_t^{+\infty}dt_1\int_{t_1}^{+\infty}dt_2\hat{V}_{\rm in}(t_2)\hat{V}_{\rm in}(t_1)+...\bigg)\hat{\phi}_{\rm in}(x)\\
&\times \bigg(1-i\int_{-\infty}^{t}dt_1\hat{V}_{\rm in}(t_1)+(-i)^2\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\hat{V}_{\rm in}(t_1)\hat{V}_{\rm in}(t_2)+...\bigg)\\
&= S^{-1}\bigg(\hat{\phi}_{\rm in}(x)-i\int_t^{+\infty}dt_1\hat{V}_{\rm in}(t_1)\hat{\phi}_{\rm in}(x)+(-i)^2\int_t^{+\infty}dt_1\int_{t_1}^{+\infty}dt_2\hat{V}_{\rm in}(t_2)\hat{V}_{\rm in}(t_1)\hat{\phi}_{\rm in}(x)\\
&- i\hat{\phi}_{\rm in}(x)\int_{-\infty}^{t}dt_1\hat{V}_{\rm in}(t_1)+(-i)^2\int_t^{+\infty}dt_1\int_{-\infty}^{t}dt_2\hat{V}_{\rm in}(t_1)\hat{\phi}_{\rm in}(x)\hat{V}_{\rm in}(t_2)\\
&+ (-i)^2\hat{\phi}_{\rm in}(x)\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\hat{V}_{\rm in}(t_1)\hat{V}_{\rm in}(t_2)+...\bigg).
\end{aligned} \tag{4.106}$$

We use the identities

$$\int_{-\infty}^{+\infty}dt_1T(\hat{\phi}_{\rm in}(x)\hat{V}_{\rm in}(t_1))=\hat{\phi}_{\rm in}(x)\int_{-\infty}^{t}dt_1\hat{V}_{\rm in}(t_1)+\int_t^{+\infty}dt_1\hat{V}_{\rm in}(t_1)\hat{\phi}_{\rm in}(x). \tag{4.107}$$

$$\int_t^{+\infty}dt_1\int_{t_1}^{+\infty}dt_2T(\hat{V}_{\rm in}(t_2)\hat{V}_{\rm in}(t_1))=\int_t^{+\infty}dt_1\int_t^{t_1}dt_2T(\hat{V}_{\rm in}(t_1)\hat{V}_{\rm in}(t_2)). \tag{4.108}$$

$$\begin{aligned}
\int_{-\infty}^{+\infty}dt_1\int_{-\infty}^{t_1}dt_2T(\hat{\phi}_{\rm in}(x)\hat{V}_{\rm in}(t_1)\hat{V}_{\rm in}(t_2)) &= \int_t^{+\infty}dt_1\int_t^{t_1}dt_2\hat{V}_{\rm in}(t_1)\hat{V}_{\rm in}(t_2)\hat{\phi}_{\rm in}(x)\\
&+ \int_t^{+\infty}dt_1\int_{-\infty}^{t}dt_2\hat{V}_{\rm in}(t_1)\hat{\phi}_{\rm in}(x)\hat{V}_{\rm in}(t_2)\\
&+ \hat{\phi}_{\rm in}(x)\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\hat{V}_{\rm in}(t_1)\hat{V}_{\rm in}(t_2).
\end{aligned} \tag{4.109}$$

We get

$$\begin{aligned}
\hat{\phi}(x) &= S^{-1}T\bigg(\hat{\phi}_{\rm in}(x)\bigg(1-i\int_{-\infty}^{+\infty}dt_1\hat{V}_{\rm in}(t_1)+(-i)^2\int_{-\infty}^{+\infty}dt_1\int_{-\infty}^{t_1}dt_2\hat{V}_{\rm in}(t_1)\hat{V}_{\rm in}(t_2)+...\bigg)\bigg)\\
&= S^{-1}T\bigg(\hat{\phi}_{\rm in}(x)S\bigg).
\end{aligned} \tag{4.110}$$

This result holds to all orders in perturbation theory. A straightforward generalization is

$$T(\hat{\phi}(x)\hat{\phi}(y)...) = S^{-1}T\bigg(\hat{\phi}_{\rm in}(x)\hat{\phi}_{\rm in}(y)...S\bigg). \tag{4.111}$$

This is known as the Gell-Mann Low formula.

4.2.4 LSZ Reduction Formulae and Green's Functions

We start by writing equations (4.80) and (4.81) in the form

$$e^{iE_{\vec{p}}t}(i\partial_t+E_{\vec{p}})\hat{Q}_{\rm in}(t,\vec{p})=\sqrt{2E_{\vec{p}}}\;\hat{a}_{\rm in}(\vec{p}). \tag{4.112}$$

$$e^{iE_{\vec{p}}t}(i\partial_t+E_{\vec{p}})\hat{Q}_{\rm out}(t,\vec{p})=\sqrt{2E_{\vec{p}}}\;\hat{a}_{\rm out}(\vec{p}). \tag{4.113}$$

Now we compute trivially the integral

$$\int_{-\infty}^{+\infty}dt\partial_t\bigg(e^{iE_{\vec{p}}t}(i\partial_t+E_{\vec{p}})\hat{Q}(t,\vec{p})\bigg)=\sqrt{2E_{\vec{p}}}\;(\hat{a}_{\rm out}(\vec{p})-\hat{a}_{\rm in}(\vec{p})). \tag{4.114}$$

From the other hand we compute

$$\begin{aligned}\int_{-\infty}^{+\infty}dt\partial_t\bigg(e^{iE_{\vec{p}}t}(i\partial_t+E_{\vec{p}})\hat{Q}(t,\vec{p})\bigg) &= i\int_{-\infty}^{+\infty}dte^{iE_{\vec{p}}t}(\partial_t^2+E_{\vec{p}}^2)\hat{Q}(t,\vec{p})\\ &= i\int d^4x\frac{\delta\mathcal{L}_{\rm int}}{\delta\phi}e^{ipx}.\end{aligned} \tag{4.115}$$

We obtain then the identity

$$i\int_{-\infty}^{+\infty}dte^{iE_{\vec{p}}t}(\partial_t^2+E_{\vec{p}}^2)\hat{Q}(t,\vec{p})=\sqrt{2E_{\vec{p}}}\;(\hat{a}_{\rm out}(\vec{p})-\hat{a}_{\rm in}(\vec{p})). \tag{4.116}$$

This is the first instance of LSZ reduction formulae. Generalizations of this result read

$$\begin{aligned}i\int_{-\infty}^{+\infty}dte^{iE_{\vec{p}}t}(\partial_t^2+E_{\vec{p}}^2)T(\hat{Q}(t,\vec{p})\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...) &=\\ \sqrt{2E_{\vec{p}}}\;\bigg(\hat{a}_{\rm out}(\vec{p})T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)-T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)\hat{a}_{\rm in}(\vec{p})\bigg).\end{aligned} \tag{4.117}$$

Next we put to use these LSZ reduction formulae. We are interested in calculating the matrix elements of the $S-$matrix. We consider an arbitrary "in" state $|\vec{p}_1\vec{p}_2...\ {\rm in}>$ and an arbitrary "out" state $|\vec{q}_1\vec{q}_2...\ {\rm out}>$. The matrix elements of interest are

$$<\vec{q}_1\vec{q}_2...\ {\rm out}|\vec{p}_1\vec{p}_2...\ {\rm in}>=<\vec{q}_1\vec{q}_2...\ {\rm in}|S|\vec{p}_1\vec{p}_2...\ {\rm in}>. \tag{4.118}$$

We recall that

$$|\vec{p}_1\vec{p}_2...\ {\rm in}> = a_{\rm in}(\vec{p}_1)^+a_{\rm in}(\vec{p}_2)^+...|0>. \tag{4.119}$$

$$|\vec{q}_1\vec{q}_2...\ {\rm out}> = a_{\rm out}(\vec{q}_1)^+a_{\rm out}(\vec{q}_2)^+...|0>. \tag{4.120}$$

We also recall the commutation relations (using box normalization)

$$[\hat{a}(\vec{p}),\hat{a}(\vec{q})^+]=V\delta_{\vec{p},\vec{q}}\;,\;[\hat{a}(\vec{p}),\hat{a}(\vec{q})]=[\hat{a}(\vec{p})^+,\hat{a}(\vec{q})^+]=0. \tag{4.121}$$

We compute by using the LSZ reduction formula (4.116) and assuming that the $\vec{p}_i$ are different from the $\vec{q}_i$ the result

$$\begin{aligned}<\vec{q}_1\vec{q}_2...\ {\rm out}|\vec{p}_1\vec{p}_2...\ {\rm in}> &= <\vec{q}_2...\ {\rm out}|\hat{a}_{\rm out}(\vec{q}_1)|\vec{p}_1\vec{p}_2...\ {\rm in}>\\ &= <\vec{q}_2..\ {\rm out}|\bigg(\hat{a}_{\rm in}(\vec{q}_1)+\frac{i}{\sqrt{2E_{\vec{q}_1}}}\int_{-\infty}^{+\infty}dt_1e^{iE_{\vec{q}_1}t_1}(\partial_{t_1}^2+E_{\vec{q}_1}^2)\hat{Q}(t_1,\vec{q}_1)\bigg)\\ &\times\ |\vec{p}_1\vec{p}_2..\ {\rm in}>\\ &= \frac{1}{\sqrt{2E_{\vec{q}_1}}}\int_{-\infty}^{+\infty}dt_1e^{iE_{\vec{q}_1}t_1}i(\partial_{t_1}^2+E_{\vec{q}_1}^2)<\vec{q}_2...\ {\rm out}|\hat{Q}(t_1,\vec{q}_1)|\vec{p}_1\vec{p}_2...\ {\rm in}>.\end{aligned} \tag{4.122}$$

From the LSZ reduction formula (4.117) we have

$$i\int_{-\infty}^{+\infty} dt_2 e^{iE_{\vec{q}_2}t_2}(\partial_{t_2}^2+E_{\vec{q}_2}^2)T(\hat{Q}(t_2,\vec{q}_2)\hat{Q}(t_1,\vec{q}_1)) = \sqrt{2E_{\vec{q}_2}}\,\bigg(\hat{a}_{\rm out}(\vec{q}_2)\hat{Q}(t_1,\vec{q}_1)-\hat{Q}(t_1,\vec{q}_1)\hat{a}_{\rm in}(\vec{q}_2)\bigg). \quad (4.123)$$

Thus immediately

$$i\int_{-\infty}^{+\infty} dt_2 e^{iE_{\vec{q}_2}t_2}(\partial_{t_2}^2+E_{\vec{q}_2}^2) < \vec{q}_3..\text{ out}|T(\hat{Q}(t_2,\vec{q}_2)\hat{Q}(t_1,\vec{q}_1))|\vec{p}_1\vec{p}_2..\text{ in}> \; = \; \sqrt{2E_{\vec{q}_2}} \; < \vec{q}_2..\text{ out}|\hat{Q}(t_1,\vec{q}_1)|\vec{p}_1\vec{p}_2..\text{ in}> . \quad (4.124)$$

Hence

$$< \vec{q}_1\vec{q}_2...\text{ out}|\vec{p}_1\vec{p}_2...\text{ in}> \; = \; \frac{1}{\sqrt{2E_{\vec{q}_1}}}\frac{1}{\sqrt{2E_{\vec{q}_2}}}\int_{-\infty}^{+\infty}dt_1 e^{iE_{\vec{q}_1}t_1}i(\partial_{t_1}^2+E_{\vec{q}_1}^2)\int_{-\infty}^{+\infty}dt_2 e^{iE_{\vec{q}_2}t_2}i(\partial_{t_2}^2+E_{\vec{q}_2}^2) \times \; < \vec{q}_3...\text{ out}|T(\hat{Q}(t_1,\vec{q}_1)\hat{Q}(t_2,\vec{q}_2))|\vec{p}_1\vec{p}_2...\text{ in}> . \quad (4.125)$$

By continuing this reduction of all "out" operators we end up with the expression

$$< \vec{q}_1\vec{q}_2...\text{ out}|\vec{p}_1\vec{p}_2...\text{ in}> \; = \; \frac{1}{\sqrt{2E_{\vec{q}_1}}}\frac{1}{\sqrt{2E_{\vec{q}_2}}}...\int_{-\infty}^{+\infty}dt_1 e^{iE_{\vec{q}_1}t_1}i(\partial_{t_1}^2+E_{\vec{q}_1}^2)\int_{-\infty}^{+\infty}dt_2 e^{iE_{\vec{q}_2}t_2}i(\partial_{t_2}^2+E_{\vec{q}_2}^2)... \times \; < 0|T(\hat{Q}(t_1,\vec{q}_1)\hat{Q}(t_2,\vec{q}_2)...)|\vec{p}_1\vec{p}_2...\text{ in}> . \quad (4.126)$$

In order to reduce the "in" operators we need other LSZ reduction formulae which involve the creation operators instead of the annihilation operators. The result we need is essentially the Hermitian conjugate of (4.117) given by

$$-i\int_{-\infty}^{+\infty} dt e^{-iE_{\vec{p}}t}(\partial_t^2+E_{\vec{p}}^2)T(\hat{Q}(t,\vec{p})^+\hat{Q}(t_1,\vec{p}_1)^+\hat{Q}(t_2,\vec{p}_2)^+...) \; = \; \sqrt{2E_{\vec{p}}}\,\bigg(\hat{a}_{\rm out}(\vec{p})^+T(\hat{Q}(t_1,\vec{p}_1)^+\hat{Q}(t_2,\vec{p}_2)^+...)-T(\hat{Q}(t_1,\vec{p}_1)^+\hat{Q}(t_2,\vec{p}_2)^+...)\hat{a}_{\rm in}(\vec{p})^+\bigg). \quad (4.127)$$

By using these LSZ reduction formulae we compute

$$< 0|T(\hat{Q}(t_1,\vec{q}_1)\hat{Q}(t_2,\vec{q}_2)...)|\vec{p}_1\vec{p}_2...\text{ in}> \; = \; \frac{1}{\sqrt{2E_{\vec{p}_1}}}\int_{-\infty}^{+\infty}dt_1^{'} e^{-iE_{\vec{p}_1}t_1^{'}}i(\partial_{t_1^{'}}^2+E_{\vec{p}_1}^2) < 0|T(\hat{Q}(t_1,\vec{q}_1)\hat{Q}(t_2,\vec{q}_2)...\hat{Q}(t_1^{'},\vec{p}_1)^+)|\vec{p}_2...\text{ in}> . \quad (4.128)$$

Full reduction of the "in" operators leads to the expression

$$< 0|T(\hat{Q}(t_1,\vec{q}_1)\hat{Q}(t_2,\vec{q}_2)...)|\vec{p}_1\vec{p}_2...\text{ in}> \; = \; \frac{1}{\sqrt{2E_{\vec{p}_1}}}\frac{1}{\sqrt{2E_{\vec{p}_2}}}...\int_{-\infty}^{+\infty}dt_1^{'} e^{-iE_{\vec{p}_1}t_1^{'}}i(\partial_{t_1^{'}}^2+E_{\vec{p}_1}^2)\int_{-\infty}^{+\infty}dt_2^{'} e^{-iE_{\vec{p}_2}t_2^{'}}i(\partial_{t_2^{'}}^2+E_{\vec{p}_2}^2)... \times < 0|T(\hat{Q}(t_1,\vec{q}_1)\hat{Q}(t_2,\vec{q}_2)...\hat{Q}(t_1^{'},\vec{p}_1)^+\hat{Q}(t_2^{'},\vec{p}_2)^+...)|0> . \quad (4.129)$$

Hence by putting the two partial results (4.126) and (4.129) together we obtain

$$
\begin{aligned}
< \vec{q}_1 ... \text{ out}|\vec{p}_1 ... \text{ in} > &= \frac{1}{\sqrt{2E_{\vec{q}_1}}} ... \frac{1}{\sqrt{2E_{\vec{p}_1}}} ... \int_{-\infty}^{+\infty} dt_1 e^{iE_{\vec{q}_1} t_1} i(\partial_{t_1}^2 + E_{\vec{q}_1}^2) ... \int_{-\infty}^{+\infty} dt_1^{'} e^{-iE_{\vec{p}_1} t_1^{'}} i(\partial_{t_1^{'}}^2 + E_{\vec{p}_1}^2) ... \\
&\times < 0|T(\hat{Q}(t_1, \vec{q}_1) ... \hat{Q}(t_1^{'}, \vec{p}_1)^+ ...)|0 > . \qquad (4.130)
\end{aligned}
$$

The final (fundamental) result is that $S-$matrix elements $< \vec{q}_1 ... \text{ out}|\vec{p}_1 ... \text{ in} >$ can be reconstructed from the so-called Green's functions $< 0|T(\hat{\phi}(x_1) ... \hat{\phi}(x_1^{'}) ...)|0 >$. Indeed we can rewrite equation (4.130) as

$$
\begin{aligned}
< \vec{q}_1 ... \text{ out}|\vec{p}_1 ... \text{ in} > &= \frac{1}{\sqrt{2E_{\vec{q}_1}}} ... \frac{1}{\sqrt{2E_{\vec{p}_1}}} ... \int d^4 x_1 e^{iq_1 x_1} i(\partial_1^2 + m^2) ... \int d^4 x_1^{'} e^{-ip_1 x_1^{'}} i(\partial_1^{'2} + m^2) ... \\
&\times < 0|T(\hat{\phi}(x_1) ... \hat{\phi}(x_1^{'}) ...)|0 > . \qquad (4.131)
\end{aligned}
$$

The factor $1/\sqrt{2E_{\vec{q}_1}} ... 1/\sqrt{2E_{\vec{p}_1}}$ is only due to our normalization of the one-particle states given in equations (4.119) and (4.120).

4.3 Feynman Diagrams For $\phi-$Four Theory

4.3.1 Perturbation Theory

We go back to our most fundamental result (4.111) and write it in the form (with $\mathcal{L}_{\text{int}}(\hat{\phi}_{\text{in}}(x)) = \mathcal{L}_{\text{int}}(x)$)

$$
\begin{aligned}
< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2) ...)|0 > &= < 0|T\bigg(\hat{\phi}_{\text{in}}(x_1)\hat{\phi}_{\text{in}}(x_2) ... S\bigg)|0 > \\
&= < 0|T\bigg(\hat{\phi}_{\text{in}}(x_1)\hat{\phi}_{\text{in}}(x_2) ... e^{i\int d^4 y \mathcal{L}_{\text{int}}(y)}\bigg)|0 > \\
&= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4 y_1 .. \int d^4 y_n < 0|T\bigg(\hat{\phi}_{\text{in}}(x_1)\hat{\phi}_{\text{in}}(x_2) .. \mathcal{L}_{\text{int}}(y_1) .. \mathcal{L}_{\text{int}}(y_n)\bigg)|0 > . \qquad (4.132)
\end{aligned}
$$

These are the Green's functions we need in order to compute the $S-$matrix elements. They are written solely in terms of free fields and the interaction Lagrangian density. This expansion is the key perturbative series in quantum field theory.

Another quantity of central importance to perturbation theory is the vacuum-to-vacuum amplitude given by

$$
< 0|0 > = < 0|S|0 > = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int d^4 y_1 .. \int d^4 y_n < 0|T\bigg(\mathcal{L}_{\text{int}}(y_1) .. \mathcal{L}_{\text{int}}(y_n)\bigg)|0 > . \qquad (4.133)
$$

Naively we would have thought that this norm is equal to 1. However it turns out that this is not the case and taking this fact into account will simplify considerably our perturbative calculations.

4.3.2 Wick's Theorem For Green's Functions

From the above discussion it is clear that the remaining task is to evaluate terms of the generic form

$$< 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)...\hat{\phi}_{\rm in}(x_{2n})\bigg)|0 > . \tag{4.134}$$

To this end we rewrite the Wick's theorem (4.75) in the form

$$< 0|T\bigg(e^{i\int d^4x J(x)\hat{\phi}_{\rm in}(x)}\bigg)|0 >= e^{-\frac{1}{2}\int d^4x\int d^4x^{'} J(x)J(x^{'})D_F(x-x^{'})}. \tag{4.135}$$

Because the scalar field is real we also have

$$< 0|T\bigg(e^{-i\int d^4x J(x)\hat{\phi}_{\rm in}(x)}\bigg)|0 >= e^{-\frac{1}{2}\int d^4x\int d^4x^{'} J(x)J(x^{'})D_F(x-x^{'})}. \tag{4.136}$$

This means that only even powers of J appear. We expand both sides in powers of J we get

$$\begin{aligned}\sum_{n=0}\frac{i^{2n}}{2n!}\int d^4x_1...d^4x_{2n}J(x_1)...J(x_{2n}) < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\bigg)|0 > &= \\ \sum_{n=0}\frac{1}{n!}(-\frac{1}{2})^n\int d^4x_1\int d^4x_2...\int d^4x_{2n-1}\int d^4x_{2n} &\times \\ J(x_1)J(x_2)...J(x_{2n-1})J(x_{2n})D_F(x_1-x_2)...D_F(x_{2n-1}-x_{2n}).\end{aligned} \tag{4.137}$$

Let us look at few examples. The first non-trivial term is

$$\begin{aligned}\frac{i^2}{2!}\int d^4x_1d^4x_2J(x_1)J(x_2) < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\bigg)|0 > &= \\ \frac{1}{1!}(-\frac{1}{2})^1\int d^4x_1\int d^4x_2J(x_1)J(x_2)D_F(x_1-x_2).\end{aligned} \tag{4.138}$$

Immediately we get the known result

$$< 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\bigg)|0 >= D_F(x_1-x_2). \tag{4.139}$$

The second non-trivial term is

$$\begin{aligned}\frac{i^4}{4!}\int d^4x_1d^4x_2d^4x_3d^4x_4J(x_1)J(x_2)J(x_3)J(x_4) < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\hat{\phi}_{\rm in}(x_3)\hat{\phi}_{\rm in}(x_4)\bigg)|0 > &= \\ \frac{1}{2!}(-\frac{1}{2})^2\int d^4x_1\int d^4x_2\int d^4x_3\int d^4x_4 &\times \\ J(x_1)J(x_2)J(x_3)J(x_4)D_F(x_1-x_2)D_F(x_3-x_4).\end{aligned} \tag{4.140}$$

Equivalently

$$\begin{aligned}\frac{i^4}{4!}\int d^4x_1d^4x_2d^4x_3d^4x_4J(x_1)J(x_2)J(x_3)J(x_4) < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\hat{\phi}_{\rm in}(x_3)\hat{\phi}_{\rm in}(x_4)\bigg)|0 > &= \\ \frac{1}{2!}(-\frac{1}{2})^2\frac{1}{3}\int d^4x_1\int d^4x_2\int d^4x_3\int d^4x_4 &\times \\ J(x_1)J(x_2)J(x_3)J(x_4)\bigg(D_F(x_1-x_2)D_F(x_3-x_4) &+ \\ D_F(x_1-x_3)D_F(x_2-x_4)+D_F(x_1-x_4)D_F(x_2-x_3)\bigg).\end{aligned} \tag{4.141}$$

In the last equation we have symmetrized the right-hand side under the permutations of the spacetime points x_1, x_2, x_3 and x_4 and then divided by $1/3$ where 3 is the number of independent permutations in this case. This is needed because the left-hand side is already symmetric under the permutations of the x_i's. By comparing the two sides we then obtain

$$< 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\hat{\phi}_{\rm in}(x_3)\hat{\phi}_{\rm in}(x_4)\bigg)|0> \;=\; D_F(x_1-x_2)D_F(x_3-x_4)+D_F(x_1-x_3)D_F(x_2-x_4) + D_F(x_1-x_4)D_F(x_2-x_3). \quad (4.142)$$

The independent permutations are called contractions and we write

$$< 0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\hat{\phi}_{\rm in}(x_3)\hat{\phi}_{\rm in}(x_4)\bigg)|0> \;=\; \sum_{\rm contraction}\prod D_F(x_i-x_j). \quad (4.143)$$

This generalizes to any Green's function. In equation (4.137) we need to symmetrize the right-hand side under the permutations of the spacetime points x_i's before comparing with the left-hand side. Thus we need to count the number of independent permutations or contractions. Since we have $2n$ points we have $(2n)!$ permutations not all of them independent. Indeed we need to divide by 2^n since $D_F(x_i-x_j)=D_F(x_j-x_i)$ and we have n such propagators. Then we need to divide by $n!$ since the order of the n propagators $D_F(x_1-x_2)$,...,$D_F(x_{2n-1}-x_{2n})$ is irrelevant. We get then $(2n)!/(2^n n!)$ independent permutations. Equation (4.137) becomes

$$\sum_{n=0}\frac{i^{2n}}{2n!}\int d^4x_1...d^4x_{2n}J(x_1)...J(x_{2n}) < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\bigg)|0> \;=\; \sum_{n=0}\frac{1}{n!}(-\frac{1}{2})^n\frac{2^n n!}{(2n)!}\int d^4x_1\int d^4x_2...\int d^4x_{2n-1}\int d^4x_{2n} \;\times\; J(x_1)J(x_2)...J(x_{2n-1})J(x_{2n}) \sum_{\rm contraction}\prod D_F(x_i-x_j). \quad (4.144)$$

By comparison we obtain

$$< 0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\bigg)|0>= \sum_{\rm contraction}\prod D_F(x_i-x_j). \quad (4.145)$$

This is Wick's theorem for Green's functions.

An alternative more systematic way of obtaining all contractions goes as follows. First let us define

$$< 0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\bigg)|0>=< 0|T\bigg(F(\hat{\phi}_{\rm in})\bigg)|0> . \quad (4.146)$$

We introduce the functional Fourier transform

$$F(\hat{\phi}_{\rm in})=\int \mathcal{D}J\tilde{F}(J)\; e^{i\int d^4x J(x)\hat{\phi}_{\rm in}(x)}. \quad (4.147)$$

Thus

$$\begin{aligned} < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\bigg)|0> &= < 0|T\bigg(\int \mathcal{D}J\tilde{F}(J)\; e^{i\int d^4x J(x)\hat{\phi}_{\rm in}(x)}\bigg)|0> \\ &= \int \mathcal{D}J\tilde{F}(J) < 0|T\bigg(e^{i\int d^4x J(x)\hat{\phi}_{\rm in}(x)}\bigg)|0> \\ &= \int \mathcal{D}J\tilde{F}(J)e^{-\frac{1}{2}\int d^4x\int d^4x^{'} J(x)D_F(x-x^{'})J(x^{'})}. \quad (4.148)\end{aligned}$$

We use the identity (starting from here we only deal with classical fields instead of field operators)

$$f\bigg(\frac{\delta}{\delta\phi}\bigg)e^{i\int d^4x J(x)\phi(x)} = f(iJ)e^{i\int d^4x J(x)\phi(x)} \tag{4.149}$$

In particular we have

$$e^{\frac{1}{2}\int d^4x\int d^4x' \frac{\delta}{\delta\phi(x)}D_F(x-x')\frac{\delta}{\delta\phi(x')}}e^{i\int d^4x J(x)\phi(x)} = e^{-\frac{1}{2}\int d^4x\int d^4x' J(x)D_F(x-x')J(x')}e^{i\int d^4x J(x)\phi(x)} \tag{4.150}$$

Thus

$$\begin{aligned}<0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\bigg)|0> &= \int \mathcal{D}J\tilde{F}(J)\bigg[e^{\frac{1}{2}\int d^4x\int d^4x' \frac{\delta}{\delta\phi(x)}D_F(x-x')\frac{\delta}{\delta\phi(x')}}e^{i\int d^4x J(x)\phi(x)}\bigg]_{\phi=0}\\ &= \bigg[e^{\frac{1}{2}\int d^4x\int d^4x' \frac{\delta}{\delta\phi(x)}D_F(x-x')\frac{\delta}{\delta\phi(x')}}F(\phi)\bigg]_{\phi=0}. \end{aligned} \tag{4.151}$$

We think of F as a function in several variables which are the classical fields $\phi(x_i)$. Thus we have

$$\frac{\delta F}{\delta\phi(x)} = \delta^4(x-x_1)\frac{\partial F}{\partial\phi(x_1)} + \delta^4(x-x_2)\frac{\partial F}{\partial\phi(x_2)} + ... \tag{4.152}$$

Hence

$$\begin{aligned}<0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\bigg)|0> &= \bigg[e^{\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial\phi(x_i)}D_F(x_i-x_j)\frac{\partial}{\partial\phi(x_j)}}F(\phi)\bigg]_{\phi=0}\\ &= \bigg[e^{\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial\phi(x_i)}D_F(x_i-x_j)\frac{\partial}{\partial\phi(x_j)}}\bigg(\phi(x_1)...\phi(x_{2n})\bigg)\bigg]_{\phi=0} \end{aligned} \tag{4.153}$$

This is our last version of the Wick's theorem.

4.3.3 The 2−Point Function

We have

$$\begin{aligned}<0|T(\hat{\phi}(x_1)\hat{\phi}(x_2))|0> &= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4y_1..\int d^4y_n <0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\mathcal{L}_{\rm int}(y_1)..\mathcal{L}_{\rm int}(y_n)\bigg)|0>\\ &= <0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\bigg)|0> + i\int d^4y_1 <0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\mathcal{L}_{\rm int}(y_1)\bigg)|0>\\ &+ \frac{i^2}{2!}\int d^4y_1\int d^4y_2 <0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\mathcal{L}_{\rm int}(y_1)\mathcal{L}_{\rm int}(y_2)\bigg)|0> + ... \end{aligned} \tag{4.154}$$

By using the result (4.153) we have (since we are considering only polynomial interactions)

$$<0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\mathcal{L}_{\rm int}(y_1)..\mathcal{L}_{\rm int}(y_n)\bigg)|0> = \bigg[e^{\partial D_F\partial}\bigg(\phi(x_1)\phi(x_2)\mathcal{L}_{\rm int}(y_1)..\mathcal{L}_{\rm int}(y_n)\bigg)\bigg]_{\phi=0}. \tag{4.155}$$

$$\partial D_F \partial = \frac{1}{2}\sum_{i,j}\frac{\partial}{\partial\phi(x_i)}D_F(x_i-x_j)\frac{\partial}{\partial\phi(x_j)}+\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial\phi(y_i)}D_F(y_i-y_j)\frac{\partial}{\partial\phi(y_j)} + \sum_{i,j}\frac{\partial}{\partial\phi(x_i)}D_F(x_i-y_j)\frac{\partial}{\partial\phi(y_j)}. \quad (4.156)$$

The 0th order term is the free propagator, viz

$$<0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\bigg)|0>=D_F(x_1-x_2). \quad (4.157)$$

We represent this amplitude by a line joining the external points x_1 and x_2 (figure 1). This is our first Feynman diagram. Physically this represents a scalar particle created at x_2 then propagates in spacetime before it gets annihilated at x_1.

The first order is given by

$$i\int d^4y_1 <0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\mathcal{L}_{\rm int}(y_1)\bigg)|0> \quad = \quad i(-\frac{\lambda}{4!})\int d^4y_1 <0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\hat{\phi}_{\rm in}(y_1)^4\bigg)|0>. \quad (4.158)$$

We apply the Wick's theorem. There are clearly many possible contractions. For six operators we can have in total 15 contractions which can be counted as follows. The first operator can be contracted in 5 different ways. The next operator can be contracted in 3 different ways and finally the remaining two operators can only be contracted in one way. Thus we get $5.3.1=15$. However there are only two distinct contractions among these 15 contractions. They are as follows

$a)-$ We can contract the two external points x_1 and x_2 together. The internal point $z=y_1$ which we will call a vertex since it corresponds to an interaction corresponds to 4 internal points (operators) which can be contracted in $3.1=3$ different ways. We have therefore three identical contributions coming from these three contractions. We get

$$3\times i(-\frac{\lambda}{4!})D_F(x_1-x_2)\int d^4z D_F(0)^2=\frac{1}{8}(-i\lambda)\int d^4z D_F(x_1-x_2)D_F(0)^2. \quad (4.159)$$

$b)-$ We can contract one of the external points with one of the internal points. There are four different ways for doing this. The remaining external point must then be contracted with one of the remaining three internal points. There are three different ways for doing this. In total we have $4.3=12$ contractions which lead to the same contribution. We have

$$12\times i(-\frac{\lambda}{4!})\int d^4z D_F(x_1-z)D_F(x_2-z)D_F(0)=\frac{1}{2}(-i\lambda)\int d^4z D_F(x_1-z)D_F(x_2-z)D_F(0). \quad (4.160)$$

The two amplitudes (4.159) and (4.160) stand for the 15 possible contractions which we found at first order. These contractions split into two topologically distinct sets represented by the two Feynman diagrams $a)$ and $b)$ on figure 2 with attached values given precisely by (4.159) and (4.160). We observe in constructing these diagrams the following

- Each line (internal or external) joining two spacetime points x and y is associated with a propagator $D_F(x-y)$.

- Interaction is represented by a vertex. Each vertex is associated with a factor $-i\lambda$.
- We multiply the propagators and vertices together then we integrate over the internal point.
- We divide by a so-called symmetry factor S. The symmetry factor is equal to the number of independent permutations which leave the diagram invariant.

  A diagram containing a line which starts and ends on the same vertex will be symmetric under the permutation of the two ends of such a line. This is clear from the identity

  $$\int d^4z D_F(0) = \int d^4z \int d^4u D_F(z-u)\delta^4(z-u). \tag{4.161}$$

  Diagram $b)$ contains such a factor and thus the symmetry factor in this case is $S=2$. Diagram $a)$ contains two such factors and thus one must divide by 2.2. Since this diagram is also invariant under the permutation of the two $D_F(0)$ we must divide by an extra factor of 2. The symmetry factor for diagram $a)$ is therefore $S = 2.2.2 = 8$.

The second order in perturbation theory is given by

$$\begin{aligned}\frac{i^2}{2!}\int d^4y_1 \int d^4y_2 <0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\mathcal{L}_{\rm int}(y_1)\mathcal{L}_{\rm int}(y_2)\bigg)|0> &= \\ -\frac{1}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 <0|T\bigg(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)\hat{\phi}_{\rm in}(y_1)^4\hat{\phi}_{\rm in}(y_2)^4\bigg)|0>.\end{aligned} \tag{4.162}$$

Again we apply Wick's theorem. There are in total $9.7.5.3 = 9.105$ contractions which can be divided into three different classes (figure 3) as follows

1) The first class corresponds to the contraction of the two external points x_1 and x_2 to the same vertex y_1 or y_2. These contractions correspond to the two topologically different contractions $a)_1$ and $b)_1$ on figure 3.

   In $a)_1$ we contract x_1 with one of the internal points in 8 different ways, then x_2 can be contracted in 3 different ways to the same internal point (say y_1). If the two remaining y_1 points are contracted together the remaining internal points y_2 can then be contracted together in 3 different ways. There are in total 8.3.3 contractions. The analytic expression is

   $$\begin{aligned}-\frac{8.3.3}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_1)D_F(0)^3 &= \\ \frac{(-i\lambda)^2}{16}\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_1)D_F(0)^3.\end{aligned} \tag{4.163}$$

   In $b)_1$ we consider the case where one of the remaining y_1 points is contracted with one of the internal points y_2 in 4 different ways. The last y_1 must then also be contracted with one of the y_2 in 3 different ways. This possibility corresponds to 8.3.4.3 contractions. The analytic expression is

   $$\begin{aligned}-\frac{8.3.4.3}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_1)D_F(y_1-y_2)^2 D_F(0) &= \\ \frac{(-i\lambda)^2}{4}\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_1)D_F(y_1-y_2)^2 D_F(0).\end{aligned} \tag{4.164}$$

2) The second class corresponds to the contraction of the external point x_1 to one of the vertices whereas the external point x_2 is contracted to the other vertex. These contractions correspond to the two topologically different contractions $a)_2$ and $b)_2$ on figure 3.

   In $a)_2$ we contract x_1 with one of the internal points (say y_1) in 8 different ways, then x_2 can be contracted in 4 different ways to the other internal point (i.e. y_2). There remains three internal points y_1 and three internal points y_2. Two of the y_1 can be contracted in 3 different ways. The remaining y_1 must be contracted with one of the y_2 in 3 different ways. Thus we have in total 8.4.3.3 contractions. The expression is

$$\begin{aligned}-\frac{8.4.3.3}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)D_F(0)^2 &= \\ \frac{(-i\lambda)^2}{4}\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)D_F(0)^2. & \qquad (4.165)\end{aligned}$$

   In $b)_2$ we consider the case where the three remaining y_1 are paired with the three remaining y_2. The first y_1 can be contracted with one of the y_2 in 3 different ways, the second y_1 can be contracted with one of the remaining y_2 in 2 different ways. Thus we have in total 8.4.3.2 contractions. The expression is

$$\begin{aligned}-\frac{8.4.3.2}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)^3 &= \\ \frac{(-i\lambda)^2}{6}\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)^3. & \qquad (4.166)\end{aligned}$$

3) The third class corresponds to the contraction of the two external points x_1 and x_2 together. These contractions correspond to the three topologically different contractions $a)_3$, $b)_3$ and $c)_2$ on figure 3.

   In $a)_3$ we can contract the y_1 among themselves in 3 different ways and contract the y_2 among themselves in 3 different ways. Thus we have 3.3 contractions. The expression is

$$\begin{aligned}-\frac{3.3}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-x_2)D_F(0)^4 &= \\ \frac{(-i\lambda)^2}{128}\int d^4y_1\int d^4y_2 D_F(x_1-x_2)D_F(0)^4. & \qquad (4.167)\end{aligned}$$

   In $b)_3$ we can contract two of the y_1 together in 6 different ways, then contract one of the remaining y_1 with one of the y_2 in 4 different ways, and then contract the last y_1 with one of the y_2 in 3 different ways. Thus we have 6.4.3 contractions. The expression is

$$\begin{aligned}-\frac{6.4.3}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-x_2)D_F(y_1-y_2)^2D_F(0)^2 &= \\ \frac{(-i\lambda)^2}{16}\int d^4y_1\int d^4y_2 D_F(x_1-x_2)D_F(y_1-y_2)^2D_F(0)^2. & \qquad (4.168)\end{aligned}$$

   In $c)_3$ we can contract the first y_1 with one of the y_2 in 4 different ways, then contract the second y_1 with one of the y_2 in 3 different ways, then contract the third y_1 with one of the y_2 in 2 different ways. We get 4.3.2 contractions. The expression is

$$\begin{aligned}-\frac{4.3.2}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-x_2)D_F(y_1-y_2)^4 &= \\ \frac{(-i\lambda)^2}{48}\int d^4y_1\int d^4y_2 D_F(x_1-x_2)D_F(y_1-y_2)^4. & \qquad (4.169)\end{aligned}$$

The above seven amplitudes (4.163), (4.164), (4.165), (4.166), (4.167), (4.168) and (4.169) can be represented by the seven Feynman diagrams $a)_1$, $b)_1$, $a)_2$, $b)_2$, $a)_3$, $b)_3$ and $c)_3$ respectively. We use in constructing these diagrams the same rules as before. We will only comment here on the symmetry factor S for each diagram. We have

- The symmetry factor for the first diagram is $S = (2.2.2).2 = 16$ where the first three factors of 2 are associated with the three $D_F(0)$ and the last factor of 2 is associated with the interchange of the two $D_F(0)$ in the figure of eight.
- The symmetry factor for the second diagram is $S = 2.2 = 4$ where the first factor of 2 is associated with $D_F(0)$ and the second factor is associated with the interchange of the two internal lines $D_F(y_1 - y_2)$.
- The symmetry factor for the third diagram is $S = 2.2$ where the two factors of 2 are associated with the two $D_F(0)$.
- The symmetry factor of the 4th diagram is $S = 3! = 6$ which is associated with the permutations of the three internal lines $D_F(y_1 - y_2)$.
- The symmetry factor of the 5th diagram is $S = 2^7 = 128$. Four factors of 2 are associated with the four $D_F(0)$. Two factors of 2 are associated with the permutations of the two $D_F(0)$ in the two figures of eight. Another factor of 2 is associated with the interchange of the two figures of eight.
- The symmetry factor of the 6th diagram is $S = 2^4 = 16$. Two factors of 2 comes from the two $D_F(0)$. A factor of 2 comes from the interchange of the two internal lines $D_F(y_1 - y_2)$. Another factor comes from the interchange of the two internal points y_1 and y_2.
- The symmetry factor of the last diagram is $S = 4!.2 = 48$. The factor 4! comes from the permutations of the four internal lines $D_F(y_1 - y_2)$ and the factor of two comes from the interchange of the two internal points y_1 and y_2.

4.3.4 Connectedness and Vacuum Energy

From the above discussion we observe that there are two types of Feynman diagrams. These are

- Connected Diagrams: These are diagrams in which every piece is connected to the external points. Examples of connected diagrams are diagram $b)$ on figure 2) and diagrams $b)_1$, $a)_2$ and $b)_2$ on figure 4.
- Disconnected Diagrams: These are diagrams in which there is at least one piece which is not connected to the external points. Examples of disconnected diagrams are diagram $a)$ on figure 2) and diagrams $a)_1$, $a)_3$, $b)_3$ and $c)_3$ on figure 4.

We write the 2−point function up to the second order in perturbation theory as

$$
\begin{aligned}
< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2))|0 > &= D_0(x_1 - x_2)[V_1 + \frac{1}{2}V_1^2 + V_2 + V_3] + D_1(x_1 - x_2)[1 + V_1] \\
&\quad + D_2^1(x_1 - x_2) + D_2^2(x_1 - x_2) + D_2^3(x_1 - x_2).
\end{aligned}
\tag{4.170}
$$

The "connected" 2−point function at the 0th and 1st orders is given respectively by

$$
D_0(x_1 - x_2) = \text{diagram } 1) = D_F(x_1 - x_2). \tag{4.171}
$$

$$D_1(x_1-x_2) = \text{diagram } 2b) = \frac{1}{2}(-i\lambda)\int d^4y_1 D_F(x_1-y_1)D_F(x_2-y_1)D_F(0). \quad (4.172)$$

The "connected" 2−point function at the 2nd order is given by the sum of the three propagators D_2^1, D_2^2 and D_2^3. Explicitly they are given by

$$D_2^1(x_1-x_2) = \text{diagram } 4b)_1 = \frac{(-i\lambda)^2}{4}\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_1)D_F(y_1-y_2)^2 D_F(0). \quad (4.173)$$

$$D_2^2(x_1-x_2) = \text{diagram } 4a)_2 = \frac{(-i\lambda)^2}{4}\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)D_F(0)^2. \quad (4.174)$$

$$D_2^3(x_1-x_2) = \text{diagram } 4b)_2 = \frac{(-i\lambda)^2}{6}\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)^3. \quad (4.175)$$

The connected 2−point function up to the second order in perturbation theory is therefore

$$< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2))|0>_{\rm conn} = D_0(x_1-x_2)+D_1(x_1-x_2)+D_2^1(x_1-x_2)+D_2^2(x_1-x_2)+D_2^3(x_1-x_2). \quad (4.176)$$

The corresponding Feynman diagrams are shown on figure 5. The disconnected diagrams are obtained from the product of these connected diagrams with the so-called vacuum graphs which are at this order in perturbation theory given by V_1, V_2 and V_3 (see (4.170)). The vacuum graphs are given explicitly by

$$V_1 = \frac{-i\lambda}{8}\int d^4y_1 D_F(0)^2. \quad (4.177)$$

$$V_2 = \frac{(-i\lambda)^2}{16}\int d^4y_1\int d^4y_2 D_F(y_1-y_2)^2 D_F(0)^2. \quad (4.178)$$

$$V_3 = \frac{(-i\lambda)^2}{48}\int d^4y_1\int d^4y_2 D_F(y_1-y_2)^4. \quad (4.179)$$

The corresponding Feynman diagrams are shown on figure 6. Clearly the "full" and the "connected" 2−point functions can be related at this order in perturbation theory as

$$< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2))|0> = < 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2))|0>_{\rm conn} \exp(\text{vacuum graphs}). \quad (4.180)$$

We now give a more general argument for this identity. We will label the various vacuum graphs by V_i, $i=1,2,3,....$ A generic Feynman diagram will contain a connected piece attached to the external points x_1 and x_2 call it W_j, n_1 disconnected pieces given by V_1, n_2 disconnected pieces given by V_2, and so on. The value of this Feynman diagram is clearly

$$W_j\prod_i \frac{1}{n_i!}V_i^{n_i}. \quad (4.181)$$

The factor $1/n_i!$ is a symmetry factor coming from the permutations of the n_i pieces V_i among themselves. Next by summing over all Feynman diagrams (i.e, all possible connected diagrams and all possible values of n_i) we obtain

$$\begin{aligned}\sum_j \sum_{n_1,...,n_i,...} W_j \prod_i \frac{1}{n_i!} V_i^{n_i} &= \sum_j W_j \sum_{n_1,...,n_i,...} \prod_i \frac{1}{n_i!} V_i^{n_i} \\ &= \sum_j W_j \prod_i \sum_{n_i} \frac{1}{n_i!} V_i^{n_i} \\ &= \sum_j W_j \prod_i \exp(V_i) \\ &= \sum_j W_j \exp(\sum_i V_i). \end{aligned} \quad (4.182)$$

This is the desired result. This result holds also for any other Green's function, viz

$$< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)|0 >=< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)|0 >_{\rm conn} \exp(\text{vacuum graphs}). \quad (4.183)$$

Let us note here that the set of all vacuum graphs is the same for all Green's functions. In particular the $0-$point function (the vacuum-to-vacuum amplitude) will be given by

$$< 0|0 >= \exp(\text{vacuum graphs}). \quad (4.184)$$

We can then observe that

$$\begin{aligned}< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)|0 >_{\rm conn} &= \frac{< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)|0 >}{< 0|0 >} \\ &= \text{sum of connected diagrams with } n \text{ external points.}\end{aligned} \quad (4.185)$$

We write this as

$$< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)|0 >_{\rm conn} = < \Omega|T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)|\Omega > . \quad (4.186)$$

$$|\Omega >= \frac{|0 >}{\sqrt{< 0|0 >}} = e^{-\frac{1}{2}(\text{vacuum graphs})}|0 > . \quad (4.187)$$

The vacuum state $|\Omega >$ will be interpreted as the ground state of the full Hamiltonian $\hat{H}$ in contrast to the vacuum state $|0 >$ which is the ground state of the free Hamiltonian $\hat{H}_0$. The vector state $|\Omega >$ has non-zero energy $\hat{E}_0$. Thus $\hat{H}|\Omega >= \hat{E}_0|\Omega >$ as opposed to $\hat{H}_0|0 >= 0$. Let $|n >$ be the other vector states of the Hamiltonian $\hat{H}$, viz $\hat{H}|n >= \hat{E}_n|n >$.

The evolution operator $\Omega(t)$ is a solution of the differential equation $i\partial_t\Omega(t) = \hat{V}_I(t)\Omega(t)$ which satisfies the boundary condition $\Omega(-\infty) = 1$. A generalization of $\Omega(t)$ is given by the evolution operator

$$\Omega(t,t^{'}) = T\bigg(e^{-i\int_{t^{'}}^t ds\hat{V}_I(s)}\bigg). \quad (4.188)$$

This solves essentially the same differential equation as $\Omega(t)$, viz

$$i\partial_t\Omega(t,t^{'}) = \hat{V}_I(t,t_0)\Omega(t). \quad (4.189)$$

$$\hat{V}_I(t,t_0) = e^{i\hat{H}_0(t-t_0)}\hat{V}e^{-i\hat{H}_0(t-t_0)}. \tag{4.190}$$

This evolution operator $\Omega(t,t^{'})$ satisfies obviously the boundary condition $\Omega(t,t)=1$. Furthermore it is not difficult to verify that an equivalent expression for $\Omega(t,t^{'})$ is given by

$$\Omega(t,t^{'}) = e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t^{'})}e^{-i\hat{H}_0(t^{'}-t_0)}. \tag{4.191}$$

We compute

$$\begin{aligned} e^{-i\hat{H}T}|0> &= e^{-i\hat{H}T}|\Omega><\Omega|0> + \sum_{n\neq 0} e^{-i\hat{H}T}|n><n|0> \\ &= e^{-i\hat{E}_0T}|\Omega><\Omega|0> + \sum_{n\neq 0} e^{-i\hat{E}_nT}|n><n|0>\,. \end{aligned} \tag{4.192}$$

In the limit $T\longrightarrow\infty(1-i\epsilon)$ the second term drops since $\hat{E}_n>\hat{E}_0$ and we obtain

$$e^{-i\hat{H}T}|0> = e^{-i\hat{E}_0T}|\Omega><\Omega|0>\,. \tag{4.193}$$

Equivalently

$$e^{-i\hat{H}(t_0-(-T))}|0> = e^{-i\hat{E}_0(t_0+T)}|\Omega><\Omega|0>\,. \tag{4.194}$$

Thus

$$|\Omega> = \frac{e^{i\hat{E}_0(t_0+T)}}{<\Omega|0>}\Omega(t_0,-T)|0>\,. \tag{4.195}$$

By choosing $t_0=T$ and using the fact that $\Omega(T,-T)=S$ we obtain

$$|\Omega> = \frac{e^{i\hat{E}_0(2T)}}{<\Omega|0>}|0>\,. \tag{4.196}$$

Finally by using the definition of $|\Omega>$ in terms of $|0>$ and assuming that the sum of vacuum graphs is pure imaginary we get

$$\frac{\hat{E}_0}{\text{vol}} = i\frac{\text{vacuum graphs}}{2T.\text{vol}}. \tag{4.197}$$

Every vacuum graph will contain a factor $(2\pi)^4\delta^4(0)$ which in the box normalization is equal exactly to $2T$.vol where vol is the volume of the three dimensional space. Hence the normalized sum of vacuum graphs is precisely equal to the vacuum energy density.

4.3.5 Feynman Rules For $\Phi-$Four Theory

We use Feynman rules for perturbative $\phi-$four theory to calculate the nth order contributions to the Green's function $<0|T(\hat{\phi}(x_1)...\hat{\phi}(x_N))|0>$. They are given as follows

1) We draw all Feynman diagrams with N external points x_i and n internal points (vertices) y_i.

2) The contribution of each Feynman diagram to the Green's function $<0|T(\hat{\phi}(x_1)...\hat{\phi}(x_N))|0>$ is equal to the product of the following three factors

   – Each line (internal or external) joining two spacetime points x and y is associated with a propagator $D_F(x-y)$. This propagator is the amplitude for propagation between the two points x and y.

   – Each vertex is associated with a factor $-i\lambda$. Interaction is represented by a vertex and thus there are always 4 lines meeting at a given vertex. The factor $-i\lambda$ is the amplitude for the emission and/or absorption of scalar particles at the vertex.

   – We divide by the symmetry factor S of the diagram which is the number of permutations which leave the diagram invariant.

3) We integrate over the internal points y_i, i.e. we sum over all places where the underlying process can happen. This is the superposition principle of quantum mechanics.

These are Feynman rules in position space. We will also need Feynman rules in momentum space. Before we state them it is better we work out explicitly few concrete examples. Let us go back to the Feynman diagram $b)$ on figure 2. It is given by

$$\frac{1}{2}(-i\lambda)\int d^4z D_F(x_1-z)D_F(x_2-z)D_F(0). \tag{4.198}$$

We will use the following expression of the Feynman scalar propagator

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-y)}. \tag{4.199}$$

We compute immediately

$$\begin{aligned}\frac{1}{2}(-i\lambda)\int d^4z D_F(x_1-z)D_F(x_2-z)D_F(0) &= \int \frac{d^4p_1}{(2\pi)^4}\int \frac{d^4p_2}{(2\pi)^4}\int \frac{d^4q}{(2\pi)^4}\bigg(\frac{1}{2}(-i\lambda)(2\pi)^4\delta^4(p_1+p_2)\\ &\times\ e^{-ip_1x_1}e^{-ip_2x_2}\Delta(p_1)\Delta(p_2)\Delta(q)\bigg).\end{aligned} \tag{4.200}$$

$$\Delta(p) = \frac{i}{p^2-m^2+i\epsilon}. \tag{4.201}$$

In the above equation p_1 and p_2 are the external momenta and q is the internal momentum. We integrate over all these momenta. Clearly we still have to multiply with the vertex $-i\lambda$ and divide by the symmetry factor which is here 2. In momentum space we attach to any line which carries a momentum p a propagator $\Delta(p)$. The new features are two things 1) we attach a plane wave e^{-ipx} to each external point x into which a momentum p is flowing and 2) we impose momentum conservation at each vertex which in this case is $(2\pi)^4\delta^4(p_1+p_2+q-q) = (2\pi)^4\delta^4(p_1+p_2)$. See figure 7.

We consider another example given by the Feynman diagram $b)_2$ on figure 4). We find

$$\begin{aligned}&\frac{(-i\lambda)^2}{6}\int d^4y_1\int d^4y_2 D_F(x_1-y_1)D_F(x_2-y_2)D_F(y_1-y_2)^3 =\\ &\int \frac{d^4p_1}{(2\pi)^4}\int \frac{d^4p_2}{(2\pi)^4}\int \frac{d^4q_1}{(2\pi)^4}\int \frac{d^4q_2}{(2\pi)^4}\int \frac{d^4q_3}{(2\pi)^4}\bigg(\frac{1}{6}(-i\lambda)^2(2\pi)^4\delta^4(p_1+p_2)(2\pi)^4\delta^4(p_1-q_1-q_2-q_3)\ \times\\ &e^{-ip_1x_1}e^{-ip_2x_2}\Delta(p_1)\Delta(p_2)\Delta(q_1)\Delta(q_2)\Delta(q_3)\bigg)\end{aligned} \tag{4.202}$$

This expression can be reconstructed from the same rules we have discussed in the previous case. See figure 8.

In summary Feynman rules in momentum space read

1) We draw all Feynman diagrams with N external points x_i and n internal points (vertices) y_i.

2) The contribution of each Feynman diagram to the Green's function $<0|T(\hat{\phi}(x_1)...\hat{\phi}(x_N))|0>$ is equal to the product of the following five factors

   - Each line (internal or external) joining two spacetime points x and y is associated with a propagator $\Delta(p)$ where p is the momentum carried by the line.
   - Each vertex is associated with a factor $-i\lambda$.
   - We attach a plane wave $\exp(-ipx)$ to each external point x where p is the momentum flowing into x.
   - We impose momentum conservation at each vertex.
   - We divide by the symmetry factor S of the diagram.

3) We integrate over all internal and external momenta.

4.4 Exercises and Problems

Asymptotic Solutions

- Show that

$$\hat{Q}(t,\vec{p}) = \hat{Q}_{\rm in}(t,\vec{p}) + \frac{1}{E_{\vec{p}}}\int_{-\infty}^{t} dt^{'} \sin E_{\vec{p}}(t-t^{'})j(t^{'},\vec{p}),$$

  is a solution of the equation of motion

$$(\partial_t^2 + E_{\vec{p}}^2)Q(t,\vec{p}) = j(t,\vec{p}).$$

- Show that

$$\hat{Q}(t,\vec{p}) = \hat{Q}^{+}_{\rm in}(t,\vec{p}) + \hat{Q}^{-}_{\rm out}(t,\vec{p}) + i\int_{-\infty}^{+\infty} dt^{'} G_{\vec{p}}(t-t^{'})j(t^{'},\vec{p}),$$

  is also a solution of the above differential equation.

- Express the Feynman scalar propagator $D_F(x-x^{'})$ in terms of $G_{\vec{p}}(t-t^{'})$.

- Show that this solution leads to

$$\hat{\phi}(x) \quad = \quad \hat{\phi}^{+}_{\rm in}(x) + \hat{\phi}^{-}_{\rm out}(x) + i\int d^4x^{'} D_F(x-x^{'})J(x^{'}).$$

Hint: Use

$$\frac{d}{dt}\int_{-\infty}^{t} dt^{'} f(t^{'},t) = \int_{-\infty}^{t} dt^{'} \frac{\partial f(t^{'},t)}{\partial t} + f(t,t).$$

$$(\partial_t^2 + E_{\vec{p}}^2)G_{\vec{p}}(t-t^{'}) = -i\delta(t-t^{'}).$$

Feynman Scalar Propagator Verify that the Feynman propagator in one-dimension is given by

$$G_{\vec{p}}(t-t^{'}) = \int \frac{dE}{2\pi}\frac{i}{E^2-E_{\vec{p}}^2+i\epsilon}e^{-iE(t-t^{'})} = \frac{e^{-iE_{\vec{p}}|t-t^{'}|}}{2E_{\vec{p}}}.$$

Fourier Transform Show that the Fourier transform of the Klein-Gordon equation of motion

$$(\partial_\mu\partial^\mu+m^2)\phi = J$$

is given by

$$(\partial_t^2+E_{\vec{p}}^2)Q(t,\vec{p}) = j(t,\vec{p}).$$

Forced Harmonic Oscillator We consider a single forced harmonic oscillator given by the equation of motion

$$(\partial_t^2+E^2)Q(t) = J(t).$$

- Show that the $S-$matrix defined by the matrix elements $S_{mn} =< m \text{ out}|n \text{ in} >$ is unitary.
- Determine S from solving the equation

$$S^{-1}\hat{a}_{\rm in}S = \hat{a}_{\rm out} = \hat{a}_{\rm in} + \frac{i}{\sqrt{2E}}j(E).$$

- Compute the probability $| < n\ out|0\ in > |^2$.
- Determine the evolution operator in the interaction picture $\Omega(t)$ from solving the Schrodinger equation

$$i\partial_t\Omega(t) = \hat{V}_I(t)\Omega(t)\ ,\ \hat{V}_I(t) = -J(t)\hat{Q}_I(t).$$

- Deduce from the fourth question the $S-$matrix and compare with the result of the second question.

Interaction Picture Show that the fields $\hat{Q}_I(t,\vec{p})$ and $\hat{P}_I(t,\vec{p})$ are free fields.

Time Ordering Operator Show that

$$\frac{1}{3!}\int_{-\infty}^t dt_1\int_{-\infty}^t dt_2\int_{-\infty}^t dt_3 T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3)) \quad = \quad \int_{-\infty}^t dt_1\int_{-\infty}^{t_1} dt_2\int_{-\infty}^{t_2} dt_3 \hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3).$$

Wick's Theorem For Forced Scalar Field Show that

$$i\sum_{\vec{p}}\beta_{\vec{p}}(+\infty) \quad = \quad \frac{1}{2}\int d^4x\int d^4x^{'}J(x)J(x^{'})\Big(\frac{\theta(t-t^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{ip(x-x^{'})} - \frac{\theta(t-t^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{-ip(x-x^{'})}\Big).$$

Unitarity of The $S-$Matrix

- Show that

$$S^{-1} = \bar{T}\bigg(e^{i\int_{-\infty}^{+\infty}ds\hat{V}_I(s)}\bigg).$$

- Use the above result to verify that S is unitary.

Evolution Operator $\Omega(t)$ and Gell-Mann Low Formula Verify up to the third order in perturbation theory the following equations

$$\Omega(t) = \bar{T}\bigg(e^{i\int_t^{+\infty}ds\hat{V}_I(s)}\bigg)S.$$

$$\hat{\phi}(x) = S^{-1}\bigg(T\hat{\phi}_{\rm in}(x)S\bigg).$$

Interaction Fields are Free Fields Show that the interaction fields $\hat{\phi}_I(t,\vec{x})$ and $\hat{\pi}_I(t,\vec{x})$ are free fields.

LSZ Reduction Formulae

- Show the LSZ reduction formulae

$$i\int_{-\infty}^{+\infty}dte^{iE_{\vec{p}}t}(\partial_t^2+E_{\vec{p}}^2)T(\hat{Q}(t,\vec{p})\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...) =$$
$$\sqrt{2E_{\vec{p}}}\ \bigg(\hat{a}_{\rm out}(\vec{p})T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)-T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)\hat{a}_{\rm in}(\vec{p})\bigg).$$

- Show that

$$i\int d^4xe^{ipx}(\partial_\mu\partial^\mu+m^2)T(\hat{\phi}(x)\hat{\phi}(x_1)\hat{\phi}(x_2)...) =$$
$$\sqrt{2E_{\vec{p}}}\ \bigg(\hat{a}_{\rm out}(\vec{p})T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)-T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)\hat{a}_{\rm in}(\vec{p})\bigg).$$

- Derive the LSZ reduction formulae

$$-i\int_{-\infty}^{+\infty}dte^{-iE_{\vec{p}}t}(\partial_t^2+E_{\vec{p}}^2)T(\hat{Q}(t,\vec{p})^+\hat{Q}(t_1,\vec{p}_1)^+\hat{Q}(t_2,\vec{p}_2)^+...) =$$
$$\sqrt{2E_{\vec{p}}}\ \bigg(\hat{a}_{\rm out}(\vec{p})^+T(\hat{Q}(t_1,\vec{p}_1)^+\hat{Q}(t_2,\vec{p}_2)^+...)-T(\hat{Q}(t_1,\vec{p}_1)^+\hat{Q}(t_2,\vec{p}_2)^+...)\hat{a}_{\rm in}(\vec{p})^+\bigg).$$

  Hint: Start from

$$e^{-iE_{\vec{p}}t}(-i\partial_t+E_{\vec{p}})\hat{Q}_{\rm in}(t,\vec{p})^+ = \sqrt{2E_{\vec{p}}}\ \hat{a}_{\rm in}(\vec{p})^+.$$

$$e^{-iE_{\vec{p}}t}(-i\partial_t+E_{\vec{p}})\hat{Q}_{\rm out}(t,\vec{p})^+ = \sqrt{2E_{\vec{p}}}\ \hat{a}_{\rm out}(\vec{p})^+.$$

Wick's Theorem Show that

$$\left[e^{\frac{1}{2}\sum_{i,j}\frac{\partial}{\partial\phi(x_i)}D_F(x_i-x_j)\frac{\partial}{\partial\phi(x_j)}}\bigg(\phi(x_1)...\phi(x_{2n})\bigg)\right]_{\phi=0}=\sum_{\text{contraction}}\prod D_F(x_i-x_j).$$

The 4−**Point Function in** Φ−**Four Theory** Calculate the 4−point function in $\phi-$four theory up to the second order in preturbation theory.

Evolution Operator $\Omega(t,t^{'})$ Show that the evolution operators

$$\Omega(t,t^{'})=T\bigg(e^{-i\int_{t^{'}}^{t}ds\hat{V}_I(s)}\bigg),$$

and

$$\Omega(t,t^{'})=e^{i\hat{H}_0(t-t_0)}e^{-i\hat{H}(t-t^{'})}e^{-i\hat{H}_0(t^{'}-t_0)}.$$

solve the differential equation

$$i\partial_t\Omega(t,t^{'})=\hat{V}_I(t,t_0)\Omega(t).$$

Determine $\hat{V}_I(t,t_0)$.

Φ−**Cube Theory** The $\phi-$cube theory is defined by the interaction Lagrangian density

$$\mathcal{L}_{\rm int}=-\frac{\lambda}{3!}\phi^3.$$

Derive Feynman rules for this theory by considering the 2−point and 4−point functions up to the second order in perturbation theory.

Part II

Quantum Electrodynamics

5

The Electromagnetic Field

5.1 Covariant Formulation of Classical Electrodynamics

The Field Tensor The electric and magnetic fields $\vec{E}$ and $\vec{B}$ generated by a charge density ρ and a current density $\vec{J}$ are given by the Maxwell's equations written in the Heaviside-Lorentz system as

$$\vec{\nabla}\vec{E} = \rho \ , \ \mathrm{Gauss}'\mathrm{s\ Law}. \tag{5.1}$$

$$\vec{\nabla}\vec{B} = 0 \ , \ \mathrm{No-Magnetic\ Monopole\ Law}. \tag{5.2}$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t} \ , \ \mathrm{Faraday}'\mathrm{s\ Law}. \tag{5.3}$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}(\vec{J}+\frac{\partial\vec{E}}{\partial t}) \ , \ \mathrm{Ampere-Maxwell}'\mathrm{s\ Law}. \tag{5.4}$$

The Lorentz force law expresses the force exerted on a charge q moving with a velocity $\vec{u}$ in the presence of an electric and magnetic fields $\vec{E}$ and $\vec{B}$. This is given by

$$\vec{F} = q(\vec{E}+\frac{1}{c}\vec{u}\times\vec{B}). \tag{5.5}$$

The continuity equation expresses local conservation of the electric charge. It reads

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\vec{J} = 0. \tag{5.6}$$

We consider now the following Lorentz transformation

$$\begin{aligned} x^{'} &= \gamma(x-vt) \\ y^{'} &= y \\ z^{'} &= z \\ t^{'} &= \gamma(t-\frac{v}{c^2}x). \end{aligned} \tag{5.7}$$

In other words (with $x^0=ct$, $x^1=x$, $x^2=y$, $x^3=z$ and signature $(+---)$)

$$x^{\mu'}=\Lambda^\mu{}_\nu x^\nu\ ,\ \Lambda=\begin{pmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{5.8}$$

The transformation laws of the electric and magnetic fields $\vec{E}$ and $\vec{B}$ under this Lorentz transformation are given by

$$\begin{aligned}E_x^{'}=E_x\ ,\ E_y^{'}=\gamma(E_y-\frac{v}{c}B_z)\ ,\ E_z^{'}=\gamma(E_z+\frac{v}{c}B_y)\\ B_x^{'}=B_x\ ,\ B_y^{'}=\gamma(B_y+\frac{v}{c}E_z)\ ,\ B_z^{'}=\gamma(B_z-\frac{v}{c}E_y).\end{aligned} \tag{5.9}$$

Clearly $\vec{E}$ and $\vec{B}$ do not transform like the spatial part of a 4−vector. In fact $\vec{E}$ and $\vec{B}$ are the components of a second-rank antisymmetric tensor. Let us recall that a second-rank tensor $F^{\mu\nu}$ is an abject carrying two indices which transforms under a Lorentz transformation Λ as

$$F^{\mu\nu'}=\Lambda^\mu{}_\lambda\Lambda^\nu{}_\sigma F^{\lambda\sigma}. \tag{5.10}$$

This has 16 components. An antisymmetric tensor will satisfy the extra condition $F_{\mu\nu}=-F_{\mu\nu}$ so the number of independent components is reduced to 6. Explicitly we write

$$F^{\mu\nu}=\begin{pmatrix}0 & F^{01} & F^{02} & F^{03}\\ -F^{01} & 0 & F^{12} & F^{13}\\ -F^{02} & -F^{12} & 0 & F^{23}\\ -F^{03} & -F^{13} & -F^{23} & 0\end{pmatrix}. \tag{5.11}$$

The transformation laws (5.10) can then be rewritten as

$$\begin{aligned}F^{01'}=F^{01}\ ,\ F^{02'}=\gamma(F^{02}-\beta F^{12})\ ,\ F^{03'}=\gamma(F^{03}+\beta F^{31})\\ F^{23'}=F^{23}\ ,\ F^{31'}=\gamma(F^{31}+\beta F^{03})\ ,\ F^{12'}=\gamma(F^{12}-\beta F^{02}).\end{aligned} \tag{5.12}$$

By comparing (5.9) and (5.12) we obtain

$$F^{01}=-E_x\ ,\ F^{02}=-E_y\ ,\ F^{03}=-E_z\ ,\ F^{12}=-B_z\ ,\ F^{31}=-B_y\ ,\ F^{23}=-B_x. \tag{5.13}$$

Thus

$$F^{\mu\nu}=\begin{pmatrix}0 & -E_x & -E_y & -E_z\\ E_x & 0 & -B_z & B_y\\ E_y & B_z & 0 & -B_x\\ E_z & -B_y & B_x & 0\end{pmatrix}. \tag{5.14}$$

Let us remark that (5.9) remains unchanged under the duality transformation

$$\vec{E}\longrightarrow\vec{B}\ ,\ \vec{B}\longrightarrow-\vec{E}. \tag{5.15}$$

The tensor (9.69) changes under the above duality transformation to the tensor

$$\tilde{F}^{\mu\nu}=\begin{pmatrix}0 & -B_x & -B_y & -B_z\\ B_x & 0 & E_z & -E_y\\ B_y & -E_z & 0 & E_x\\ B_z & E_y & -E_x & 0\end{pmatrix}. \tag{5.16}$$

It is not difficult to show that

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F^{\alpha\beta}. \tag{5.17}$$

The 4−dimensional Levi-Civita antisymmetric tensor $\epsilon^{\mu\nu\alpha\beta}$ is defined in an obvious way.

The second-rank antisymmetric tensor $\tilde{F}$ is called the field tensor while the second-rank antisymmetric tensor $\tilde{F}$ is called the dual field tensor.

Covariant Formulation The proper charge density ρ_0 is the charge density measured in the inertial reference frame $O^{'}$ where the charge is at rest. This is given by $\rho_0 = Q/V_0$ where V_0 is the proper volume. Because the dimension along the direction of the motion is Lorentz contracted the volume V measured in the reference frame O is given by $V = \sqrt{1-u^2/c^2}V_0$. Thus the charge density measured in O is

$$\rho = \frac{Q}{V} = \frac{\rho_0}{\sqrt{1-\frac{u^2}{c^2}}}. \tag{5.18}$$

The current density $\vec{J}$ measured in O is proportional to the velocity $\vec{u}$ and to the current density ρ, viz

$$\vec{J} = \rho\vec{u} = \frac{\rho_0\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}. \tag{5.19}$$

The 4−vector velocity η^μ is defined by

$$\eta^\mu = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}(c,\vec{u}). \tag{5.20}$$

Hence we can define the current density 4−vector J^μ by

$$J^\mu = \rho_0\eta^\mu = (c\rho, J_x, J_y, J_z). \tag{5.21}$$

The continuity equation $\vec{\nabla}\vec{J} = -\partial\rho/\partial t$ which expresses charge conservation will take the form

$$\partial_\mu J^\mu = 0. \tag{5.22}$$

In terms of $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ Maxwell's equations will take the form

$$\partial_\mu F^{\mu\nu} = \frac{1}{c}J^\nu \ , \ \partial_\mu\tilde{F}^{\mu\nu} = 0. \tag{5.23}$$

The first equation yields Gauss's and Ampere-Maxwell's laws whereas the second equation yields Maxwell's third equation $\vec{\nabla}\vec{B} = 0$ and Faraday's law.

It remains to write down a covariant Lorentz force. We start with the 4−vector proper force given by

$$K^\mu = \frac{q}{c}\eta_\nu F^{\mu\nu}. \tag{5.24}$$

This is called the Minkowski force. The spatial part of this force is

$$\vec{K} = \frac{q}{\sqrt{1-\frac{u^2}{c^2}}}(\vec{E} + \frac{1}{c}\vec{u}\times\vec{B}). \tag{5.25}$$

We have also

$$K^{\mu} = \frac{dp^{\mu}}{d\tau}. \tag{5.26}$$

In other words

$$\vec{K} = \frac{d\vec{p}}{d\tau} = \frac{dt}{d\tau}\vec{F} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\vec{F}. \tag{5.27}$$

This leads precisely to the Lorentz force law

$$\vec{F} = q(\vec{E} + \frac{1}{c}\vec{u}\times\vec{B}). \tag{5.28}$$

5.2 Gauge Potentials and Gauge Transformations

The electric and magnetic fields $\vec{E}$ and $\vec{B}$ can be expressed in terms of a scalar potential V and a vector potential $\vec{A}$ as

$$\vec{B} = \vec{\nabla}\times\vec{A}. \tag{5.29}$$

$$\vec{E} = -\frac{1}{c}(\vec{\nabla}V + \frac{\partial\vec{A}}{\partial t}). \tag{5.30}$$

We construct the 4−vector potential A^{μ} as

$$A^{\mu} = (V/c, \vec{A}). \tag{5.31}$$

The field tensor $F_{\mu\nu}$ can be rewritten in terms of A_{μ} as

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}. \tag{5.32}$$

This equation is actually equivalent to the two equations (9.74) and (9.75). The homogeneous Maxwell's equation $\partial_{\mu}\tilde{F}^{\mu\nu} = 0$ is automatically solved by this ansatz. The inhomogeneous Maxwell's equation $\partial_{\mu}F^{\mu\nu} = J^{\nu}/c$ becomes

$$\partial_{\mu}\partial^{\mu}A^{\nu} - \partial^{\nu}\partial_{\mu}A^{\mu} = \frac{1}{c}J^{\nu}. \tag{5.33}$$

We have a gauge freedom in choosing A^{μ} given by local gauge transformations of the form (with λ any scalar function)

$$A^{\mu} \longrightarrow A^{'\mu} = A^{\mu} + \partial^{\mu}\lambda. \tag{5.34}$$

Indeed under this transformation we have

$$F^{\mu\nu} \longrightarrow F^{'\mu\nu} = F^{\mu\nu}. \tag{5.35}$$

These local gauge transformations form a (gauge) group. In this case the group is just the abelian $U(1)$ unitary group. The invariance of the theory under these transformations is termed a gauge invariance. The 4−vector potential A^{μ} is called a gauge potential or a gauge field. We make use of the invariance under gauge transformations by working with a gauge potential A^{μ} which satisfies some extra conditions. This procedure is known as gauge fixing. Some of the gauge conditions so often used are

$$\partial_\mu A^\mu = 0 \ , \ \text{Lorentz Gauge.} \tag{5.36}$$

$$\partial_i A^i = 0 \ , \ \text{Coulomb Gauge.} \tag{5.37}$$

$$A^0 = 0 \ , \ \text{Temporal Gauge.} \tag{5.38}$$

$$A^3 = 0 \ , \ \text{Axial Gauge.} \tag{5.39}$$

In the Lorentz gauge the equations of motion (9.78) become

$$\partial_\mu \partial^\mu A^\nu = \frac{1}{c} J^\nu. \tag{5.40}$$

Clearly we still have a gauge freedom $A^\mu \longrightarrow A^{'\mu} = A^\mu + \partial^\mu \phi$ where $\partial_\mu \partial^\mu \phi = 0$. In other words if A^μ satisfies the Lorentz gauge $\partial_\mu A^\mu = 0$ then $A^{'\mu}$ will also satisfy the Lorentz gauge, i.e. $\partial_\mu A^{'\mu} = 0$ iff $\partial_\mu \partial^\mu \phi = 0$. This residual gauge symmetry can be fixed by imposing another condition such as the temporal gauge $A^0 = 0$. We have therefore 2 constraints imposed on the components of the gauge potential A^μ which means that only two of them are really independent.

5.3 Maxwell's Lagrangian Density

The equations of motion of the gauge field A^μ is

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{1}{c} J^\nu. \tag{5.41}$$

These equations of motion should be derived from a local Lagrangian density $\mathcal{L}$, i.e. a Lagrangian which depends only on the fields and their first derivatives at the point $\vec{x}$. We have then

$$\mathcal{L} = \mathcal{L}(A_\mu, \partial_\nu A_\mu). \tag{5.42}$$

The Lagrangian is the integral over $\vec{x}$ of the Lagrangian density, viz

$$L = \int d\vec{x} \mathcal{L}. \tag{5.43}$$

The action is the integral over time of L, namely

$$S = \int dt L = \int d^4x \mathcal{L}. \tag{5.44}$$

We compute

$$\begin{aligned} \delta S &= \int d^4x \delta \mathcal{L} \\ &= \int d^4x \Big[\delta A_\nu \frac{\delta \mathcal{L}}{\delta A_\nu} - \delta A_\nu \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu} + \partial_\mu \Big(\delta A_\nu \frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu} \Big) \Big]. \end{aligned} \tag{5.45}$$

The surface term is zero because the field A_ν at infinity is assumed to be zero and thus

$$\delta A_\nu = 0\ ,\ x^\mu \longrightarrow \pm\infty. \tag{5.46}$$

We get

$$\delta S = \int d^4x \delta A_\nu \bigg[\frac{\delta \mathcal{L}}{\delta A_\nu} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu}\bigg]. \tag{5.47}$$

The principle of least action $\delta S = 0$ yields therefore the Euler-Lagrange equations

$$\frac{\delta \mathcal{L}}{\delta A_\nu} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu} = 0. \tag{5.48}$$

Firstly the Lagrangian density $\mathcal{L}$ is a Lorentz scalar. Secondly the equations of motion (5.41) are linear in the field A^μ and hence the Lagrangian density $\mathcal{L}$ can at most be quadratic in A^μ. The most general form of $\mathcal{L}$ which is quadratic in A^μ is

$$\mathcal{L}_{\rm Maxwell} = \alpha(\partial_\mu A^\mu)^2 + \beta(\partial_\mu A^\nu)(\partial^\mu A_\nu) + \gamma(\partial_\mu A^\nu)(\partial_\nu A^\mu) + \delta A_\mu A^\mu + \epsilon J_\mu A^\mu. \tag{5.49}$$

We calculate

$$\frac{\delta \mathcal{L}_{\rm Maxwell}}{\delta A_\rho} = 2\delta A^\rho + \epsilon J^\rho. \tag{5.50}$$

$$\frac{\delta \mathcal{L}_{\rm Maxwell}}{\delta \partial_\sigma A_\rho} = 2\alpha \eta^{\sigma\rho}\partial_\mu A^\mu + 2\beta\partial^\sigma A^\rho + 2\gamma \partial^\rho A^\sigma. \tag{5.51}$$

Thus

$$\frac{\delta \mathcal{L}_{\rm Maxwell}}{\delta A_\rho} - \partial_\sigma \frac{\delta \mathcal{L}_{\rm Maxwell}}{\delta \partial_\sigma A_\rho} = 0 \Leftrightarrow 2\beta\partial_\sigma\partial^\sigma A^\rho + 2(\alpha+\gamma)\partial^\rho\partial_\sigma A^\sigma - 2\delta A^\rho = \epsilon J^\rho. \tag{5.52}$$

By comparing with the equations of motion (5.41) we obtain immediately (with ζ an arbitrary parameter)

$$2\beta = -\zeta\ ,\ 2(\alpha+\gamma) = \zeta\ ,\ \delta = 0\ ,\ \epsilon = -\frac{1}{c}\zeta. \tag{5.53}$$

We get the Lagrangian density

$$\begin{aligned}\mathcal{L}_{\rm Maxwell} &= \alpha\bigg((\partial_\mu A^\mu)^2 - \partial_\mu A_\nu \partial^\nu A^\mu\bigg) - \frac{\zeta}{2}\bigg(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\bigg) - \frac{1}{c}\zeta J_\mu A^\mu \\ &= \alpha\partial_\mu\bigg(A^\mu\partial_\nu A^\nu - A^\nu \partial_\nu A^\mu\bigg) - \frac{\zeta}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}\zeta J_\mu A^\mu.\end{aligned} \tag{5.54}$$

The first term is a total derivative which vanishes since the field A_ν vanishes at infinity. Thus we end up with the Lagrangian density

$$\mathcal{L}_{\rm Maxwell} = -\frac{\zeta}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}\zeta J_\mu A^\mu. \tag{5.55}$$

In order to get a correctly normalized Hamiltonian density from this Lagrangian density we choose $\zeta = 1$. We get finally the result

$$\mathcal{L}_{\rm Maxwell} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}J_\mu A^\mu. \tag{5.56}$$

5.4 Polarization Vectors

In this section we will consider a free electromagnetic gauge field A^μ, i.e. we take $J^\mu = 0$. In the Feynman gauge (see next section for detail) the equations of motion of the gauge field A^μ read

$$\partial_\mu \partial^\mu A^\nu = 0. \tag{5.57}$$

These are 4 massless Klein-Gordon equations. The solutions are plane-waves of the form

$$A^\mu = e^{\pm \frac{i}{\hbar} px} \epsilon^\mu_\lambda(\vec{p}). \tag{5.58}$$

The 4−momentum p^μ is such that

$$p_\mu p^\mu = 0. \tag{5.59}$$

There are 4 independent polarization vectors $\epsilon^\mu_\lambda(\vec{p})$. The polarization vectors for $\lambda = 1, 2$ are termed transverse , the polarization vector for $\lambda = 3$ is termed longitudinal and the polarization vector for $\lambda = 0$ is termed scalar.

In the case of the Lorentz condition $\partial_\mu A^\mu = 0$ the polarization vectors $\epsilon^\mu_\lambda(\vec{p})$ are found to satisfy $p_\mu \epsilon^\mu_\lambda(\vec{p}) = 0$. By imposing also the temporal gauge condition $A^0 = 0$ we get $\epsilon^0_\lambda(\vec{p}) = 0$ and the Lorentz condition becomes the Coulomb gauge $\vec{p}.\vec{\epsilon}_\lambda(\vec{p}) = 0$.

Motivated by this we choose the polarization vectors $\epsilon^\mu_\lambda(\vec{p})$ as follows. We pick a fixed Lorentz frame in which the time axis is along some timelike unit 4−vector n^μ, viz

$$n_\mu n^\mu = 1 \ , \ n^0 > 0. \tag{5.60}$$

The transverse polarization vectors will be chosen in the plane orthogonal to n^μ and to the 4−momentum p^μ. The second requirement is equivalent to the Lorentz condition:

$$p_\mu \epsilon^\mu_\lambda(\vec{p}) = 0 \ , \ \lambda = 1, 2. \tag{5.61}$$

The first requirement means that

$$n_\mu \epsilon^\mu_\lambda(\vec{p}) = 0 \ , \ \lambda = 1, 2. \tag{5.62}$$

The transverse polarization vectors will furthermore be chosen to be spacelike (which is equivalent to the temporal gauge condition) and orthonormal, i.e.

$$\epsilon^\mu_1(\vec{p}) = (0, \vec{\epsilon}_1(\vec{p})) \ , \ \epsilon^\mu_2(\vec{p}) = (0, \vec{\epsilon}_2(\vec{p})), \tag{5.63}$$

and

$$\vec{\epsilon}_i(\vec{p}).\vec{\epsilon}_j(\vec{p}) = \delta_{ij}. \tag{5.64}$$

The longitudinal polarization vector is chosen in the plane (n^μ, p^μ) orthogonal to n^μ. More precisely we choose

$$\epsilon^\mu_3(\vec{p}) = \frac{p^\mu - (np)n^\mu}{np}. \tag{5.65}$$

For $n^\mu = (1, 0, 0, 0)$ we get $\epsilon^\mu_3(\vec{p}) = (0, \vec{p}/|\vec{p}|)$. This longitudinal polarization vector satisfies

$$\epsilon^\mu_3(\vec{p})\epsilon_{3\mu}(\vec{p}) = -1 \ , \ \epsilon^\mu_3(\vec{p}) n_\mu = 0 \ , \ \epsilon^\mu_3(\vec{p})\epsilon_{\lambda\mu}(\vec{p}) = 0 \ , \ \lambda = 1, 2. \tag{5.66}$$

Let us also remark

$$p_\mu \epsilon_3^\mu(\vec{p}) = -n^\mu p_\mu. \tag{5.67}$$

Indeed for a massless vector field it is impossible to choose a third polarization vector which is transevrse. A massless particle can only have two polarization states regardless of its spin whereas a massive particle with spin j can have $2j+1$ polarization states.

The scalar polarization vector is chosen to be n^μ itself, namely

$$\epsilon_0^\mu(\vec{p}) = n^\mu. \tag{5.68}$$

In summary the polarization vectors $\epsilon_\lambda^\mu(\vec{p})$ are chosen such that they satisfy the orthonormalization condition

$$\epsilon_\lambda^\mu(\vec{p})\epsilon_{\lambda' \mu}(\vec{p}) = \eta_{\lambda\lambda'}. \tag{5.69}$$

They also satisfy

$$p_\mu \epsilon_1^\mu(\vec{p}) = p_\mu \epsilon_2^\mu(\vec{p}) = 0 \ , \ - p_\mu \epsilon_3^\mu(\vec{p}) = p_\mu \epsilon_0^\mu(\vec{p}) = n^\mu p^\mu. \tag{5.70}$$

By choosing $n^\mu = (1,0,0,0)$ and $\vec{p} = (0,0,p)$ we obtain $\epsilon_0^\mu(\vec{p}) = (1,0,0,0)$, $\epsilon_1^\mu(\vec{p}) = (0,1,0,0)$, $\epsilon_2^\mu(\vec{p}) = (0,0,1,0)$ and $\epsilon_3^\mu(\vec{p}) = (0,0,0,1)$.

We compute in the reference frame in which $n^\mu = (1,0,0,0)$ the completeness relations

$$\sum_{\lambda=0}^{3} \eta_{\lambda\lambda} \epsilon_\lambda^0(\vec{p})\epsilon_\lambda^0(\vec{p}) = \epsilon_0^0(\vec{p})\epsilon_0^0(\vec{p}) = 1. \tag{5.71}$$

$$\sum_{\lambda=0}^{3} \eta_{\lambda\lambda} \epsilon_\lambda^0(\vec{p})\epsilon_\lambda^i(\vec{p}) = \epsilon_0^0(\vec{p})\epsilon_0^i(\vec{p}) = 0. \tag{5.72}$$

$$\sum_{\lambda=0}^{3} \eta_{\lambda\lambda} \epsilon_\lambda^i(\vec{p})\epsilon_\lambda^j(\vec{p}) = -\sum_{\lambda=1}^{3} \epsilon_\lambda^i(\vec{p})\epsilon_\lambda^j(\vec{p}). \tag{5.73}$$

The completeness relation for a 3−dimensional orthogonal dreibein is

$$\sum_{\lambda=1}^{3} \epsilon_\lambda^i(\vec{p})\epsilon_\lambda^j(\vec{p}) = \delta^{ij}. \tag{5.74}$$

This can be checked for example by going to the reference frame in which $\vec{p} = (0,0,p)$. Hence we get

$$\sum_{\lambda=0}^{3} \eta_{\lambda\lambda} \epsilon_\lambda^i(\vec{p})\epsilon_\lambda^j(\vec{p}) = \eta^{ij}. \tag{5.75}$$

In summary we get the completeness relations

$$\sum_{\lambda=0}^{3} \eta_{\lambda\lambda} \epsilon_\lambda^\mu(\vec{p})\epsilon_\lambda^\nu(\vec{p}) = \eta^{\mu\nu}. \tag{5.76}$$

From this equation we derive that the sum over the transverse polarization states is given by

$$\sum_{\lambda=1}^{2} \epsilon_\lambda^\mu(\vec{p})\epsilon_\lambda^\nu(\vec{p}) = -\eta^{\mu\nu} - \frac{p^\mu p^\nu}{(np)^2} + \frac{p^\mu n^\nu + p^\nu n^\mu}{np}. \tag{5.77}$$

5.5 Quantization of The Electromagnetic Gauge Field

We start with the Lagrangian density

$$\mathcal{L}_{\rm Maxwell} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}J_\mu A^\mu. \qquad (5.78)$$

The field tensor is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The equations of motion of the gauge field A^μ derived from the Lagrangian density $\mathcal{L}_{\rm Maxwell}$ are given by

$$\partial_\mu\partial^\mu A^\nu - \partial^\nu\partial_\mu A^\mu = \frac{1}{c}J^\nu. \qquad (5.79)$$

There is a freedom in the definition of the gauge field A^μ given by the gauge transformations

$$A^\mu \longrightarrow A^{'\mu} = A^\mu + \partial^\mu\lambda. \qquad (5.80)$$

The form of the equations of motion (5.79) strongly suggest the Lorentz condition

$$\partial^\mu A_\mu = 0. \qquad (5.81)$$

We incorporate this constraint via a Lagrange multiplier ζ in order to obtain a gauge-fixed Lagrangian density, viz

$$\mathcal{L}_{\rm gauge-fixed} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\zeta(\partial^\mu A_\mu)^2 - \frac{1}{c}J_\mu A^\mu. \qquad (5.82)$$

The added extra term is known as a gauge-fixing term. This modification was proposed first by Fermi. The equations of motion derived from this Lagrangian density are

$$\partial_\mu\partial^\mu A^\nu - (1-\zeta)\partial^\nu\partial_\mu A^\mu = \frac{1}{c}J^\nu. \qquad (5.83)$$

These are equivalent to Maxwell's equations in the Lorentz gauge. To see this we remark first that

$$\partial_\nu\bigg(\partial_\mu\partial^\mu A^\nu - (1-\zeta)\partial^\nu\partial_\mu A^\mu\bigg) = \frac{1}{c}\partial_\nu J^\nu. \qquad (5.84)$$

Gauge invariance requires current conservation, i.e. we must have $\partial_\nu J^\nu = 0$. Thus we obtain

$$\partial_\mu\partial^\mu\phi = 0 \ , \ \phi = \partial_\mu A^\mu. \qquad (5.85)$$

This is a Cauchy initial-value problem for $\partial_\mu A^\mu$. In other words if $\partial_\mu A^\mu = 0$ and $\partial_0(\partial_\mu A^\mu) = 0$ at an initial time $t = t_0$ then $\partial_\mu A^\mu = 0$ at all times. Hence (5.83) are equivalent to Maxwell's equations in the Lorentz gauge.

We will work in the so-called Feynman gauge which corresponds to $\zeta = 1$ and for simplicity we will set $J^\mu = 0$. The equations of motion become the massless Klein-Gordon equations

$$\partial_\mu\partial^\mu A^\nu = 0. \qquad (5.86)$$

These can be derived from the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu\partial^\mu A^\nu. \qquad (5.87)$$

This Lagrangian density is equal to the gauge-fixed Lagrangian density $\mathcal{L}_{\rm gauge-fixed}$ modulo a total derivative term, viz

$$\mathcal{L}_{\rm gauge-fixed} = \mathcal{L} + \text{total derivative term}. \tag{5.88}$$

The conjugate momentum field is defined by

$$\begin{aligned}\pi_\mu &= \frac{\delta\mathcal{L}}{\delta\partial_t A^\mu}\\ &= -\frac{1}{c^2}\partial_t A_\mu.\end{aligned} \tag{5.89}$$

The Hamiltonian density is then given by

$$\begin{aligned}\mathcal{H} &= \pi_\mu\partial_t A^\mu - \mathcal{L}\\ &= \frac{1}{2}\partial_i A_\mu\partial^i A^\mu - \frac{1}{2}\partial_0 A_\mu\partial^0 A^\mu\\ &= \frac{1}{2}(\partial_0\vec{A})^2 + \frac{1}{2}(\vec{\nabla}\vec{A})^2 - \frac{1}{2}(\partial_0 A^0)^2 - \frac{1}{2}(\vec{\nabla}A^0)^2.\end{aligned} \tag{5.90}$$

The contribution of the zero-component A^0 of the gauge field is negative. Thus the Hamiltonian density is not positive definite as it should be. This is potentially a severe problem which will be solved by means of the gauge condition.

We have already found that there are 4 independent polarization vectors $\epsilon^\mu_\lambda(\vec{p})$ for each momentm $\vec{p}$. The 4−momentum p^μ satisfies $p^\mu p_\mu = 0$, i.e. $(p^0)^2 = \vec{p}^2$. We define $\omega(\vec{p}) = \frac{c}{\hbar}p^0 = \frac{c}{\hbar}|\vec{p}|$. The most general solution of the classical equations of motion in the Lorentz gauge can be put in the form

$$A^\mu = c\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{\sqrt{2\omega(\vec{p})}}\sum_{\lambda=0}^{3}\bigg(e^{-\frac{i}{\hbar}px}\epsilon^\mu_\lambda(\vec{p})a(\vec{p},\lambda) + e^{\frac{i}{\hbar}px}\epsilon^\mu_\lambda(\vec{p})a(\vec{p},\lambda)^*\bigg)_{p^0=|\vec{p}|}. \tag{5.91}$$

We compute

$$\begin{aligned}\frac{1}{2}\int\partial_i A^\mu\partial^i A^\mu &= -c^2\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{4\omega(\vec{p})}\frac{p^ip^i}{\hbar^2}\sum_{\lambda,\lambda'=0}^{3}\epsilon^\mu_\lambda(\vec{p})\epsilon_{\lambda'\mu}(\vec{p})\bigg(a(\vec{p},\lambda)a(\vec{p},\lambda')^* + a(\vec{p},\lambda)^*a(\vec{p},\lambda')\bigg)\\ &- c^2\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{4\omega(\vec{p})}\frac{p^ip^i}{\hbar^2}\sum_{\lambda,\lambda'=0}^{3}\epsilon^\mu_\lambda(\vec{p})\epsilon_{\lambda'\mu}(-\vec{p})\bigg(e^{-\frac{2i}{\hbar}p^0x^0}a(\vec{p},\lambda)a(-\vec{p},\lambda')\\ &+ e^{+\frac{2i}{\hbar}p^0x^0}a(\vec{p},\lambda)^*a(-\vec{p},\lambda')^*\bigg).\end{aligned} \tag{5.92}$$

$$\begin{aligned}\frac{1}{2}\int\partial_0 A^\mu\partial^0 A^\mu &= c^2\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{4\omega(\vec{p})}\frac{p^0p^0}{\hbar^2}\sum_{\lambda,\lambda'=0}^{3}\epsilon^\mu_\lambda(\vec{p})\epsilon_{\lambda'\mu}(\vec{p})\bigg(a(\vec{p},\lambda)a(\vec{p},\lambda')^* + a(\vec{p},\lambda)^*a(\vec{p},\lambda')\bigg)\\ &- c^2\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{4\omega(\vec{p})}\frac{p^0p^0}{\hbar^2}\sum_{\lambda,\lambda'=0}^{3}\epsilon^\mu_\lambda(\vec{p})\epsilon_{\lambda'\mu}(-\vec{p})\bigg(e^{-\frac{2i}{\hbar}p^0x^0}a(\vec{p},\lambda)a(-\vec{p},\lambda')\\ &+ e^{+\frac{2i}{\hbar}p^0x^0}a(\vec{p},\lambda)^*a(-\vec{p},\lambda')^*\bigg).\end{aligned} \tag{5.93}$$

The Hamiltonian becomes (since $p^0p^0=p^ip^i$)

$$\begin{aligned} H &= \int d^3x\bigg(\frac{1}{2}\partial_i A_\mu\partial^i A^\mu-\frac{1}{2}\partial_0 A_\mu\partial^0 A^\mu\bigg)\\ &= -c^2\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{2\omega(\vec{p})}\frac{p^0p^0}{\hbar^2}\sum_{\lambda,\lambda^{'}=0}^{3}\epsilon^\mu_\lambda(\vec{p})\epsilon_{\lambda^{'}\mu}(\vec{p})\bigg(a(\vec{p},\lambda)a(\vec{p},\lambda^{'})^*+a(\vec{p},\lambda)^*a(\vec{p},\lambda^{'})\bigg)\\ &= -\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{\omega(\vec{p})}{2}\sum_{\lambda,\lambda^{'}=0}^{3}\epsilon^\mu_\lambda(\vec{p})\epsilon_{\lambda^{'}\mu}(\vec{p})\bigg(a(\vec{p},\lambda)a(\vec{p},\lambda^{'})^*+a(\vec{p},\lambda)^*a(\vec{p},\lambda^{'})\bigg)\\ &= -\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{\omega(\vec{p})}{2}\sum_{\lambda=0}^{3}\eta_{\lambda\lambda}\bigg(a(\vec{p},\lambda)a(\vec{p},\lambda)^*+a(\vec{p},\lambda)^*a(\vec{p},\lambda)\bigg). \end{aligned} \tag{5.94}$$

In the quantum theory A^μ becomes the operator

$$\hat{A}^\mu=c\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{\sqrt{2\omega(\vec{p})}}\sum_{\lambda=0}^{3}\bigg(e^{-\frac{i}{\hbar}px}\epsilon^\mu_\lambda(\vec{p})\hat{a}(\vec{p},\lambda)+e^{\frac{i}{\hbar}px}\epsilon^\mu_\lambda(\vec{p})\hat{a}(\vec{p},\lambda)^+\bigg)_{p^0=|\vec{p}|}. \tag{5.95}$$

The conjugate momentum π^μ becomes the operator

$$\begin{aligned} \hat{\pi}^\mu &= -\frac{1}{c^2}\partial_t\hat{A}^\mu\\ &= \int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{i}{c}\sqrt{\frac{\omega(\vec{p})}{2}}\sum_{\lambda=0}^{3}\bigg(e^{-\frac{i}{\hbar}px}\epsilon^\mu_\lambda(\vec{p})\hat{a}(\vec{p},\lambda)-e^{\frac{i}{\hbar}px}\epsilon^\mu_\lambda(\vec{p})\hat{a}(\vec{p},\lambda)^+\bigg)_{p^0=|\vec{p}|}. \end{aligned} \tag{5.96}$$

We impose the equal-time canonical commutation relations

$$[\hat{A}^\mu(x^0,\vec{x}),\hat{\pi}^\nu(x^0,\vec{y})]=i\hbar\eta^{\mu\nu}\delta^3(\vec{x}-\vec{y}). \tag{5.97}$$

$$[\hat{A}^\mu(x^0,\vec{x}),\hat{A}^\nu(x^0,\vec{y})]=[\hat{\pi}^\mu(x^0,\vec{x}),\hat{\pi}^\nu(x^0,\vec{y})]=0. \tag{5.98}$$

The operators $\hat{a}^+$ and $\hat{a}$ are expected to be precisely the creation and annihilation operators. In other words we expect that

$$[\hat{a}(\vec{p},\lambda),\hat{a}(\vec{q},\lambda^{'})]=[\hat{a}(\vec{p},\lambda)^+,\hat{a}(\vec{q},\lambda^{'})^+]=0. \tag{5.99}$$

We compute then

$$\begin{aligned} [\hat{A}^\mu(x^0,\vec{x}),\hat{\pi}^\nu(x^0,\vec{y})] &= -i\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\int\frac{d^3\vec{q}}{(2\pi\hbar)^3}\frac{1}{\sqrt{2\omega(\vec{p})}}\sqrt{\frac{\omega(\vec{q})}{2}}\sum_{\lambda,\lambda^{'}=0}^{3}\epsilon^\mu_\lambda(\vec{p})\epsilon^\nu_{\lambda^{'}}(\vec{q})\bigg(\\ & e^{-\frac{i}{\hbar}px}e^{+\frac{i}{\hbar}qy}[\hat{a}(\vec{p},\lambda),\hat{a}(\vec{q},\lambda^{'})^+]+e^{+\frac{i}{\hbar}px}e^{-\frac{i}{\hbar}qy}[\hat{a}(\vec{q},\lambda^{'}),\hat{a}(\vec{p},\lambda)^+]\bigg). \end{aligned} \tag{5.100}$$

We can immediately conclude that we must have

$$[\hat{a}(\vec{p},\lambda),\hat{a}(\vec{q},\lambda^{'})^+]=-\eta_{\lambda\lambda^{'}}\hbar(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}). \tag{5.101}$$

By using (5.99) and (5.101) we can also verify the equal-time canonical commutation relations (5.98). The minus sign in (5.101) causes serious problems. For transverse ($i=1,2$) and longitudinal ($i=3$) polarizations the number operator is given as usual by $\hat{a}(\vec{p},i)^+\hat{a}(\vec{p},i)$. Indeed we compute

$$[\hat{a}(\vec{p},i)^+\hat{a}(\vec{p},i),\hat{a}(\vec{q},i)]=-\hbar(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q})\hat{a}(\vec{q},i)$$
$$[\hat{a}(\vec{p},i)^+\hat{a}(\vec{p},i),\hat{a}(\vec{q},i)^+]=\hbar(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q})\hat{a}(\vec{q},i)^+. \tag{5.102}$$

In the case of the scalar polarization ($\lambda=0$) the number operator is given by $-\hat{a}(\vec{p},0)^+\hat{a}(\vec{p},0)$ since

$$[-\hat{a}(\vec{p},0)^+\hat{a}(\vec{p},0),\hat{a}(\vec{q},0)]=-\hbar(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q})\hat{a}(\vec{q},0)$$
$$[-\hat{a}(\vec{p},0)^+\hat{a}(\vec{p},0),\hat{a}(\vec{q},0)^+]=\hbar(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q})\hat{a}(\vec{q},0)^+. \tag{5.103}$$

In the quantum theory the Hamiltonian becomes the operator

$$\hat{H} = -\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{\omega(\vec{p})}{2}\sum_{\lambda=0}^{3}\eta_{\lambda\lambda}\bigg(\hat{a}(\vec{p},\lambda)\hat{a}(\vec{p},\lambda)^++\hat{a}(\vec{p},\lambda)^+\hat{a}(\vec{p},\lambda)\bigg). \tag{5.104}$$

As before normal ordering yields the Hamiltonian operator

$$\begin{aligned}\hat{H} &= -\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\omega(\vec{p})\sum_{\lambda=0}^{3}\eta_{\lambda\lambda}\hat{a}(\vec{p},\lambda)^+\hat{a}(\vec{p},\lambda)\\ &= \int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\omega(\vec{p})\bigg(\sum_{i=1}^{3}\hat{a}(\vec{p},i)^+\hat{a}(\vec{p},i)-\hat{a}(\vec{p},0)^+\hat{a}(\vec{p},0)\bigg).\end{aligned} \tag{5.105}$$

Since $-\hat{a}(\vec{p},0)^+\hat{a}(\vec{p},0)$ is the number operator for scalar polarization the Hamiltonian $\hat{H}$ can only have positive eigenvalues. Let $|0>$ be the vacuum state, viz

$$\hat{a}(\vec{p},\lambda)|0>=0\ ,\ \forall\vec{p}\text{ and }\forall\lambda. \tag{5.106}$$

The one-particle states are defined by

$$|\vec{p},\lambda>=\hat{a}(\vec{p},\lambda)^+|0>\ . \tag{5.107}$$

Let us compute the expectation value

$$<\vec{p},\lambda|\hat{H}|\vec{p},\lambda>\ . \tag{5.108}$$

By using $\hat{H}|0>=0$ and $[\hat{H},\hat{a}(\vec{p},\lambda)^+]=\hbar\omega(\vec{p})\hat{a}(\vec{p},\lambda)^+$ we find

$$\begin{aligned}<\vec{p},\lambda|\hat{H}|\vec{p},\lambda> &= <\vec{p},\lambda|[\hat{H},\hat{a}(\vec{p},\lambda)^+]|0>\\ &= \hbar\omega(\vec{p})<\vec{p},\lambda|\vec{p},\lambda>\ .\end{aligned} \tag{5.109}$$

However

$$\begin{aligned}<\vec{p},\lambda|\vec{p},\lambda> &= <0|[\hat{a}(\vec{p},\lambda),\hat{a}(\vec{p},\lambda)^+]|0>\\ &= -\eta_{\lambda\lambda}\hbar(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q})<0|0>\\ &= -\eta_{\lambda\lambda}\hbar(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}).\end{aligned} \tag{5.110}$$

This is negative for the scalar polarization $\lambda=0$ which is potentially a severe problem. As a consequence the expectation value of the Hamiltonian operator in the one-particle state with scalar polarization is negative. The resolution of these problems lies in the Lorentz gauge fixing condition which needs to be taken into consideration.

5.6 Gupta-Bleuler Method

In the quantum theory the Lorentz gauge fixing condition $\partial_\mu A^\mu = 0$ becomes the operator equation

$$\partial_\mu \hat{A}^\mu = 0. \tag{5.111}$$

Explicitly we have

$$\partial_\mu \hat{A}^\mu = -c\int \frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{\sqrt{2\omega(\vec{p})}}\frac{i}{\hbar}p_\mu \sum_{\lambda=0}^{3}\bigg(e^{-\frac{i}{\hbar}px}\epsilon^\mu_\lambda(\vec{p})\hat{a}(\vec{p},\lambda) - e^{\frac{i}{\hbar}px}\epsilon^\mu_\lambda(\vec{p})\hat{a}(\vec{p},\lambda)^+\bigg)_{p^0=|\vec{p}|} = 0. \tag{5.112}$$

However

$$\begin{aligned}
[\partial_\mu \hat{A}^\mu(x^0,\vec{x}),\hat{A}^\nu(x^0,\vec{y})] &= [\partial_0 \hat{A}^0(x^0,\vec{x}),\hat{A}^\nu(x^0,\vec{y})] + [\partial_i \hat{A}^i(x^0,\vec{x}),\hat{A}^\nu(x^0,\vec{y})]\\
&= -c[\hat{\pi}^0(x^0,\vec{x}),\hat{A}^\nu(x^0,\vec{y})] + \partial_i^x[\hat{A}^i(x^0,\vec{x}),\hat{A}^\nu(x^0,\vec{y})]\\
&= i\hbar c\eta^{0\nu}\delta^3(\vec{x}-\vec{y}).
\end{aligned} \tag{5.113}$$

In other words in the quantum theory we can not impose the Lorentz condition as the operator identity (5.111).

The problem we faced in the previous section was the fact that the Hilbert space of quantum states has an indefinite metric, i.e. the norm was not positive-definite. As we said the solution of this problem consists in imposing the Lorentz gauge condition but clearly this can not be done in the operator form (5.111). Obviously there are physical states in the Hilbert space associated with the photon transverse polarization states and unphysical states associated with the longitudinal and scalar polarization states. It is therefore natural to impose the Lorentz gauge condition only on the physical states $|\phi>$ associated with the transverse photons. We may require for example that the expectation value $<\phi|\partial_\mu \hat{A}^\mu|\phi>$ vanishes, viz

$$<\phi|\partial_\mu \hat{A}^\mu|\phi>= 0. \tag{5.114}$$

Let us recall that the gauge field operator is given by

$$\hat{A}^\mu = c\int \frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{\sqrt{2\omega(\vec{p})}}\sum_{\lambda=0}^{3}\bigg(e^{-\frac{i}{\hbar}px}\epsilon^\mu_\lambda(\vec{p})\hat{a}(\vec{p},\lambda) + e^{\frac{i}{\hbar}px}\epsilon^\mu_\lambda(\vec{p})\hat{a}(\vec{p},\lambda)^+\bigg)_{p^0=|\vec{p}|}. \tag{5.115}$$

This is the sum of a positive-frequency part $\hat{A}^\mu_+$ and a negative-frequency part $\hat{A}^\mu_-$, viz

$$\hat{A}^\mu = \hat{A}^\mu_+ + \hat{A}^\mu_-. \tag{5.116}$$

These parts are given respectively by

$$\hat{A}^\mu_+ = c\int \frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{\sqrt{2\omega(\vec{p})}}\sum_{\lambda=0}^{3}e^{-\frac{i}{\hbar}px}\epsilon^\mu_\lambda(\vec{p})\hat{a}(\vec{p},\lambda). \tag{5.117}$$

$$\hat{A}^\mu_- = c\int \frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{\sqrt{2\omega(\vec{p})}}\sum_{\lambda=0}^{3}e^{\frac{i}{\hbar}px}\epsilon^\mu_\lambda(\vec{p})\hat{a}(\vec{p},\lambda)^+. \tag{5.118}$$

Instead of (5.114) we choose to impose the Lorentz gauge condition as the eigenvalue equation

$$\partial_\mu \hat{A}^\mu_+ |\phi>= 0. \tag{5.119}$$

This is equivalent to

$$< \phi|\partial_\mu \hat{A}^\mu_- = 0. \tag{5.120}$$

The condition (5.119) is stronger than (5.114). Indeed we can check that $< \phi|\partial_\mu \hat{A}^\mu|\phi >=< \phi|\partial_\mu \hat{A}^\mu_+|\phi > + < \phi|\partial_\mu \hat{A}^\mu_-|\phi >= 0$. In this way the physical states are defined precisely as the eigenvectors of the operator $\partial_\mu \hat{A}^\mu_+$ with eigenvalue 0. In terms of the annihilation operators $\hat{a}(\vec{p}, \lambda)$ the condition (5.119) reads

$$c\int \frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{\sqrt{2\omega(\vec{p})}}\sum_{\lambda=0}^{3} e^{-\frac{i}{\hbar}px}(-\frac{i}{\hbar}p_\mu \epsilon^\mu_\lambda(\vec{p}))\hat{a}(\vec{p}, \lambda)|\phi >= 0. \tag{5.121}$$

Since $p_\mu \epsilon^\mu_i(\vec{p}) = 0$, $i = 1, 2$ and $p_\mu \epsilon^\mu_3(\vec{p}) = -p_\mu \epsilon^\mu_0(\vec{p}) = -n^\mu p_\mu$ we get

$$c\int \frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{\sqrt{2\omega(\vec{p})}} e^{-\frac{i}{\hbar}px}\frac{i}{\hbar}p_\mu n^\mu\bigg(\hat{a}(\vec{p}, 3) - \hat{a}(\vec{p}, 0)\bigg)|\phi >= 0. \tag{5.122}$$

We immediately conclude that

$$\bigg(\hat{a}(\vec{p}, 3) - \hat{a}(\vec{p}, 0)\bigg)|\phi >= 0. \tag{5.123}$$

Hence we deduce the crucial identity

$$< \phi|\hat{a}(\vec{p}, 3)^+\hat{a}(\vec{p}, 3)|\phi >=< \phi|\hat{a}(\vec{p}, 0)^+\hat{a}(\vec{p}, 0)|\phi > . \tag{5.124}$$

$$\begin{aligned} < \phi|\hat{H}|\phi > &= \int \frac{d^3\vec{p}}{(2\pi\hbar)^3}\omega(\vec{p})\bigg(\sum_{i=1}^{2} < \phi|\hat{a}(\vec{p}, i)^+\hat{a}(\vec{p}, i)|\phi > + < \phi|\hat{a}(\vec{p}, 3)^+\hat{a}(\vec{p}, 3)|\phi > \\ &- < \phi|\hat{a}(\vec{p}, 0)^+\hat{a}(\vec{p}, 0)|\phi >\bigg) \\ &= \int \frac{d^3\vec{p}}{(2\pi\hbar)^3}\omega(\vec{p})\sum_{i=1}^{2} < \phi|\hat{a}(\vec{p}, i)^+\hat{a}(\vec{p}, i)|\phi > . \end{aligned} \tag{5.125}$$

This is always positive definite and only transverse polarization states contribute to the expectation value of the Hamiltonian operator. This same thing will happen for all other physical observables such as the momentum operator and the angular momentum operator. Let us define

$$L(\vec{p}) = \hat{a}(\vec{p}, 3) - \hat{a}(\vec{p}, 0). \tag{5.126}$$

We have

$$L(\vec{p})|\phi >= 0. \tag{5.127}$$

It is trivial to show that

$$[L(\vec{p}), L(\vec{p}^{'})^+] = 0. \tag{5.128}$$

Thus

$$L(\vec{p})|\phi_c>=0, \quad (5.129)$$

where $|\phi_c>$ is also a physical state defined by

$$|\phi_c>=f_c(L^+)|\phi>. \quad (5.130)$$

The operator $f_c(L^+)$ can be expanded as

$$f_c(L^+)=1+\int d^3\vec{p}^{'}c(\vec{p}^{'})L(\vec{p}^{'})^++\int d^3\vec{p}^{'}\int d^3\vec{p}^{''}c(\vec{p}^{'},\vec{p}^{''})L(\vec{p}^{'})^+L(\vec{p}^{''})^++... \quad (5.131)$$

It is also trivial to show that

$$[f_c(L^+)^+,f_{c'}(L^+)]=0. \quad (5.132)$$

The physical state $|\phi_c>$ is completely equivalent to the state $|\phi>$ although $|\phi_c>$ contains longitudinal and scalar polarization states while $|\phi>$ contains only transverse polarization states. Indeed

$$\begin{aligned}<\phi_c|\phi_{c'}> &= <\phi|f_c(L^+)^+f_{c'}(L^+)|\phi> \\ &= <\phi|f_{c'}(L^+)f_c(L^+)^+|\phi> \\ &= <\phi|\phi>. \end{aligned} \quad (5.133)$$

Thus the scalar product between any two states $|\phi_c>$ and $|\phi_{c'}>$ is fully determined by the norm of the state $|\phi>$. The state $|\phi_c>$ constructed from a given physical state $|\phi>$ defines an equivalence class. Clearly the state $|\phi>$ can be taken to be the representative of this equivalence class. The members of this equivalence class are related by gauge transformations. This can be checked explicitly as follows. We compute

$$<\phi_c|\hat{A}_\mu|\phi_c> = <\phi|f_c(L^+)^+[\hat{A}_\mu,f_c(L^+)]|\phi>+<\phi|[f_c(L^+)^+,\hat{A}_\mu]|\phi>+<\phi|\hat{A}_\mu|\phi>. \quad (5.134)$$

By using the fact that the commutators of $\hat{A}^\mu$ with $L(\vec{p})$ and $L(\vec{p})^+$ are $c-$numbers we obtain

$$<\phi_c|\hat{A}_\mu|\phi_c> = \int d^3\vec{p}c(\vec{p})[\hat{A}_\mu,L(\vec{p})^+]+\int d^3\vec{p}c(\vec{p})^*[L(\vec{p}),\hat{A}_\mu]+<\phi|\hat{A}_\mu|\phi>. \quad (5.135)$$

We compute

$$[\hat{A}^\mu,L(\vec{p})^+]=\frac{\hbar c}{\sqrt{2\omega(\vec{p})}}e^{-\frac{i}{\hbar}px}(\epsilon_3^\mu(\vec{p})+\epsilon_0^\mu(\vec{p})). \quad (5.136)$$

Thus

$$\begin{aligned}<\phi_c|\hat{A}^\mu|\phi_c> &= \hbar c\int\frac{d^3\vec{p}}{\sqrt{2\omega(\vec{p})}}\bigg(\epsilon_3^\mu(\vec{p})+\epsilon_0^\mu(\vec{p})\bigg)\bigg(c(\vec{p})e^{-\frac{i}{\hbar}px}+c(\vec{p})^*e^{\frac{i}{\hbar}px}\bigg)+<\phi|\hat{A}^\mu|\phi> \\ &= \hbar c\int\frac{d^3\vec{p}}{\sqrt{2\omega(\vec{p})}}\bigg(\frac{p^\mu}{n.p}\bigg)\bigg(c(\vec{p})e^{-\frac{i}{\hbar}px}+c(\vec{p})^*e^{\frac{i}{\hbar}px}\bigg)+<\phi|\hat{A}^\mu|\phi> \\ &= \hbar c(-\frac{\hbar}{i}\partial^\mu)\int\frac{d^3\vec{p}}{\sqrt{2\omega(\vec{p})}}\bigg(\frac{1}{n.p}\bigg)\bigg(c(\vec{p})e^{-\frac{i}{\hbar}px}-c(\vec{p})^*e^{\frac{i}{\hbar}px}\bigg)+<\phi|\hat{A}^\mu|\phi> \\ &= \partial^\mu\Lambda+<\phi|\hat{A}^\mu|\phi>. \end{aligned} \quad (5.137)$$

$$\Lambda = i\hbar^2 c\int \frac{d^3\vec{p}}{\sqrt{2\omega(\vec{p})}}\bigg(\frac{1}{n.p}\bigg)\bigg(c(\vec{p})e^{-\frac{i}{\hbar}px} - c(\vec{p})^* e^{\frac{i}{\hbar}px}\bigg). \quad (5.138)$$

Since $p^0 = |\vec{p}|$ we have $\partial_\mu\partial^\mu\Lambda = 0$, i.e. the gauge function Λ is consistent with the Lorentz gauge condition.

5.7 Propagator

The probability amplitudes for a gauge particle to propagate from the spacetime point y to the spacetime x is

$$iD^{\mu\nu}(x-y) =< 0|\hat{A}^\mu(x)\hat{A}^\nu(y)|0> . \quad (5.139)$$

We compute

$$\begin{aligned} iD^{\mu\nu}(x-y) &= c^2\int\frac{d^3\vec{q}}{(2\pi\hbar)^3}\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{\sqrt{2\omega(\vec{q})}}\frac{1}{\sqrt{2\omega(\vec{p})}}e^{-\frac{i}{\hbar}qx}e^{+\frac{i}{\hbar}py}\sum_{\lambda^{'},\lambda=0}^{3}\epsilon^\mu_{\lambda^{'}}(\vec{q})\epsilon^\nu_\lambda(\vec{p}) \\ &\times\ < 0|[\hat{a}(\vec{q},\lambda^{'}),\hat{a}(\vec{p},\lambda)^+]|0> \\ &= c^2\hbar^2\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}p(x-y)}\sum_{\lambda=0}^{3}(-\eta_{\lambda\lambda}\epsilon^\mu_\lambda(\vec{q})\epsilon^\nu_\lambda(\vec{p})) \\ &= c^2\hbar^2\int\frac{d^3\vec{p}}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}p(x-y)}(-\eta^{\mu\nu}) \\ &= \hbar^2 D(x-y)(-\eta^{\mu\nu}). \end{aligned} \quad (5.140)$$

The function $D(x-y)$ is the probability amplitude for a massless real scalar particle to propagate from y to x. The retarded Green's function of the gauge field can be defined by

$$\begin{aligned} iD^{\mu\nu}_R(x-y) &= \hbar^2 D_R(x-y)(-\eta^{\mu\nu}) \\ &= \theta(x^0-y^0) < 0|[\hat{A}^\mu(x),\hat{A}^\nu(y)]|0> . \end{aligned} \quad (5.141)$$

The second line follows from the fact that $D_R(x-y) = \theta(x^0-y^0) < 0|[\hat{\phi}(x),\hat{\phi}(y)]|0 >$. In momentum space this retarded Green's function reads

$$iD^{\mu\nu}_R(x-y) = \hbar^2\bigg(c\hbar\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i}{p^2}e^{-\frac{i}{\hbar}p(x-y)}\bigg)(-\eta^{\mu\nu}). \quad (5.142)$$

Since $\partial_\alpha\partial^\alpha D_R(x-y) = (-ic/\hbar)\delta^4(x-y)$ we must have

$$\bigg(\partial_\alpha\partial^\alpha\eta_{\mu\nu}\bigg)D^{\nu\lambda}_R(x-y) = \hbar c\delta^4(x-y)\eta^\lambda_\mu. \quad (5.143)$$

Another solution of this equation is the so-called Feynman propagator for a gauge field given by

$$\begin{aligned} iD^{\mu\nu}_F(x-y) &= \hbar^2 D_F(x-y)(-\eta^{\mu\nu}) \\ &= < 0|T\hat{A}^\mu(x)\hat{A}^\nu(y)|0> . \end{aligned} \quad (5.144)$$

In momentum space this reads

$$iD^{\mu\nu}_F(x-y) = \hbar^2\bigg(c\hbar\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i}{p^2+i\epsilon}e^{-\frac{i}{\hbar}p(x-y)}\bigg)(-\eta^{\mu\nu}). \quad (5.145)$$

5.8 Exercises and Problems

Maxwell's Equations

1) Derive Maxwell's equations from

$$\partial_\mu F^{\mu\nu} = \frac{1}{c} J^\nu \ , \ \partial_\mu \tilde{F}^{\mu\nu} = 0. \tag{5.146}$$

2) Derive from the expression of the field tensor $F_{\mu\nu}$ in terms of A^μ the electric and magnetic fields in terms of the scalar and vector potentials.

Noether's Theorem

1) Prove Noether's theorem for an infinitesimal transformation of the form

$$\phi(x) \longrightarrow \phi^{'}(x) = \phi(x) + \delta\phi(x). \tag{5.147}$$

2) Determine the conserved current of the Dirac Lagrangian density under the local gauge transformation

$$\psi \longrightarrow \psi^{'} = e^{i\alpha}\psi. \tag{5.148}$$

3) What is the significance of the corresponding conserved charge.

Polarization Vectors

1) Write down the polarization vectors in the reference frame where $n^\mu = (1, 0, 0, 0)$.

2) Verify that

$$\sum_{\lambda=1}^{2} \epsilon_\lambda^\mu(\vec{p})\epsilon_\lambda^\nu(\vec{p}) = -\eta^{\mu\nu} - \frac{p^\mu p^\nu}{(np)^2} + \frac{p^\mu n^\nu + p^\nu n^\mu}{np}. \tag{5.149}$$

Gauge Invariance and Current Conservation

1) Show that current conservation $\partial^\mu J_\mu = 0$ is a necessary and sufficient condition for gauge invariance. Consider the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + J_\mu A^\mu. \tag{5.150}$$

2) The gauge-fixed equations of motion are given by

$$\partial_\mu \partial^\mu A^\nu - (1-\zeta)\partial^\nu \partial_\mu A^\mu = \frac{1}{c} J^\nu. \tag{5.151}$$

Show that for $\zeta \neq 0$ these equations of motion are equivalent to Maxwell's equations in the Lorentz gauge.

Commutation Relations Verify

$$[\hat{a}(\vec{p},\lambda),\hat{a}(\vec{q},\lambda^{'})^{+}]=-\eta_{\lambda\lambda'}\hbar(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}). \quad (5.152)$$

Hamiltonian Operator

1) Show that the classical Hamiltonian of the electromagnetic field is given by

$$H = \int d^3x\bigg(\frac{1}{2}\partial_i A_\mu \partial^i A^\mu - \frac{1}{2}\partial_0 A_\mu \partial^0 A^\mu\bigg). \quad (5.153)$$

2) Show that in the quantum theory the Hamiltonian operator is of the form

$$\hat{H} = \int \frac{d^3\vec{p}}{(2\pi\hbar)^3}\omega(\vec{p})\bigg(\sum_{i=1}^{3}\hat{a}(\vec{p},i)^{+}\hat{a}(\vec{p},i) - \hat{a}(\vec{p},0)^{+}\hat{a}(\vec{p},0)\bigg). \quad (5.154)$$

3) Impose the Lorentz gauge condition using the Gupta-Bleuler method. What are the physical states. What happens to the expectation values of $\hat{H}$.

Physical States Let us define

$$L(\vec{p})=\hat{a}(\vec{p},3)-\hat{a}(\vec{p},0). \quad (5.155)$$

Physical states are defined by

$$L(\vec{p})|\phi>=0. \quad (5.156)$$

Define

$$|\phi_c>=f_c(L^+)|\phi>. \quad (5.157)$$

1) Show that the physical state $|\phi_c>$ is completely equivalent to the physical state $|\phi>$.

2) Show that the two states $|\phi>$ and $|\phi_c>$ are related by a gauge transformation. Determine the gauge parameter.

Photon Propagator

1) Compute the photon amplitude $iD^{\mu\nu}(x-y)=<0|\hat{A}^\mu(x)\hat{A}^\nu(y)|0>$ in terms of the scalar amplitude $D(x-y)$.

2) Derive the photon propagator in a general gauge ξ.

6
Quantum Electrodynamics

6.1 Lagrangian Density

The Dirac Lagrangian density which describes a free propagating fermion of mass m is given by the term

$$\mathcal{L}_{\rm Dirac} = \bar{\psi}(i\gamma^{\mu}\partial_{\mu} - m)\psi. \tag{6.1}$$

The Maxwell's Lagrangian density describing a free propagating photon is given by the term

$$\mathcal{L}_{\rm Maxwell} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{6.2}$$

This density gives Maxwell's equations in vacuum. It is therefore clear that the Lagrangian density describing a photon interacting with a fermion of mass m is of the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_{\mu}A^{\mu} + \mathcal{L}_{\rm Dirac}. \tag{6.3}$$

The term $-J_{\mu}A^{\mu}$ is dictated by the requirement that this Lagrangian density must give Maxwell's equations in the presence of sources. The corresponding current J_{μ} is a conserved $4-$vector which will clearly depend on the spinors ψ and $\bar{\psi}$. A solution is given by

$$J_{\mu} = e\bar{\psi}\gamma^{\mu}\psi. \tag{6.4}$$

The first term in the above Lagrangian density (6.3) is invariant under the gauge transformation

$$A^{\mu} \longrightarrow A^{'\mu} = A^{\mu} + \partial^{\mu}\lambda. \tag{6.5}$$

The second term will transform under this gauge transformation as

$$-J_{\mu}A^{\mu} \longrightarrow -J_{\mu}A^{'\mu} = -J_{\mu}A^{\mu} - J_{\mu}\partial^{\mu}\lambda. \tag{6.6}$$

The Lagrangian density (6.3) is gauge invariant only if the spinor transforms under the gauge transformation (6.5) in such a way that *a*) the current remains invariant and *b*) to cancel the term $-J_\mu\partial^\mu\lambda$.

In order to find the transformation law of the spinor we recall that the current J_μ is the Noether's current associated with the following transformation

$$\psi \longrightarrow \psi^{'} = \exp(-ie\lambda)\psi. \tag{6.7}$$

Indeed

$$\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \longrightarrow \bar{\psi}^{'}(i\gamma^\mu\partial_\mu - m)\psi^{'} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \partial_\mu\lambda J^\mu. \tag{6.8}$$

We remark that if we simultaneously transform the the photon and the Dirac fields according to (6.5) and (6.7) respectively we find that the Lagrangian density (6.3) is invariant. We also remark that the 0 component of the Noether's current J^μ is the volume density of the electric charge and hence gauge symmetry underlies the principle of conservation of electric charge.

The gauge-fixed Lagrangian density is then given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\zeta(\partial^\mu A_\mu)^2 + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma_\mu\psi A^\mu. \tag{6.9}$$

The propagator of the photon field in a general gauge ζ is given by the formula

$$iD_F^{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}\bigg(-\eta^{\mu\nu} + (1-\frac{1}{\zeta})\frac{p^\mu p^\nu}{p^2}\bigg)\exp(-ip(x-y)). \tag{6.10}$$

The propagator of the fermion field is given by

$$(S_F)_{ab}(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{i(\gamma^\mu p_\mu + m)_{ab}}{p^2-m^2+i\epsilon}\exp(-ip(x-y)). \tag{6.11}$$

6.2 Review of ϕ^4 Theory

The primary objects of interest are the probability amplitudes $< \vec{q}_1...\ \text{out}|\vec{p}_1...\ \text{in} >$ which are equal to the $S-$matrix elements $< \vec{q}_1...\ \text{out}|\vec{p}_1...\ \text{in} >$. They can be reconstructed from the Green's functions $< 0|T(\hat{\phi}(x_1)...\hat{\phi}(x_1^{'})...)|0 >$ using the formula

$$\begin{aligned} < \vec{q}_1...\ \text{out}|\vec{p}_1...\ \text{in} > &= < \vec{q}_1...\ \text{in}|S|\vec{p}_1...\ \text{in} > \\ &= \int d^4x_1 e^{iq_1x_1} i(\partial_1^2+m^2)...\int d^4x_1^{'} e^{-ip_1x_1^{'}} i(\partial_1^{'2}+m^2)... < 0|T(\hat{\phi}(x_1)...\hat{\phi}(x_1^{'})...)|0 > . \end{aligned} \tag{6.12}$$

The "in" states are defined by

$$|\vec{p}_1\vec{p}_2...\ \text{in} > = \sqrt{2E_{\vec{p}_1}}\sqrt{2E_{\vec{p}_2}}...a_{\text{in}}(\vec{p}_1)^+a_{\text{in}}(\vec{p}_2)^+...|0 > . \tag{6.13}$$

The Green's functions $< 0|T(\hat{\phi}(x_1)...\hat{\phi}(x_1^{'})...)|0 >$ are calculated using the Gell-Mann Low formula and the $S-$matrix given by

$$T(\hat{\phi}(x)\hat{\phi}(y)...) = S^{-1}T\bigg(\hat{\phi}_{\text{in}}(x)\hat{\phi}_{\text{in}}(y)...S\bigg). \tag{6.14}$$

$$S = T\left(e^{i\int d^4x\mathcal{L}_{\rm int}(\hat{\phi}_{\rm in}(x))}\right). \tag{6.15}$$

We obtain

$$\begin{aligned}
<0|T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)|0> &= <0|T\left(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)...S\right)|0> \\
&= <0|T\left(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)...e^{i\int d^4y\mathcal{L}_{\rm int}(y)}\right)|0> \\
&= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^4y_1..\int d^4y_n <0|T\left(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)..\mathcal{L}_{\rm int}(y_1)..\mathcal{L}_{\rm int}(y_n)\right)|0> .
\end{aligned} \tag{6.16}$$

Clearly we need to evaluate terms of the generic form

$$<0|T\left(\hat{\phi}_{\rm in}(x_1)\hat{\phi}_{\rm in}(x_2)...\hat{\phi}_{\rm in}(x_{2n})\right)|0> . \tag{6.17}$$

To this end we use Wick's theorem

$$<0|T\left(e^{i\int d^4xJ(x)\hat{\phi}_{\rm in}(x)}\right)|0>= e^{-\frac{1}{2}\int d^4x\int d^4x^{'}J(x)J(x^{'})D_F(x-x^{'})}. \tag{6.18}$$

This is equivalent to the statement

$$<0|T\left(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\right)|0>= \sum_{\rm contraction}\prod D_F(x_i-x_j). \tag{6.19}$$

6.3 Wick's Theorem for Forced Spinor Field

6.3.1 Generating Function

We will construct a Wick's theorem for fermions by analogy with the scalar case. First we recall Wick's theorem for scalar fields given by

$$<0|T\left(e^{i\int d^4xJ(x)\hat{\phi}_{\rm in}(x)}\right)|0>= e^{-\frac{1}{2}\int d^4x\int d^4x^{'}J(x)J(x^{'})D_F(x-x^{'})}. \tag{6.20}$$

This leads to the result

$$<0|T\left(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_{2n})\right)|0> = \sum_{\rm contraction}\prod D_F(x_i-x_j). \tag{6.21}$$

Let us now consider the evolution operator

$$\Omega(t) = T\left(e^{-i\int_{-\infty}^t ds\hat{V}_I(s)}\right). \tag{6.22}$$

We take the potential

$$\begin{aligned}
V &= -\int d^3x\mathcal{L}_{\rm int} \\
&= -\int d^3x\left(\bar{\eta}(x)\psi(x)+\bar{\psi}(x)\eta(x)\right) \\
&= -\int\frac{d^3p}{(2\pi)^3}\left(\bar{\eta}(t,\vec{p})\chi(t,\vec{p})+\bar{\chi}(t,\vec{p})\eta(t,\vec{p})\right).
\end{aligned} \tag{6.23}$$

We have used the Fourier expansions

$$\psi(x)=\int\frac{d^3p}{(2\pi)^3}\chi(t,\vec{p})e^{i\vec{p}\vec{x}}\ ,\ \eta(x)=\int\frac{d^3p}{(2\pi)^3}\eta(t,\vec{p})e^{i\vec{p}\vec{x}}. \tag{6.24}$$

We will assume that η_α and $\bar{\eta}_\alpha=(\eta^+\gamma^0)_\alpha$ are anticommuting $c-$numbers. We note that for $\eta,\bar{\eta}\longrightarrow 0$ the spinor χ becomes free given by

$$\hat{\chi}_{\rm in}(t,\vec{p})=\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_i\bigg(e^{-iE_{\vec{p}}t}u^i(\vec{p})\hat{b}_{\rm in}(\vec{p},i)+e^{iE_{\vec{p}}t}v^i(-\vec{p})\hat{d}_{\rm in}(-\vec{p},i)^+\bigg). \tag{6.25}$$

The potential $\hat{V}_I$ actually depends on Heisenberg fields which are precisely the free fields "in". We compute then

$$-i\int_{-\infty}^t ds\hat{V}_I(s)=\sum_{\vec{p}}\sum_i\bigg(\alpha_{\vec{p},i}(t)\hat{b}_{\rm in}(\vec{p},i)+\alpha^*_{\vec{p},i}(t)\hat{b}_{\rm in}(\vec{p},i)^++\gamma_{\vec{p},i}(t)\hat{d}_{\rm in}(-\vec{p},i)^++\gamma^*_{\vec{p},i}(t)\hat{d}_{\rm in}(-\vec{p},i)\bigg). \tag{6.26}$$

$$\alpha_{\vec{p},i}(t)=\frac{1}{V}\frac{i}{\sqrt{2E_{\vec{p}}}}\int_{-\infty}^t dse^{-iE_{\vec{p}}s}\bar{\eta}(s,\vec{p})u^i(\vec{p}). \tag{6.27}$$

$$\gamma_{\vec{p},i}(t)=\frac{1}{V}\frac{i}{\sqrt{2E_{\vec{p}}}}\int_{-\infty}^t dse^{iE_{\vec{p}}s}\bar{\eta}(s,\vec{p})v^i(-\vec{p}). \tag{6.28}$$

We recall the anticommutation relations

$$[\hat{b}(\vec{p},i),\hat{b}(\vec{q},j)^+]_+=[\hat{d}(\vec{p},i),\hat{d}(\vec{q},j)^+]=\delta_{ij}V\delta_{\vec{p},\vec{q}}. \tag{6.29}$$

$$[\hat{b}(\vec{p},i),\hat{d}(\vec{q},j)]_+=[\hat{b}(\vec{p},i),\hat{d}(\vec{q},j)^+]=0. \tag{6.30}$$

We immediately compute

$$\begin{aligned}\Omega(t) &= T\bigg(\prod_{\vec{p}}\prod_i e^{\alpha^*_{\vec{p},i}(t)\hat{b}_{\rm in}(\vec{p},i)^+}e^{\alpha_{\vec{p},i}(t)\hat{b}_{\rm in}(\vec{p},i)}e^{\gamma_{\vec{p},i}(t)\hat{d}_{\rm in}(-\vec{p},i)^+}e^{\gamma^*_{\vec{p},i}(t)\hat{d}_{\rm in}(-\vec{p},i)}e^{\frac{V}{2}(\alpha^*_{\vec{p},i}(t)\alpha_{\vec{p},i}(t)+\gamma_{\vec{p},i}(t)\gamma^*_{\vec{p},i}(t))}\bigg)\\ &= \prod_{\vec{p}}\prod_i\bigg(e^{\alpha^*_{\vec{p},i}(t)\hat{b}_{\rm in}(\vec{p},i)^+}e^{\alpha_{\vec{p},i}(t)\hat{b}_{\rm in}(\vec{p},i)}e^{\gamma_{\vec{p},i}(t)\hat{d}_{\rm in}(-\vec{p},i)^+}e^{\gamma^*_{\vec{p},i}(t)\hat{d}_{\rm in}(-\vec{p},i)}e^{\frac{V}{2}(\alpha^*_{\vec{p},i}(t)\alpha_{\vec{p},i}(t)+\gamma_{\vec{p},i}(t)\gamma^*_{\vec{p},i}(t))}e^{\beta_{\vec{p},i}(t)}\bigg).\end{aligned} \tag{6.31}$$

Define

$$\Omega_{\vec{p}}(t) = \prod_i\bigg(e^{\alpha^*_{\vec{p},i}(t)\hat{b}_{\rm in}(\vec{p},i)^+}e^{\alpha_{\vec{p},i}(t)\hat{b}_{\rm in}(\vec{p},i)}e^{\gamma_{\vec{p},i}(t)\hat{d}_{\rm in}(-\vec{p},i)^+}e^{\gamma^*_{\vec{p},i}(t)\hat{d}_{\rm in}(-\vec{p},i)}e^{\frac{V}{2}(\alpha^*_{\vec{p},i}(t)\alpha_{\vec{p},i}(t)+\gamma_{\vec{p},i}(t)\gamma^*_{\vec{p},i}(t))}e^{\beta_{\vec{p},i}(t)}\bigg). \tag{6.32}$$

We have

$$\begin{aligned}\partial_t\Omega_{\vec{p}}(t).\Omega^{-1}_{\vec{p}}(t) &= \sum_i\bigg(\partial_t\alpha^*_{\vec{p},i}(t).\hat{b}_{\rm in}(\vec{p},i)^++e^{\alpha^*_{\vec{p},i}(t)\hat{b}_{\rm in}(\vec{p},i)^+}\partial_t\alpha_{\vec{p},i}(t).\hat{b}_{\rm in}(\vec{p},i)e^{-\alpha^*_{\vec{p},i}(t)\hat{b}_{\rm in}(\vec{p},i)^+}\\ &+ \partial_t\gamma_{\vec{p},i}(t).\hat{d}_{\rm in}(-\vec{p},i)^++e^{\gamma_{\vec{p},i}(t)\hat{d}_{\rm in}(-\vec{p},i)^+}\partial_t\gamma^*_{\vec{p},i}(t).\hat{d}_{\rm in}(-\vec{p},i)e^{-\gamma_{\vec{p},i}(t)\hat{d}_{\rm in}(-\vec{p},i)^+}\\ &+ \frac{V}{2}\big(\partial_t\alpha^*_{\vec{p},i}(t).\alpha_{\vec{p},i}(t)+\alpha^*_{\vec{p},i}(t)\partial_t\alpha_{\vec{p},i}(t)+\partial_t\gamma_{\vec{p},i}(t).\gamma^*_{\vec{p},i}(t)+\gamma_{\vec{p},i}(t)\partial_t\gamma^*_{\vec{p},i}(t)\big)\\ &+ \partial_t\beta_{\vec{p},i}(t)\bigg).\end{aligned} \tag{6.33}$$

We use the identities

$$e^{\alpha^*_{\vec{p},i}(t)\hat{b}_{\rm in}(\vec{p},i)^+}\partial_t\alpha_{\vec{p},i}(t).\hat{b}_{\rm in}(\vec{p},i)=\bigg(\partial_t\alpha_{\vec{p},i}(t).\hat{b}_{\rm in}(\vec{p},i)-V\alpha^*_{\vec{p},i}(t)\partial_t\alpha_{\vec{p},i}(t)\bigg)e^{\alpha^*_{\vec{p},i}(t)\hat{b}_{\rm in}(\vec{p},i)^+}. \quad (6.34)$$

$$e^{\gamma_{\vec{p},i}(t)\hat{d}_{\rm in}(-\vec{p},i)^+}\partial_t\gamma^*_{\vec{p},i}(t).\hat{d}_{\rm in}(-\vec{p},i)=\bigg(\partial_t\gamma^*_{\vec{p},i}(t).\hat{d}_{\rm in}(-\vec{p},i)-V\gamma_{\vec{p},i}(t)\partial_t\gamma^*_{\vec{p},i}(t)\bigg)e^{\gamma_{\vec{p},i}(t)\hat{d}_{\rm in}(-\vec{p},i)^+}. \quad (6.35)$$

We get then

$$\begin{aligned}\partial_t\Omega_{\vec{p}}(t).\Omega^{-1}_{\vec{p}}(t) &= \sum_i\bigg(\partial_t\alpha^*_{\vec{p},i}(t).\hat{b}_{\rm in}(\vec{p},i)^++\partial_t\alpha_{\vec{p},i}(t).\hat{b}_{\rm in}(\vec{p},i)+\partial_t\gamma_{\vec{p},i}(t).\hat{d}_{\rm in}(-\vec{p},i)^++\partial_t\gamma^*_{\vec{p},i}(t).\hat{d}_{\rm in}(-\vec{p},i)\\ &+ \frac{V}{2}\big(\partial_t\alpha^*_{\vec{p},i}(t).\alpha_{\vec{p},i}(t)-\alpha^*_{\vec{p},i}(t)\partial_t\alpha_{\vec{p},i}(t)+\partial_t\gamma_{\vec{p},i}(t).\gamma^*_{\vec{p},i}(t)-\gamma_{\vec{p},i}(t)\partial_t\gamma^*_{\vec{p},i}(t)\big)\\ &+ \partial_t\beta_{\vec{p},i}(t)\bigg). \end{aligned} \quad (6.36)$$

Let us recall that

$$i\partial_t\Omega(t)=\hat{V}_I(t)\Omega(t). \quad (6.37)$$

This leads to

$$\begin{aligned}i\partial_t\Omega_{\vec{p}}(t).\Omega^{-1}_{\vec{p}}(t) &= \hat{V}_I(t,\vec{p})\\ &= -\frac{1}{V}\bigg(\bar{\eta}(t,\vec{p})\hat{\chi}_{\rm in}(t,\vec{p})+\bar{\hat{\chi}}_{\rm in}(t,\vec{p})\eta(t,\vec{p})\bigg)\\ &= -\frac{1}{V}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_i\bigg(e^{-iE_{\vec{p}}t}\bar{\eta}(t,\vec{p})u^i(\vec{p})\hat{b}_{\rm in}(\vec{p},i)-e^{iE_{\vec{p}}t}\bar{u}^i(\vec{p})\eta(t,\vec{p})\hat{b}^+_{\rm in}(\vec{p},i)\\ &+ e^{iE_{\vec{p}}t}\bar{\eta}(t,\vec{p})v^i(-\vec{p})\hat{d}_{\rm in}(-\vec{p},i)^+-e^{-iE_{\vec{p}}t}\bar{v}^i(-\vec{p})\eta(t,\vec{p})\hat{d}_{\rm in}(-\vec{p},i)\bigg). \end{aligned} \quad (6.38)$$

By comparison we must have

$$\partial_t\alpha_{\vec{p},i}(t)=\frac{i}{V}\frac{1}{\sqrt{2E_{\vec{p}}}}e^{-iE_{\vec{p}}t}\bar{\eta}(t,\vec{p})u^i(\vec{p}). \quad (6.39)$$

$$\partial_t\gamma_{\vec{p},i}(t)=\frac{i}{V}\frac{1}{\sqrt{2E_{\vec{p}}}}e^{iE_{\vec{p}}t}\bar{\eta}(t,\vec{p})v^i(-\vec{p}). \quad (6.40)$$

These equations are already satisfied by (6.27) and (6.28). By comparing (6.36) and (6.38) we also obtain

$$\partial_t\beta_{\vec{p},i}(t)=-\frac{V}{2}\big(\partial_t\alpha^*_{\vec{p},i}(t).\alpha_{\vec{p},i}(t)-\alpha^*_{\vec{p},i}(t)\partial_t\alpha_{\vec{p},i}(t)+\partial_t\gamma_{\vec{p},i}(t).\gamma^*_{\vec{p},i}(t)-\gamma_{\vec{p},i}(t)\partial_t\gamma^*_{\vec{p},i}(t)\big). \quad (6.41)$$

In other words

$$\beta_{\vec{p},i}(t)=-\frac{V}{2}\int_{-\infty}^t ds\bigg(\partial_s\alpha^*_{\vec{p},i}(s).\alpha_{\vec{p},i}(s)-\alpha^*_{\vec{p},i}(s)\partial_s\alpha_{\vec{p},i}(s)+\partial_s\gamma_{\vec{p},i}(s).\gamma^*_{\vec{p},i}(s)-\gamma_{\vec{p},i}(s)\partial_s\gamma^*_{\vec{p},i}(s)\bigg). \quad (6.42)$$

We compute in the limit $t \longrightarrow \infty$ the following

$$\frac{V}{2}\sum_i\bigg(\alpha^*_{\vec{p},i}(t)\alpha_{\vec{p},i}(t)+\gamma_{\vec{p},i}(t)\gamma^*_{\vec{p},i}(t)\bigg) = \frac{1}{2V}\int_{-\infty}^{+\infty}ds\int_{-\infty}^{+\infty}ds^{'}\bigg[-\frac{1}{2E_{\vec{p}}}e^{iE_{\vec{p}}(s-s^{'})}\bar{\eta}(s^{'},\vec{p}) \times (\gamma^0E_{\vec{p}}-\gamma^ip^i+m)\eta(s,\vec{p})+\frac{1}{2E_{\vec{p}}}e^{iE_{\vec{p}}(s-s^{'})}\bar{\eta}(s,\vec{p}) \times (\gamma^0E_{\vec{p}}+\gamma^ip^i-m)\eta(s^{'},\vec{p})\bigg]. \quad (6.43)$$

Also

$$-\frac{V}{2}\int_{-\infty}^{t}ds\sum_i\bigg(\partial_s\alpha^*_{\vec{p},i}(s).\alpha_{\vec{p},i}(s)-\alpha^*_{\vec{p},i}(s)\partial_s\alpha_{\vec{p},i}(s)\bigg) = \frac{1}{2V}\int_{-\infty}^{+\infty}ds\int_{-\infty}^{s}ds^{'}\bigg[\frac{1}{2E_{\vec{p}}}e^{iE_{\vec{p}}(s-s^{'})}\bar{\eta}(s^{'},\vec{p}) \times (\gamma^0E_{\vec{p}}-\gamma^ip^i+m)\eta(s,\vec{p})-\frac{1}{2E_{\vec{p}}}e^{-iE_{\vec{p}}(s-s^{'})}\bar{\eta}(s,\vec{p}) \times (\gamma^0E_{\vec{p}}-\gamma^ip^i+m)\eta(s^{'},\vec{p})\bigg]. \quad (6.44)$$

$$-\frac{V}{2}\int_{-\infty}^{t}ds\sum_i\bigg(\partial_s\gamma_{\vec{p},i}(s).\gamma^*_{\vec{p},i}(s)-\gamma_{\vec{p},i}(s)\partial_s\gamma^*_{\vec{p},i}(s)\bigg) = \frac{1}{2V}\int_{-\infty}^{+\infty}ds\int_{-\infty}^{s}ds^{'}\bigg[-\frac{1}{2E_{\vec{p}}}e^{iE_{\vec{p}}(s-s^{'})}\bar{\eta}(s,\vec{p}) \times (\gamma^0E_{\vec{p}}+\gamma^ip^i-m)\eta(s^{'},\vec{p})+\frac{1}{2E_{\vec{p}}}e^{-iE_{\vec{p}}(s-s^{'})}\bar{\eta}(s^{'},\vec{p}) \times (\gamma^0E_{\vec{p}}+\gamma^ip^i-m)\eta(s,\vec{p})\bigg]. \quad (6.45)$$

Thus

$$\frac{V}{2}\sum_{\vec{p}}\sum_i\bigg(\alpha^*_{\vec{p},i}(t)\alpha_{\vec{p},i}(t)+\gamma_{\vec{p},i}(t)\gamma^*_{\vec{p},i}(t)\bigg) = -\frac{1}{2}\int d^4x\int d^4x^{'}\,\bar{\eta}(x^{'})\frac{1}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}(\gamma.p+m)e^{ip(x-x^{'})}\eta(x) + \frac{1}{2}\int d^4x\int d^4x^{'}\,\bar{\eta}(x^{'})\frac{1}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}(\gamma.p-m)e^{-ip(x-x^{'})}\eta(x). \quad (6.46)$$

$$\sum_{\vec{p}}\sum_i\beta_{\vec{p},i}(t) = \frac{1}{2}\int d^4x\int d^4x^{'}\,\bar{\eta}(x^{'})\frac{\theta(s-s^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}\bigg((\gamma.p+m)e^{ip(x-x^{'})}+(\gamma.p-m)e^{-ip(x-x^{'})}\bigg)\eta(x) - \frac{1}{2}\int d^4x\int d^4x^{'}\,\bar{\eta}(x^{'})\frac{\theta(s^{'}-s)}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}\bigg((\gamma.p+m)e^{ip(x-x^{'})}+(\gamma.p-m)e^{-ip(x-x^{'})}\bigg)\eta(x). \quad (6.47)$$

Hence by using $\theta(s-s^{'})-\theta(s^{'}-s)-1=-2\theta(s^{'}-s)$ and $\theta(s-s^{'})-\theta(s^{'}-s)+1=2\theta(s-s^{'})$ we get

$$\begin{aligned}
\frac{V}{2}\sum_{\vec{p}}\sum_{i}\bigg(\alpha^*_{\vec{p},i}(t)\alpha_{\vec{p},i}(t)+\gamma_{\vec{p},i}(t)\gamma^*_{\vec{p},i}(t)\bigg)+\sum_{\vec{p}}\sum_{i}\beta_{\vec{p},i}(t) &= \\
\int d^4x\int d^4x^{'}\bar{\eta}(x^{'})\bigg[\frac{\theta(s-s^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}(\gamma.p-m)e^{-ip(x-x^{'})}-\frac{\theta(s^{'}-s)}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}(\gamma.p+m)e^{ip(x-x^{'})}\bigg]\eta(x) &= \\
\int d^4x\int d^4x^{'}\bar{\eta}(x^{'})\bigg[\frac{\theta(s-s^{'})}{V}(i\gamma^{\mu}\partial_{\mu}^x-m)\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{-ip(x-x^{'})}+\frac{\theta(s^{'}-s)}{V}(i\gamma^{\mu}\partial_{\mu}^x-m)\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{ip(x-x^{'})}\bigg]\eta(x) &= \\
\int d^4x\int d^4x^{'}\bar{\eta}(x^{'})(i\gamma^{\mu}\partial_{\mu}^x-m)\bigg[\frac{\theta(s-s^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{-ip(x-x^{'})}+\frac{\theta(s^{'}-s)}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{ip(x-x^{'})}\bigg]\eta(x). & \quad (6.48)
\end{aligned}$$

The Feynamn scalar and spinor propagators are given respectively by

$$D_F(x-x^{'})=\frac{\theta(s-s^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{-ip(x-x^{'})}+\frac{\theta(s^{'}-s)}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{ip(x-x^{'})}. \tag{6.49}$$

$$S_F(x-x^{'})=(i\gamma^{\mu}\partial_{\mu}^x+m)D_F(x-x^{'}). \tag{6.50}$$

We have

$$\begin{aligned}
S_F(x^{'}-x) &= (i\gamma^{\mu}\partial_{\mu}^{x^{'}}+m)D_F(x^{'}-x) \\
&= -(i\gamma^{\mu}\partial_{\mu}^x-m)D_F(x-x^{'}).
\end{aligned} \tag{6.51}$$

We obtain therefore

$$\frac{V}{2}\sum_{\vec{p}}\sum_{i}\bigg(\alpha^*_{\vec{p},i}(t)\alpha_{\vec{p},i}(t)+\gamma_{\vec{p},i}(t)\gamma^*_{\vec{p},i}(t)\bigg)+\sum_{\vec{p}}\sum_{i}\beta_{\vec{p},i}(t)=-\int d^4x\int d^4x^{'}\bar{\eta}(x^{'})S_F(x^{'}-x)\eta(x). \tag{6.52}$$

The final result is

$$T\bigg(e^{i\int d^4x(\bar{\eta}(x)\hat{\psi}_{\rm in}(x)+\bar{\hat{\psi}}_{\rm in}(x)\eta(x))}\bigg)=:e^{i\int d^4x(\bar{\eta}(x)\hat{\psi}_{\rm in}(x)+\bar{\hat{\psi}}_{\rm in}(x)\eta(x))}:e^{-\int d^4x\int d^4x^{'}\bar{\eta}(x^{'})S_F(x^{'}-x)\eta(x)}. \tag{6.53}$$

The normal ordering is as usual defined by putting the creation operators to the left of the annihilation operators. Explicitly we have in this case

$$:e^{i\int d^4x(\bar{\eta}(x)\hat{\psi}_{\rm in}(x)+\bar{\hat{\psi}}_{\rm in}(x)\eta(x))}: \;=\; \prod_{\vec{p}}\prod_{i}\bigg(e^{\alpha^*_{\vec{p},i}(t)\hat{b}_{\rm in}(\vec{p},i)^+}e^{\alpha_{\vec{p},i}(t)\hat{b}_{\rm in}(\vec{p},i)}e^{\gamma_{\vec{p},i}(t)\hat{d}_{\rm in}(-\vec{p},i)^+}e^{\gamma^*_{\vec{p},i}(t)\hat{d}_{\rm in}(-\vec{p},i)}\bigg). \tag{6.54}$$

Therefore we will have in the vacuum the identity

$$<0|T\bigg(e^{i\int d^4x(\bar{\eta}(x)\hat{\psi}_{\rm in}(x)+\bar{\hat{\psi}}_{\rm in}(x)\eta(x))}\bigg)|0>=e^{-\int d^4x\int d^4x^{'}\bar{\eta}(x^{'})S_F(x^{'}-x)\eta(x)}. \tag{6.55}$$

6.3.2 Wick's Theorem

Now we expand both sides of the above equations in η and $\bar{\eta}$. The left hand side becomes

$$e^{-\int d^4x\int d^4x^{'}\bar{\eta}(x)S_F(x-x^{'})\eta(x^{'})} = \sum_n \frac{(-1)^n}{n!}\int d^4x_1\int d^4x_1^{'}...\int d^4x_n\int d^4x_n^{'}\bar{\eta}(x_1)S_F(x_1-x_1^{'})\eta(x_1^{'})\times ...\bar{\eta}(x_n)S_F(x_n-x_n^{'})\eta(x_n^{'}). \quad (6.56)$$

It is obvious that only terms with equal numbers of η and $\bar{\eta}$ are present. Thus we conclude that only expectation values with equal numbers of $\hat{\psi}$ and $\bar{\hat{\psi}}$ are non-zero. The first few terms of the expansion in η and $\bar{\eta}$ of the right hand side of the above identity are

$$\begin{aligned} <0|T\bigg(e^{i\int d^4x(\bar{\eta}(x)\hat{\psi}_{\rm in}(x)+\bar{\hat{\psi}}_{\rm in}(x)\eta(x))}\bigg)|0> &= 1+\frac{i^2}{2!}\int d^4x_1\int d^4x_1^{'}<0|T(L(x_1)L(x_1^{'}))|0> \\ &+ \frac{i^4}{4!}\int d^4x_1\int d^4x_1^{'}\int d^4x_2\int d^4x_2^{'}<0|T(L(x_1)L(x_1^{'})L(x_2)L(x_2^{'})) \\ &\times |0>+... \end{aligned} \quad (6.57)$$

In above $L(x)=\bar{\eta}(x)\hat{\psi}_{\rm in}(x)+\bar{\hat{\psi}}_{\rm in}(x)\eta(x)$. The terms of order 1 and 3 (and in fact all terms of order $2n+1$ where n is an integer) must vanish by comparison with the left hand side. We conclude as anticipated above that all expectation values with a number of $\hat{\psi}$ not equal to the number of $\bar{\hat{\psi}}$ vanish identically. There are two contributions in the second term which are equal by virtue of the T product. Similarly there are 6 contributions in the third term which are again equal by virtue of the T product. Hence we get

$$\begin{aligned} <0|T\bigg(e^{i\int d^4x(\bar{\eta}(x)\hat{\psi}_{\rm in}(x)+\bar{\hat{\psi}}_{\rm in}(x)\eta(x))}\bigg)|0> &= 1+\frac{i^2}{2!}(2)\int d^4x_1\int d^4x_1^{'}<0|T(\bar{\eta}(x_1)\hat{\psi}_{\rm in}(x_1).\bar{\hat{\psi}}_{\rm in}(x_1^{'})\eta(x_1^{'}))|0> \\ &+ \frac{i^4}{4!}(6)\int d^4x_1\int d^4x_1^{'}\int d^4x_2\int d^4x_2^{'}<0|T(\bar{\eta}(x_1)\hat{\psi}_{\rm in}(x_1) \\ &\times \bar{\eta}(x_2)\hat{\psi}_{\rm in}(x_2).\bar{\hat{\psi}}_{\rm in}(x_1^{'})\eta(x_1^{'}).\bar{\hat{\psi}}_{\rm in}(x_2^{'})\eta(x_2^{'}))|0>+... \end{aligned} \quad (6.58)$$

In general we should obtain

$$\begin{aligned} <0|T\bigg(e^{i\int d^4x(\bar{\eta}(x)\hat{\psi}_{\rm in}(x)+\bar{\hat{\psi}}_{\rm in}(x)\eta(x))}\bigg)|0> &= \sum_n(\frac{i^n}{n!})^2\int d^4x_1\int d^4x_1^{'}...\int d^4x_n\int d^4x_n^{'}<0|T(\bar{\eta}(x_1)\hat{\psi}_{\rm in}(x_1)... \\ &\times \bar{\eta}(x_n)\hat{\psi}_{\rm in}(x_n).\bar{\hat{\psi}}_{\rm in}(x_1^{'})\eta(x_1^{'})...\bar{\hat{\psi}}_{\rm in}(x_n^{'})\eta(x_n^{'}))|0> \\ &= \sum_n(\frac{i^n}{n!})^2\int d^4x_1\int d^4x_1^{'}...\int d^4x_n\int d^4x_n^{'}\bar{\eta}(x_n)...\bar{\eta}(x_1) \\ &\times <0|T(\hat{\psi}_{\rm in}(x_1)...\hat{\psi}_{\rm in}(x_n).\bar{\hat{\psi}}_{\rm in}(x_1^{'})...\bar{\hat{\psi}}_{\rm in}(x_n^{'}))|0>\eta(x_n^{'})...\eta(x_1^{'}). \end{aligned} \quad (6.59)$$

We rewrite now the left hand side as

$$e^{-\int d^4x\int d^4x^{'}\bar{\eta}(x)S_F(x-x^{'})\eta(x^{'})} = \sum_n \frac{(-1)^n}{n!}\int d^4x_1\int d^4x_1^{'}...\int d^4x_n\int d^4x_n^{'}\bar{\eta}_{\alpha_n}(x_n)...\bar{\eta}_{\alpha_1}(x_1)S_F(x_1-x_1^{'})^{\alpha_1\beta_1}$$
$$\times \ ...S_F(x_n-x_n^{'})^{\alpha_n\beta_n}\eta_{\beta_1}(x_1^{'})...\eta_{\beta_n}(x_n^{'})$$
$$= \sum_n \frac{(-1)^n}{n!}\int d^4x_1\int d^4x_1^{'}...\int d^4x_n\int d^4x_n^{'}\bar{\eta}_{\alpha_n}(x_n)...\bar{\eta}_{\alpha_1}(x_1)S_F(x_1-x_n^{'})^{\alpha_1\beta_n}$$
$$\times \ ...S_F(x_n-x_1^{'})^{\alpha_n\beta_1}\eta_{\beta_n}(x_n^{'})...\eta_{\beta_1}(x_1^{'}). \tag{6.60}$$

There are $n!$ permutations of the indices 1, 2,...,n. Let p_1,p_2,...,p_n be a given permutation of 1, 2,...,n with a parity δ_p. We recall that $\delta_p=+1$ for permutations and $\delta_p=-1$ for odd permutations. Then because of the anticommutativity of $\eta_{\beta_1}(x_1^{'})$,...,$\eta_{\beta_n}(x_n^{'})$ we can write the above equation as

$$e^{-\int d^4x\int d^4x^{'}\bar{\eta}(x)S_F(x-x^{'})\eta(x^{'})} = \sum_n \frac{(-1)^n}{n!}\int d^4x_1\int d^4x_1^{'}...\int d^4x_n\int d^4x_n^{'}\bar{\eta}_{\alpha_n}(x_n)...\bar{\eta}_{\alpha_1}(x_1)$$
$$\times \left[\frac{1}{n!}\sum_{\text{permutations}}\delta_p\ S_F(x_1-x_{p_n}^{'})^{\alpha_1\beta_{p_n}}...S_F(x_n-x_{p_1}^{'})^{\alpha_n\beta_{p_1}}\right]\eta_{\beta_n}(x_n^{'})...\eta_{\beta_1}(x_1^{'}). \tag{6.61}$$

This is clearly true because for a given permutation we can write $\eta_{\beta_n}(x_n^{'})...\eta_{\beta_1}(x_1^{'})=\delta_p\eta_{\beta_{p_n}}(x_{p_n}^{'})...\eta_{\beta_{p_1}}(x_{p_1}^{'})$. By comparing (6.59) and (6.61) we get the final result

$$<0|T(\hat{\psi}_{\text{in}}^{\alpha_1}(x_1)...\hat{\psi}_{\text{in}}^{\alpha_n}(x_n).\bar{\hat{\psi}}_{\text{in}}^{\beta_1}(x_1^{'})...\bar{\hat{\psi}}_{\text{in}}^{\beta_n}(x_n^{'}))|0>=\sum_{\text{permutations}}\delta_p\ S_F(x_1-x_{p_n}^{'})^{\alpha_1\beta_{p_n}}...S_F(x_n-x_{p_1}^{'})^{\alpha_n\beta_{p_1}}. \tag{6.62}$$

6.4 Wick's Theorem for Forced Electromagnetic Field

The Lagrangian density for a forced electromagnetic field (in the Lorentz gauge $\zeta=1$) is given by

$$\mathcal{L}_{\text{free}} = \frac{1}{2}A_\mu(\partial.\partial)A^\mu - J_\mu A^\mu. \tag{6.63}$$

We assume that the source $J_\mu(x)$ vanishes outside a finite time interval. Thus at early and late times $J_\mu(x)\longrightarrow 0$ and A^μ becomes a free field. We have then

$$\hat{A}^\mu \longrightarrow \hat{A}^\mu_{\text{in}} = \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{\lambda=0}^{3}\left(e^{-ipx}\epsilon^\mu_\lambda(\vec{p})\hat{a}_{\text{in}}(\vec{p},\lambda)+e^{ipx}\epsilon^\mu_\lambda(\vec{p})\hat{a}_{\text{in}}(\vec{p},\lambda)^+\right),\ t\longrightarrow -\infty \tag{6.64}$$

$$\hat{A}^\mu \longrightarrow \hat{A}^\mu_{\text{out}} = \int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\sum_{\lambda=0}^{3}\left(e^{-ipx}\epsilon^\mu_\lambda(\vec{p})\hat{a}_{\text{out}}(\vec{p},\lambda)+e^{ipx}\epsilon^\mu_\lambda(\vec{p})\hat{a}_{\text{out}}(\vec{p},\lambda)^+\right),\ t\longrightarrow +\infty. \tag{6.65}$$

This is a system equivalent to 4 independent massless Klein-Gordon fields. The corresponding Wick's theorem is therefore a straightforward generalization of (6.18). We have then

$$< 0|T\bigg(e^{i\int d^4x J_\mu(x)\hat{A}^\mu_{\rm in}(x)}\bigg)|0 >= e^{-\frac{1}{2}\int d^4x\int d^4x^{'} J_\mu(x)J_\nu(x^{'}).iD_F^{\mu\nu}(x-x^{'})}. \tag{6.66}$$

As usual by expanding both sides of this equation in powers of the current J_μ we get Wick's theorem in the equivalent form

$$< 0|T\bigg(\hat{A}^{\mu_1}_{\rm in}(x_1)...\hat{A}^{\mu_{2n}}_{\rm in}(x_{2n})\bigg)|0 >= \sum_{\rm contraction}\prod iD_F^{\mu_i\mu_j}(x_i-x_j). \tag{6.67}$$

6.5 The LSZ Reduction fromulas and The $S-$Matrix

6.5.1 The LSZ Reduction fromulas

We divide the QED Lagrangian into a free part and an interaction part. The free part (in the Lorentz gauge $\zeta = 1$) is given by

$$\mathcal{L}_{\rm free} = \frac{1}{2}A_\mu(\partial.\partial)A^\mu + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi. \tag{6.68}$$

The interaction part is given by

$$\mathcal{L}_{\rm int} = -e\bar{\psi}\gamma_\mu\psi A^\mu. \tag{6.69}$$

We are going to assume that the interaction part vanishes in the limits $t \longrightarrow \pm\infty$. Therefore the spinor field in the limits $t \longrightarrow \pm\infty$ will obey the free equation of motion

$$(i\gamma^\mu\partial_\mu - m)\psi = 0. \tag{6.70}$$

As usual we expand the field as

$$\psi = \int \frac{d^3p}{(2\pi)^3}\chi(t,\vec{p})\ e^{i\vec{p}\vec{x}}. \tag{6.71}$$

Thus the field $\chi(t,\vec{p})$ will obey the equation of motion

$$(i\gamma^0\partial_t - \gamma^i p^i - m)\chi = 0. \tag{6.72}$$

In the limit $t \longrightarrow \pm\infty$ we have then

$$\hat{\chi}_{\rm in}(t,\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}\sum_s\bigg(e^{-iE_{\vec{p}}t}u^{(s)}(\vec{p})\hat{b}_{\rm in}(\vec{p},s) + e^{iE_{\vec{p}}t}v^{(s)}(-\vec{p})\hat{d}_{\rm in}(-\vec{p},s)^+\bigg)\ ,\ t \longrightarrow -\infty. \tag{6.73}$$

$$\hat{\chi}_{\rm out}(t,\vec{p}) = \frac{1}{\sqrt{2E_{\vec{p}}}}\sum_s\bigg(e^{-iE_{\vec{p}}t}u^{(s)}(\vec{p})\hat{b}_{\rm out}(\vec{p},s) + e^{iE_{\vec{p}}t}v^{(s)}(-\vec{p})\hat{d}_{\rm out}(-\vec{p},s)^+\bigg)\ ,\ t \longrightarrow +\infty. \tag{6.74}$$

The operator $\hat{b}(\vec{p},s)^+$ creates a fermion of momentum $\vec{p}$ and polarization s whereas $\hat{d}(\vec{p},s)^+$ creates an antifermion of momentum $\vec{p}$ and polarization s. From the above expressions we obtain

$$e^{iE_{\vec{p}}t}(i\partial_t + E_{\vec{p}})\bar{u}^s(p)\hat{\chi}_{\rm in,out}(t,\vec{p}) = 2m\sqrt{2E_{\vec{p}}}\ \hat{b}_{\rm in,out}(\vec{p},s). \tag{6.75}$$

$$e^{iE_{\vec{p}}t}\bar{\chi}_{\rm in,out}(t,-\vec{p})(i\overleftarrow{\partial}_t+E_{\vec{p}})v^s(p)=-2m\sqrt{2E_{\vec{p}}}\ \hat{d}_{\rm in,out}(\vec{p},s). \tag{6.76}$$

The full equations of motion obeyed by ψ and χ are

$$(i\gamma^\mu\partial_\mu-m)\psi=-\frac{\mathcal{L}_{\rm int}}{\delta\bar{\psi}}. \tag{6.77}$$

$$(i\gamma^0\partial_t-\gamma^ip^i-m)\chi(t,\vec{p})=-\int d^3x\frac{\delta\mathcal{L}_{\rm int}}{\delta\bar{\psi}}\ e^{-i\vec{p}\vec{x}}. \tag{6.78}$$

We compute

$$\int_{-\infty}^{+\infty}dt\ \partial_t\bigg(e^{iE_{\vec{p}}t}(i\partial_t+E_{\vec{p}})\bar{u}^s(p)\hat{\chi}(t,\vec{p})\bigg)=2m\sqrt{2E_{\vec{p}}}\ (\hat{b}_{\rm out}(\vec{p},s)-\hat{b}_{\rm in}(\vec{p},s)). \tag{6.79}$$

$$\int_{-\infty}^{+\infty}dt\ \partial_t\bigg(e^{iE_{\vec{p}}t}\bar{\chi}(t,-\vec{p})(i\overleftarrow{\partial}_t+E_{\vec{p}})v^s(p)\bigg)\ =\ -2m\sqrt{2E_{\vec{p}}}\ (\hat{d}_{\rm out}(\vec{p},s)-\hat{d}_{\rm in}(\vec{p},s)). \tag{6.80}$$

From the other hand we compute

$$\begin{aligned}\int_{-\infty}^{+\infty}dt\ \partial_t\bigg(e^{iE_{\vec{p}}t}(i\partial_t+E_{\vec{p}})\bar{u}^s(p)\hat{\chi}(t,\vec{p})\bigg) &= i\int_{-\infty}^{+\infty}dt\ e^{iE_{\vec{p}}t}(\partial_t^2+E_{\vec{p}}^2)\bar{u}^s(p)\hat{\chi}(t,\vec{p})\\ &= i\int d^4x\ e^{ipx}(\partial^2+m^2)\bar{u}^s(p)\hat{\psi}(x)\\ &= -i\bar{u}^s(p)(\gamma^\mu p_\mu+m)(\gamma^\mu p_\mu-m)\hat{\psi}(-p)\\ &= -2im\bar{u}^s(p)(\gamma^\mu p_\mu-m)\hat{\psi}(-p)\\ &= -2im\int d^4x\ e^{ipx}\bar{u}^s(p)(i\gamma^\mu\partial_\mu-m)\hat{\psi}(x).\end{aligned} \tag{6.81}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}dt\ \partial_t\bigg(e^{iE_{\vec{p}}t}\bar{\chi}(t,-\vec{p})(i\overleftarrow{\partial}_t+E_{\vec{p}})v^s(p)\bigg) &= i\int_{-\infty}^{+\infty}dt\ e^{iE_{\vec{p}}t}\bar{\chi}(t,-\vec{p})(\overleftarrow{\partial}_t^2+E_{\vec{p}}^2)v^s(p)\\ &= i\int d^4x\ e^{ipx}\bar{\hat{\psi}}(x)(\overleftarrow{\partial}^2+m^2)v^s(p)\\ &= -i\bar{\hat{\psi}}(-p)(\gamma^\mu p_\mu+m)(\gamma^\mu p_\mu-m)v^s(p)\\ &= 2im\bar{\hat{\psi}}(-p)(\gamma^\mu p_\mu+m)v^s(p)\\ &= 2im\int d^4x\ e^{ipx}\bar{\hat{\psi}}(x)(i\gamma^\mu\overleftarrow{\partial}_\mu+m)v^s(p).\end{aligned} \tag{6.82}$$

By comparison we obtain

$$\sqrt{2E_{\vec{p}}}\ (\hat{b}_{\rm out}(\vec{p},s)-\hat{b}_{\rm in}(\vec{p},s))\ =\ \frac{1}{i}\int d^4x\ e^{ipx}\bar{u}^s(p)(i\gamma^\mu\partial_\mu-m)\hat{\psi}(x). \tag{6.83}$$

$$\sqrt{2E_{\vec{p}}}\ (\hat{d}_{\rm in}(\vec{p},s)-\hat{d}_{\rm out}(\vec{p},s))\ =\ \frac{1}{i}\int d^4x\ e^{ipx}.\bar{\hat{\psi}}(x)(-i\gamma^\mu\overleftarrow{\partial}_\mu-m)v^s(p). \tag{6.84}$$

These are the first two examples of Lehmann-Symanzik-Zimmermann reduction formulae. The fields $\hat{\psi}(x)$ in the above equation is the interacting spinor field in the Heisenberg picture. Generalization of the above equations read

$$\sqrt{2E_{\vec{p}}}\,\bigg(\hat{b}_{\rm out}(\vec{p},s)T(...)\mp T(...)\hat{b}_{\rm in}(\vec{p},s)\bigg) = \frac{1}{i}\int d^4x\; e^{ipx}\bar{u}^s(p)(i\gamma^{\mu}\partial_{\mu}-m)T(\hat{\psi}(x)...).\tag{6.85}$$

$$\sqrt{2E_{\vec{p}}}\,\bigg(T(...)\hat{d}_{\rm in}(\vec{p},s)\mp \hat{d}_{\rm out}(\vec{p},s)T(...)\bigg) = \frac{1}{i}\int d^4x\; e^{ipx}.T(...\bar{\hat{\psi}}(x))(-i\gamma^{\mu}\overleftarrow{\partial}_{\mu}-m)v^s(p).\tag{6.86}$$

The minus sign corresponds to the case where $T(...)$ includes an even number of spinor field operators whereas the plus sign corresponds to the case where $T(...)$ includes an odd number of spinor field operators. By taking essentially the hermitian conjugate of the above equations we get the LSZ reduction formulas

$$\sqrt{2E_{\vec{p}}}\,\bigg(T(...)\hat{b}^{+}_{\rm in}(\vec{p},s)\mp \hat{b}^{+}_{\rm out}(\vec{p},s)T(...)\bigg) = \frac{1}{i}\int d^4x\; e^{-ipx}.T(...\bar{\hat{\psi}}(x))(-i\gamma^{\mu}\overleftarrow{\partial}_{\mu}-m)u^s(p).\tag{6.87}$$

$$\sqrt{2E_{\vec{p}}}\,\bigg(\hat{d}^{+}_{\rm out}(\vec{p},s)T(...)\mp T(...)\hat{d}^{+}_{\rm in}(\vec{p},s)\bigg) = \frac{1}{i}\int d^4x\; e^{-ipx}\bar{v}^s(p)(i\gamma^{\mu}\partial_{\mu}-m)T(\hat{\psi}(x)...).\tag{6.88}$$

We recall in passing the anticommutation relations (using box normalization)

$$[\hat{b}(\vec{p},i),\hat{b}(\vec{q},j)^{+}]_{\pm}=\delta_{ij}V\delta_{\vec{p},\vec{q}},\tag{6.89}$$

$$[\hat{d}(\vec{p},i)^{+},\hat{d}(\vec{q},j)]_{\pm}=\delta_{ij}V\delta_{\vec{p},\vec{q}},\tag{6.90}$$

and

$$[\hat{b}(\vec{p},i),\hat{d}(\vec{q},j)]_{\pm}=[\hat{d}(\vec{q},j)^{+},\hat{b}(\vec{p},i)]_{\pm}=0.\tag{6.91}$$

Example I: $e^- + e^+ \longrightarrow \mu^- + \mu^+$

As an example we will consider the process of annihilation of an electron-positron pair into a muon-antimuon pair given by

$$e^-(p_1)+e^+(q_1)\longrightarrow \mu^-(p_2)+\mu^+(q_2).\tag{6.92}$$

This is a process of fundamental importance in QED and collider physics. A related process of similar fundamental relevance is the annihilation of an electron-positron pair into a quark-antiquark pair given by

$$e^- + e^+ \longrightarrow Q+\bar{Q}.\tag{6.93}$$

The normalization for one-particle excited states is fixed by [1]

$$|\vec{p},s>=\sqrt{2E_{\vec{p}}}\;\hat{b}(\vec{p},s)^{+}|0>\;\;,\;|\vec{q},s>=\sqrt{2E_{\vec{q}}}\;\hat{d}(\vec{q},s)^{+}|0>.\tag{6.94}$$

[1] Here we have changed the notation compared to the previous course.

The initial and final states are given by

$$\begin{aligned} \text{initial state} &= |\vec{p}_1, s_1 > |\vec{q}_1, r_1 > \\ &= \sqrt{2E_{\vec{p}_1}}\sqrt{2E_{\vec{q}_1}}\ \hat{b}(\vec{p}_1, s_1)^+ \hat{d}(\vec{q}_1, r_1)^+ |0 > . \end{aligned} \tag{6.95}$$

$$\begin{aligned} \text{final state} &= |\vec{p}_2, s_2 > |\vec{q}_2, r_2 > \\ &= \sqrt{2E_{\vec{p}_2}}\sqrt{2E_{\vec{q}_2}}\ \hat{b}(\vec{p}_2, s_2)^+ \hat{d}(\vec{q}_2, r_2)^+ |0 > . \end{aligned} \tag{6.96}$$

These states are precisely the "in" and "out" states which we also denote by $|\vec{p}_1 s_1, \vec{q}_1 r_1 \text{ in} >$ and $|\vec{p}_2 s_2, \vec{q}_2 r_2 \text{ out} >$ respectively. The probability amplitude $< \vec{p}_2 s_2, \vec{q}_2 r_2 \text{ out}|\vec{p}_1 s_1, \vec{q}_1 r_1 \text{ in} >$ is then given by

$$< \vec{p}_2 s_2, \vec{q}_2 r_2 \text{ out}|\vec{p}_1 s_1, \vec{q}_1 r_1 \text{ in} > \;=\; \sqrt{2E_{\vec{q}}} < \vec{p}_2 s_2 \text{ out}|\hat{d}_{\text{out}}(\vec{q}_2, r_2)|\vec{p}_1 s_1, \vec{q}_1 r_1 \text{ in} > . \tag{6.97}$$

By assuming that $q_2 \neq q_1$ and $r_2 \neq r_1$ we obtain

$$\begin{aligned} < \vec{p}_2 s_2, \vec{q}_2 r_2 \text{ out}|\vec{p}_1 s_1, \vec{q}_1 r_1 \text{ in} > &= \frac{1}{i}\int d^4y_2\ e^{iq_2 y_2}. < \vec{p}_2 s_2 \text{ out}|\bar{\hat{\psi}}(y_2)|\vec{p}_1 s_1, \vec{q}_1 r_1 \text{ in} > \\ &\times\ (i\gamma^\mu \overleftarrow{\partial}_{\mu, y_2} + m_\mu) v^{r_2}(q_2). \end{aligned} \tag{6.98}$$

By also assuming that $p_2 \neq p_1$ and $s_2 \neq s_1$ we can similarly reduce the muon state. By using the appropriate LSZ reduction formula we get

$$\begin{aligned} < \vec{p}_2 s_2, \vec{q}_2 r_2 \text{ out}|\vec{p}_1 s_1, \vec{q}_1 r_1 \text{ in} > &= \frac{1}{i}\sqrt{2E_{\vec{p}_2}}\int d^4y_2\ e^{iq_2 y_2}. < 0 \text{ out}|\hat{b}_{\text{out}}(\vec{p}_2, s_2)\bar{\hat{\psi}}(y_2)|\vec{p}_1 s_1, \vec{q}_1 r_1 \text{ in} > \\ &\times\ (i\gamma^\mu \overleftarrow{\partial}_{\mu, y_2} + m_\mu) v^{r_2}(q_2) \\ &= \frac{1}{i^2}\int d^4y_2\ e^{iq_2 y_2}\int d^4x_2\ e^{ip_2 x_2}.\bar{u}^{s_2}(p_2)(i\gamma^\mu \partial_{\mu, x_2} - m_\mu) \\ &\times\ < 0 \text{ out}|T(\hat{\psi}(x_2)\bar{\hat{\psi}}(y_2))|\vec{p}_1 s_1, \vec{q}_1 r_1 \text{ in} > (i\gamma^\mu \overleftarrow{\partial}_{\mu, y_2} + m_\mu) v^{r_2}(q_2). \end{aligned} \tag{6.99}$$

Next we reduce the initial electron and positron states. Again by using the appropriate LSZ reduction formulae we obtain

$$\begin{aligned} < 0|T(\hat{\psi}(x_2)\bar{\hat{\psi}}(y_2))|\vec{p}_1 s_1, \vec{q}_1 r_1 \text{ in} > &= \sqrt{2E_{\vec{q}_1}} < 0 \text{ out}|T(\hat{\psi}(x_2)\bar{\hat{\psi}}(y_2))\hat{d}_{\text{in}}(\vec{q}_1, r_1)^+|\vec{p}_1 s_1 \text{ in} > \\ &= -\frac{1}{i}\int d^4y_1\ e^{-iq_1 y_1}.\bar{v}^{r_1}(q_1)(i\gamma^\mu \partial_{\mu, y_1} - m_e) \\ &\times\ < 0 \text{ out}|T(\hat{\psi}(y_1)\hat{\psi}(x_2)\bar{\hat{\psi}}(y_2))|\vec{p}_1 s_1 \text{ in} > \\ &= -\frac{1}{i}\sqrt{2E_{\vec{p}_1}}\int d^4y_1\ e^{-iq_1 y_1}.\bar{v}^{r_1}(q_1)(i\gamma^\mu \partial_{\mu, y_1} - m_e) \\ &\times\ < 0 \text{ out}|T(\hat{\psi}(y_1)\hat{\psi}(x_2)\bar{\hat{\psi}}(y_2))\hat{b}_{\text{in}}(\vec{p}_1, s_1)^+|0 \text{ in} > \\ &= -\frac{1}{(-i)^2}\int d^4x_1\ e^{-ip_1 x_1}\int d^4y_1\ e^{-iq_1 y_1}.\bar{v}^{r_1}(q_1)(i\gamma^\mu \partial_{\mu, y_1} - m_e) \\ &\times\ < 0 \text{ out}|T(\bar{\hat{\psi}}(x_1)\hat{\psi}(y_1)\hat{\psi}(x_2)\bar{\hat{\psi}}(y_2))|0 \text{ in} > (i\gamma^\mu \overleftarrow{\partial}_{\mu, x_1} + m_e) u^{s_1}(p_1). \end{aligned} \tag{6.100}$$

The probabilty amplitude is therefore given by

$$
\begin{aligned}
<\vec{p}_2s_2,\vec{q}_2r_2\text{ out}|\vec{p}_1s_1,\vec{q}_1r_1\text{ in}> &= \frac{-1}{i^2}\frac{1}{(-i)^2}\int d^4y_2\, e^{iq_2y_2}\int d^4x_2\, e^{ip_2x_2}\int d^4x_1\, e^{-ip_1x_1}\int d^4y_1\, e^{-iq_1y_1}.\\
&\times \bigg(\bar{u}^{s_2}(p_2)(i\gamma^\mu\partial_{\mu,x_2}-m_\mu)\bigg)_{\alpha_2}\bigg(\bar{v}^{r_1}(q_1)(i\gamma^\mu\partial_{\mu,y_1}-m_e)\bigg)_{\beta_1}\\
&\times <0\text{ out}|T\bigg(\bar{\hat{\psi}}_{\alpha_1}(x_1)\hat{\psi}_{\beta_1}(y_1)\hat{\psi}_{\alpha_2}(x_2)\bar{\hat{\psi}}_{\beta_2}(y_2)\bigg)|0\text{ in}>\\
&\times \bigg((i\gamma^\mu\overleftarrow{\partial}_{\mu,y_2}+m_\mu)v^{r_2}(q_2)\bigg)_{\beta_2}\bigg((i\gamma^\mu\overleftarrow{\partial}_{\mu,x_1}+m_e)u^{s_1}(p_1)\bigg)_{\alpha_1}. \quad (6.101)
\end{aligned}
$$

This depends on the Green's function

$$
\begin{aligned}
G_{\alpha_1,\beta_1,\alpha_2,\beta_2}(x_1,y_1,x_2,y_2) &= <0\text{ out}|T\bigg(\bar{\hat{\psi}}_{\alpha_1}(x_1)\hat{\psi}_{\beta_1}(y_1)\hat{\psi}_{\alpha_2}(x_2)\bar{\hat{\psi}}_{\beta_2}(y_2)\bigg)|0\text{ in}>\\
&= \int\frac{d^4p_1}{(2\pi)^4}\frac{d^4q_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4q_2}{(2\pi)^4}G_{\alpha_1,\beta_1,\alpha_2,\beta_2}(p_1,q_1,p_2,q_2)\, e^{ip_1x_1+iq_1y_1+ip_2x_2+iq_2y_2}.
\end{aligned}
\tag{6.102}
$$

We get

$$
\begin{aligned}
<\vec{p}_2s_2,\vec{q}_2r_2\text{ out}|\vec{p}_1s_1,\vec{q}_1r_1\text{ in}> &= -\bigg(\bar{u}^{s_2}(p_2)(\gamma^\mu p_{2,\mu}-m_\mu)\bigg)_{\alpha_2}\bigg(\bar{v}^{r_1}(q_1)(\gamma^\mu q_{1,\mu}+m_e)\bigg)_{\beta_1}\\
&\times G_{\alpha_1,\beta_1,\alpha_2,\beta_2}(p_1,q_1,-p_2,-q_2)\\
&\times \bigg((\gamma^\mu q_{2,\mu}+m_\mu)v^{r_2}(q_2)\bigg)_{\beta_2}\bigg((\gamma^\mu p_{1,\mu}-m_e)u^{s_1}(p_1)\bigg)_{\alpha_1}
\end{aligned}
\tag{6.103}
$$

The Green's function $G_{\alpha_1,\beta_1,\alpha_2,\beta_2}(p_1,q_1,-p_2,-q_2)$ must be proportional to the delta function $(2\pi)^4\delta^4(p_1+q_1+p_2+q_2)$ by energy-momentum conservation. Furthermore it will be proportional to the external propagators $1/(\gamma^\mu p_{2,\mu}-m_\mu)$, $1/(\gamma^\mu q_{1,\mu}+m_e)$, $1/(\gamma^\mu q_{2,\mu}+m_\mu)$ and $1/(\gamma^\mu p_{1,\mu}-m_e)$ and thus they will be canceled. We write therefore

$$
\begin{aligned}
<\vec{p}_2s_2,\vec{q}_2r_2\text{ out}|\vec{p}_1s_1,\vec{q}_1r_1\text{ in}> &= -\bar{u}^{s_2}_{\alpha_2}(p_2)\bar{v}^{r_1}_{\beta_1}(q_1)G^{\text{amputated}}_{\alpha_1,\beta_1,\alpha_2,\beta_2}(p_1,q_1,-p_2,-q_2)v^{r_2}_{\beta_2}(q_2)\\
&\times u^{s_1}_{\alpha_1}(p_1).
\end{aligned}
\tag{6.104}
$$

$$
\begin{aligned}
G^{\text{amputated}}_{\alpha_1,\beta_1,\alpha_2,\beta_2}(p_1,q_1,-p_2,-q_2) &= \int d^4x_1\int d^4y_1\int d^4x_2\int d^4y_2\, e^{-ip_1x_1-iq_1y_1+ip_2x_2+iq_2y_2}\\
&\times <0\text{ out}|T\bigg(\bar{\hat{\psi}}_{\alpha_1}(x_1)\hat{\psi}_{\beta_1}(y_1)\hat{\psi}_{\alpha_2}(x_2)\bar{\hat{\psi}}_{\beta_2}(y_2)\bigg)|0\text{ in}>^{\text{amputated}}.
\end{aligned}
\tag{6.105}
$$

6.5.2 The Gell-Mann Low Formula and the $S-$Matrix

The $S-$matrix and $T-$matrix elements are defined by

$$
\begin{aligned}
<\vec{p}_2s_2,\vec{q}_2r_2\text{ out}|\vec{p}_1s_1,\vec{q}_1r_1\text{ in}> &= <\vec{p}_2s_2,\vec{q}_2r_2\text{ in}|S|\vec{p}_1s_1,\vec{q}_1r_1\text{ in}>\\
&= <\vec{p}_2s_2,\vec{q}_2r_2\text{ in}|\vec{p}_1s_1,\vec{q}_1r_1\text{ in}>+<\vec{p}_2s_2,\vec{q}_2r_2\text{ in}|iT|\vec{p}_1s_1,\vec{q}_1r_1\text{ in}>.
\end{aligned}
\tag{6.106}
$$

The second term (i.e. the $T-$matrix element) is due entirely to interactions. By assuming that the initial and final states are different we obtain simply

$$\begin{aligned} <\vec{p}_2s_2,\vec{q}_2r_2\ \text{out}|\vec{p}_1s_1,\vec{q}_1r_1\ \text{in}> &= <\vec{p}_2s_2,\vec{q}_2r_2\ \text{in}|iT|\vec{p}_1s_1,\vec{q}_1r_1\ \text{in}> \\ &= (2\pi)^4\delta^4(p_1+q_1-p_2-q_2).i\mathcal{M}(p_1q_1\longrightarrow p_2q_2).(6.107) \end{aligned}$$

The matrix element $\mathcal{M}$ is by construction Lorentz invariant and it is precisley the scattering amplitude. It is almost obvious from the above discussion that

$$i\mathcal{M}(p_1q_1\longrightarrow p_2q_2)=\text{sum of all connected amputated Feynman diagrams.} \tag{6.108}$$

In the following we will explicitly prove this result in the context of the scattering process $e^-e^+\longrightarrow\mu^-\mu^+$.

First we need to express the Green's function $<0\ \text{out}|T\big(\hat{\bar{\psi}}_{\alpha_1}(x_1)\hat{\psi}_{\beta_1}(y_1)\hat{\psi}_{\alpha_2}(x_2)\hat{\bar{\psi}}_{\beta_2}(y_2)\big)|0\ \text{in}>$ in terms of free fields and the interaction. The starting point is to understand that $\hat{\psi}(x)$, $\hat{\bar{\psi}}(x)$ and also $\hat{A}(x)$ are Heisenberg operators. The Schrödinger operators are defined by

$$\hat{\psi}(t,\vec{x})=U(t)^{-1}\hat{\psi}(\vec{x})U(t)\ ,\ \hat{\bar{\psi}}(t,\vec{x})=U(t)^{-1}\hat{\bar{\psi}}(\vec{x})U(t). \tag{6.109}$$

$$\hat{A}^\mu(t,\vec{x})=U(t)^{-1}\hat{A}^\mu(\vec{x})U(t). \tag{6.110}$$

The unitary time evolution operator solves the Schrodinger equation

$$i\partial_tU(t)=\hat{H}U(t). \tag{6.111}$$

The Hamiltonian operator is

$$\hat{H}=\hat{H}_0+\hat{V}. \tag{6.112}$$

$$\hat{H}_0 = -\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}p^ip^i\ \hat{A}^*_\mu(\vec{p})\hat{A}^\mu(\vec{p})+\int\frac{d^3p}{(2\pi)^3}\hat{\chi}^+(\vec{p})\gamma^0(\gamma^ip^i+m)\hat{\chi}(\vec{p}). \tag{6.113}$$

$$\hat{V} = -\int d^3x\mathcal{L}_{\text{int}}=e\int\frac{d^3p}{(2\pi)^3}\int\frac{d^3q}{(2\pi)^3}\hat{\chi}^+(\vec{p})\gamma^0\gamma^\mu\hat{\chi}(\vec{q})\hat{A}_\mu(\vec{p}-\vec{q}). \tag{6.114}$$

Let us recall the Fourier expansions of the different fields. We expand the spinor field as

$$\psi(\vec{x})=\int\frac{d^3p}{(2\pi)^3}\chi(\vec{p})e^{i\vec{p}\vec{x}}. \tag{6.115}$$

The corresponding conjugate momentum field is

$$\Pi(\vec{x})=i\psi^+=i\int\frac{d^3p}{(2\pi)^3}\chi^+(\vec{p})e^{-i\vec{p}\vec{x}}. \tag{6.116}$$

The gauge field is expanded as follows

$$\hat{A}^\mu(\vec{x})=\int\frac{d^3p}{(2\pi)^3}\hat{A}^\mu(\vec{p})e^{i\vec{p}\vec{x}}. \tag{6.117}$$

We introduce the unitary operator Ω in the interaction picture by

$$U(t)=e^{-it\hat{H}_0}\Omega(t). \tag{6.118}$$

The operator Ω satisfies the Schrodinger equation

$$i\partial_t\Omega(t)=\hat{V}_I(t)\Omega(t)\ ,\ \hat{V}_I(t)=e^{it\hat{H}_0}\hat{V}e^{-it\hat{H}_0}. \tag{6.119}$$

The interaction and Heisenberg operators are related by

$$\hat{\psi}(x)=\Omega(t)^{-1}\hat{\psi}_I(x)\Omega(t)\ ,\ \hat{A}^\mu(x)=\Omega(t)^{-1}\hat{A}^\mu_I(x)\Omega(t). \tag{6.120}$$

The interaction and Schrodinger operators are related by

$$\hat{\psi}_I(x)=e^{it\hat{H}_0}\hat{\psi}(\vec{x})e^{-it\hat{H}_0}\ ,\ \hat{A}^\mu_I(x)=e^{it\hat{H}_0}\hat{A}^\mu(\vec{x})e^{-it\hat{H}_0}. \tag{6.121}$$

The solution of the above last differential equation is

$$\begin{aligned}\Omega(t) &= T\bigg(e^{-i\int_{-\infty}^t ds\hat{V}_I(s)}\bigg)\\ &= T\bigg(e^{i\int_{-\infty}^t ds\int d^3x\mathcal{L}_{\rm int}\big(\hat{\psi}_I(s,\vec{x}),\hat{A}_I(s,\vec{x})\big)}\bigg).\end{aligned} \tag{6.122}$$

The $S-$matrix is defined by

$$\begin{aligned}S=\Omega(+\infty) &= T\bigg(e^{-i\int_{-\infty}^{+\infty} ds\hat{V}_I(s)}\bigg)\\ &= T\bigg(e^{i\int d^4x\mathcal{L}_{\rm int}\big(\hat{\psi}_I(s,\vec{x}),\hat{A}_I(s,\vec{x})\big)}\bigg).\end{aligned} \tag{6.123}$$

This is a unitary operator, viz

$$S^+=S^{-1} = \bar{T}\bigg(e^{-i\int_{-\infty}^{+\infty} ds\hat{V}_I(s)}\bigg). \tag{6.124}$$

This operator satisfies

$$<0\ \text{out}|=<0\ \text{in}|S. \tag{6.125}$$

The "in" and "out" Hilbert spaces are related by

$$<...\text{out}|=<...\text{in}|S. \tag{6.126}$$

The interaction fields ψ_I and A^μ_I are free fields. In the limit $t\longrightarrow-\infty$ we see that $\Omega(t)\longrightarrow 1$ and hence $\hat{\psi}(x)\longrightarrow\psi_I(x)$ and $\hat{A}^\mu(x)\longrightarrow\hat{A}^\mu_I(x)$. But we know that $\hat{\psi}(x)\longrightarrow\hat{\psi}_{\rm in}(x)$ and $\hat{A}^\mu(x)\longrightarrow\hat{A}^\mu_{\rm in}(x)$ when $t\longrightarrow-\infty$. Thus

$$\hat{\psi}_I(x)=\hat{\psi}_{\rm in}(x)\ ,\ \hat{A}^\mu_I(x)=\hat{A}^\mu_{\rm in}(x). \tag{6.127}$$

Similarly to the case of the scalar field we can derive the identities

$$T(\hat{\psi}(x)...\bar{\hat{\psi}}(y)...) = S^{-1}T\bigg(\hat{\psi}_{\rm in}(x)...\bar{\hat{\psi}}_{\rm in}(y)...S\bigg). \tag{6.128}$$

In general we must have

$$T(\hat{\psi}(x)...\bar{\hat{\psi}}(y)...\hat{A}^\mu(z)...) = S^{-1}T\bigg(\hat{\psi}_{\rm in}(x)...\bar{\hat{\psi}}_{\rm in}(y)...\hat{A}^\mu_{\rm in}(z)...S\bigg). \tag{6.129}$$

6.5.3 Perturbation Theory:Tree Level

We are now in a position to compute the perturbative expansion of the Green's function $< 0\ \text{out}|T\big(\bar{\hat{\psi}}_{\alpha_1}(x_1)\hat{\psi}_{\beta_1}(y_1)\hat{\psi}_{\alpha_2}(x_2)\bar{\hat{\psi}}_{\beta_2}(y_2)\big)|0\ \text{in}>$. We have

$$< 0\ \text{out}|T\big(\bar{\hat{\psi}}^{\alpha_1}(x_1)\hat{\psi}^{\beta_1}(y_1)\hat{\psi}^{\alpha_2}(x_2)\bar{\hat{\psi}}^{\beta_2}(y_2)\big)|0\ \text{in}> \;=\; < 0\ \text{in}|T\big(\bar{\hat{\psi}}^{\alpha_1}_{\text{in}}(x_1)\hat{\psi}^{\beta_1}_{\text{in}}(y_1)\hat{\psi}^{\alpha_2}_{\text{in}}(x_2)\bar{\hat{\psi}}^{\beta_2}_{\text{in}}(y_2)S\big)|0\ \text{in}>. \tag{6.130}$$

In the following we will set $|0\ \text{out}>=|0\ \text{in}>=|0>$ for simplicity. However it should be obvious from the context which $|0>$ is $|0\ \text{out}>$ and which $|0>$ is $|0\ \text{in}>$. The first few terms are

$$\begin{aligned} S &= 1+i\int d^4z\mathcal{L}_{\text{int}}(z)+\frac{i^2}{2!}\int d^4z_1\int d^4z_2\mathcal{L}_{\text{int}}(z_1)\mathcal{L}_{\text{int}}(z_2)+\frac{i^3}{3!}\int d^4z_1\int d^4z_2\int d^4z_3\mathcal{L}_{\text{int}}(z_1)\mathcal{L}_{\text{int}}(z_2)\\ &\times\ \mathcal{L}_{\text{int}}(z_3)+\frac{i^4}{4!}\int d^4z_1\int d^4z_2\int d^4z_3\int d^4z_4\mathcal{L}_{\text{int}}(z_1)\mathcal{L}_{\text{int}}(z_2)\mathcal{L}_{\text{int}}(z_3)\mathcal{L}_{\text{int}}(z_4)+... \end{aligned} \tag{6.131}$$

Of course

$$\mathcal{L}_{\text{int}}(z)=\mathcal{L}^e_{\text{int}}(z)+\mathcal{L}^\mu_{\text{int}}(z)=-e\bigg(\bar{\hat{\psi}}_{\text{in}}(z)\gamma_\mu\hat{\psi}_{\text{in}}(z)+\bar{\hat{\psi}}_{\text{in}}(z)\gamma_\mu\hat{\psi}_{\text{in}}(z)\bigg)\hat{A}^\mu(z). \tag{6.132}$$

By using Wick's theorem for the electromagnetic field we deduce that the second and the fourth terms will lead to contributions to the probability amplitude (6.103) which vanish identically. Indeed the vacuum expectation value of the product of an odd number of gauge field operators is always zero.

By using Wick's theorem for fermions the first term will lead to

$$< 0|T\big(\bar{\hat{\psi}}^{\alpha_1}_{\text{in}}(x_1)\hat{\psi}^{\beta_1}_{\text{in}}(y_1)\hat{\psi}^{\alpha_2}_{\text{in}}(x_2)\bar{\hat{\psi}}^{\beta_2}_{\text{in}}(y_2)\big)|0>=-S_F^{\beta_1\alpha_1}(y_1-x_1)S_F^{\alpha_2\beta_2}(x_2-y_2). \tag{6.133}$$

The even contraction will allow the electron to propagate into a muon which is not possible. Recall that the electron is at x_1 with spin and momentum (s_1,p_1), the positron is at y_1 with spin and momentum (r_1,q_1), the muon is at x_2 with spin and momentum (s_2,p_2) and the antimuon is at y_2 with spin and momentum (r_2,q_2). The contribution of this term to the probability amplitude (6.103) is

$$\begin{aligned} <\vec{p}_2s_2,\vec{q}_2r_2\ \text{out}|\vec{p}_1s_1,\vec{q}_1r_1\ \text{in}> &= \bigg(\bar{v}^{r_1}(q_1)(\gamma.q_1+m_e)u^{s_1}(-q_1).(2\pi)^4\delta^4(p_1+q_1)\bigg)\\ &\times\ \bigg(\bar{u}^{s_2}(-q_2)(\gamma.q_2+m_\mu)v^{r_2}(q_2).(2\pi)^4\delta^4(p_2+q_2)\bigg)\\ &= 0. \end{aligned} \tag{6.134}$$

We have used $(\gamma.p-m)u^r(p)=0$ and $(\gamma.p+m)v^r(p)=0$.

The first two terms in the $S-$matrix which give non-vanishing contribution to the probability amplitude (6.103) are therefore given by

$$\begin{aligned} S &= \frac{i^2}{2!}\int d^4z_1\int d^4z_2\mathcal{L}_{\text{int}}(z_1)\mathcal{L}_{\text{int}}(z_2)\\ &+\ \frac{i^4}{4!}\int d^4z_1\int d^4z_2\int d^4z_3\int d^4z_4\mathcal{L}_{\text{int}}(z_1)\mathcal{L}_{\text{int}}(z_2)\mathcal{L}_{\text{int}}(z_3)\mathcal{L}_{\text{int}}(z_4)+... \end{aligned} \tag{6.135}$$

The first term corresponds to the so-called tree level contribution. This is given by

$$\begin{aligned}
<0|T\big(\hat{\bar{\psi}}^{\alpha_1}(x_1)\hat{\psi}^{\beta_1}(y_1)\hat{\psi}^{\alpha_2}(x_2)\hat{\bar{\psi}}^{\beta_2}(y_2)\big)|0> &= \frac{i^2}{2!}\int d^4z_1\int d^4z_2 <0|T\big(\hat{\bar{\psi}}_{\rm in}^{\alpha_1}(x_1)\hat{\psi}_{\rm in}^{\beta_1}(y_1)\hat{\psi}_{\rm in}^{\alpha_2}(x_2)\\
&\times \hat{\bar{\psi}}_{\rm in}^{\beta_2}(y_2)\mathcal{L}_{\rm int}(z_1)\mathcal{L}_{\rm int}(z_2)\big)|0>\\
&= 2\frac{i^2}{2!}\int d^4z_1\int d^4z_2 <0|T\big(\hat{\bar{\psi}}_{\rm in}^{\alpha_1}(x_1)\hat{\psi}_{\rm in}^{\beta_1}(y_1)\hat{\psi}_{\rm in}^{\alpha_2}(x_2)\\
&\times \hat{\bar{\psi}}_{\rm in}^{\beta_2}(y_2)\mathcal{L}^e_{\rm int}(z_1)\mathcal{L}^\mu_{\rm int}(z_2)\big)|0>\\
&= 2\frac{i^2}{2!}(-e)^2(\gamma_\mu)^{\gamma_1\delta_1}(\gamma_\nu)^{\gamma_2\delta_2}\int d^4z_1\int d^4z_2 <0|T(\hat{A}^\mu(z_1)\\
&\times \hat{A}^\nu(z_2))|0><0|T\big(\hat{\bar{\psi}}_{\rm in}^{\alpha_1}(x_1)\hat{\psi}_{\rm in}^{\beta_1}(y_1)\hat{\bar{\psi}}_{\rm in}^{\gamma_1}(z_1)\hat{\psi}_{\rm in}^{\delta_1}(z_1)\big)\\
&\times |0><0|T\big(\hat{\psi}_{\rm in}^{\alpha_2}(x_2)\hat{\bar{\psi}}_{\rm in}^{\beta_2}(y_2)\hat{\bar{\psi}}_{\rm in}^{\gamma_2}(z_2)\hat{\psi}_{\rm in}^{\delta_2}(z_2)\big)|0>. \qquad (6.136)
\end{aligned}$$

In the second line we have dropped the terms corresponding to $\mathcal{L}^e_{\rm int}(z_1)\mathcal{L}^e_{\rm int}(z_2)$ and $\mathcal{L}^\mu_{\rm int}(z_1)\mathcal{L}^\mu_{\rm int}(z_2)$ since they are zero by an argument similar to the one which led to (6.134). In the third line we have used the fact that the total Hilbert space is the tensor product of the Hilbert spaces associated with the electron, the muon and the photon. Using Wick's theorems we get

$$<0|T(\hat{A}^\mu(z_1)\hat{A}^\nu(z_2))|0>= iD_{\mu\nu}(z_1-z_2). \qquad (6.137)$$

$$<0|T\big(\hat{\bar{\psi}}_{\rm in}^{\alpha_1}(x_1)\hat{\psi}_{\rm in}^{\beta_1}(y_1)\hat{\bar{\psi}}_{\rm in}^{\gamma_1}(z_1)\hat{\psi}_{\rm in}^{\delta_1}(z_1)\big)|0>= -S_F^{\beta_1\gamma_1}(y_1-z_1)S_F^{\delta_1\alpha_1}(z_1-x_1)+S_F^{\beta_1\alpha_1}(y_1-x_1)S_F^{\delta_1\gamma_1}(0). \qquad (6.138)$$

$$<0|T\big(\hat{\psi}_{\rm in}^{\alpha_2}(x_2)\hat{\bar{\psi}}_{\rm in}^{\beta_2}(y_2)\hat{\bar{\psi}}_{\rm in}^{\gamma_2}(z_2)\hat{\psi}_{\rm in}^{\delta_2}(z_2)\big)|0>= S_F^{\alpha_2\gamma_2}(x_2-z_2)S_F^{\delta_2\beta_2}(z_2-y_2)-S_F^{\alpha_2\beta_2}(x_2-y_2)S_F^{\delta_2\gamma_2}(0). \qquad (6.139)$$

The propagator $S_F(0)$ will lead to disconnected diagrams so we will simply drop it right from the start. We get then

$$\begin{aligned}
<0|T\big(\hat{\bar{\psi}}^{\alpha_1}(x_1)\hat{\psi}^{\beta_1}(y_1)\hat{\psi}^{\alpha_2}(x_2)\hat{\bar{\psi}}^{\beta_2}(y_2)\big)|0> &= -2i\frac{i^2}{2!}(-e)^2\int d^4z_1\int d^4z_2 D^{\mu\nu}(z_1-z_2)\big[S_F(y_1-z_1)\\
&\times \gamma_\mu S_F(z_1-x_1)\big]^{\beta_1\alpha_1}\big[S_F(x_2-z_2)\gamma_\nu S_F(z_2-y_2)\big]^{\alpha_2\beta_2}. \qquad (6.140)
\end{aligned}$$

We use the free propagators

$$iD_F^{\mu\nu}(z_1-z_2)=\int\frac{d^4p}{(2\pi)^4}\frac{-i\eta^{\mu\nu}}{p^2+i\epsilon}e^{-ip(z_1-z_2)}. \qquad (6.141)$$

$$S_F^{\alpha\beta}(x-y)=\int\frac{d^4p}{(2\pi)^4}\frac{i(\gamma.p+m)^{\alpha\beta}}{p^2-m^2+i\epsilon}e^{-ip(x-y)}. \qquad (6.142)$$

Thus

$$
\begin{aligned}
<0|T\big(\bar{\hat{\psi}}^{\alpha_1}(x_1)\hat{\psi}^{\beta_1}(y_1)\hat{\psi}^{\alpha_2}(x_2)\bar{\hat{\psi}}^{\beta_2}(y_2)\big)|0> &= -2\frac{i^2}{2!}(-e)^2\int\frac{d^4q_1}{(2\pi)^4}\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4q_2}{(2\pi)^4}\\
&\times \bigg(\frac{\gamma.q_1+m_e}{q_1^2-m_e^2}\gamma_\mu\frac{-\gamma.p_1+m_e}{p_1^2-m_e^2}\bigg)^{\beta_1\alpha_1}\frac{-i\eta^{\mu\nu}}{(q_1+p_1)^2}\bigg(\frac{\gamma.p_2+m_\mu}{p_2^2-m_\mu^2}\gamma_\nu\\
&\times \frac{-\gamma.q_2+m_\mu}{q_2^2-m_\mu^2}\bigg)^{\alpha_2\beta_2}e^{-ip_1x_1-iq_1y_1-ip_2x_2-iq_2y_2}(2\pi)^4\delta^4(q_1+p_1\\
&+ p_2+q_2). \qquad (6.143)
\end{aligned}
$$

The Fourier transform of $<0|T\big(\bar{\hat{\psi}}^{\alpha_1}(x_1)\hat{\psi}^{\beta_1}(y_1)\hat{\psi}^{\alpha_2}(x_2)\bar{\hat{\psi}}^{\beta_2}(y_2)\big)|0>$ is then

$$
\begin{aligned}
G_{\alpha_1,\beta_1,\alpha_2,\beta_2}(p_1,q_1,p_2,q_2) &= \int d^4x_1\int d^4y_1\int d^4x_2\int d^4y_2 <0|T\big(\bar{\hat{\psi}}^{\alpha_1}(x_1)\hat{\psi}^{\beta_1}(y_1)\hat{\psi}^{\alpha_2}(x_2)\bar{\hat{\psi}}^{\beta_2}(y_2)\big)|0>\\
&\times e^{-ip_1x_1-iq_1y_1-ip_2x_2-iq_2y_2}\\
&= -2\frac{i^2}{2!}(-e)^2\bigg(\frac{-\gamma.q_1+m_e}{q_1^2-m_e^2}\gamma_\mu\frac{\gamma.p_1+m_e}{p_1^2-m_e^2}\bigg)^{\beta_1\alpha_1}\frac{-i\eta^{\mu\nu}}{(q_1+p_1)^2}\bigg(\frac{-\gamma.p_2+m_\mu}{p_2^2-m_\mu^2}\gamma_\nu\\
&\times \frac{\gamma.q_2+m_\mu}{q_2^2-m_\mu^2}\bigg)^{\alpha_2\beta_2}(2\pi)^4\delta^4(q_1+p_1+p_2+q_2). \qquad (6.144)
\end{aligned}
$$

The probability amplitude (6.103) at tree level becomes

$$
\begin{aligned}
<\vec{p}_2s_2,\vec{q}_2r_2\ \text{out}|\vec{p}_1s_1,\vec{q}_1r_1\ \text{in}> &= \bigg(\bar{v}^{r_1}(q_1)(-ie\gamma_\mu)u^{s_1}(p_1)\bigg)\frac{-i\eta^{\mu\nu}}{(p_1+q_1)^2}\bigg(\bar{u}^{s_2}(p_2)(-ie\gamma_\nu)v^{r_2}(q_2)\bigg)\\
&\times (2\pi)^4\delta^4(q_1+p_1-p_2-q_2). \qquad (6.145)
\end{aligned}
$$

This can be represented by the diagram TRE.

6.5.4 Perturbation Theory: One-Loop Corrections

The second term in (6.135) will lead to the first radiative corrections for the probability amplitude (6.103) of the process $e^-+e^+\longrightarrow\mu^-+\mu^+$. We have

$$
\begin{aligned}
<0|T\big(\bar{\hat{\psi}}^{\alpha_1}(x_1)\hat{\psi}^{\beta_1}(y_1)\hat{\psi}^{\alpha_2}(x_2)\bar{\hat{\psi}}^{\beta_2}(y_2)\big)|0> &= \frac{i^4}{4!}\int d^4z_1...\int d^4z_4 <0|T\big(\bar{\hat{\psi}}^{\alpha_1}_{\text{in}}(x_1)\hat{\psi}^{\beta_1}_{\text{in}}(y_1)\hat{\psi}^{\alpha_2}_{\text{in}}(x_2)\bar{\hat{\psi}}^{\beta_2}_{\text{in}}(y_2)\\
&\times \mathcal{L}_{\text{int}}(z_1)...\mathcal{L}_{\text{int}}(z_4)\big)|0>\\
&= \frac{i^4}{4!}(-e)^4\int d^4z_1...\int d^4z_4 <0|T(\hat{A}^{\mu_1}(z_1)...\hat{A}^{\mu_4}(z_4))|0>\bigg[\\
&\times 4<0|T\big(\bar{\hat{\psi}}^{\alpha_1}_{\text{in}}(x_1)\hat{\psi}^{\beta_1}_{\text{in}}(y_1)\mathcal{L}^e_{\mu_1}(z_1)\mathcal{L}^e_{\mu_2}(z_2)\mathcal{L}^e_{\mu_3}(z_3)|0>\\
&\times <0|T\big(\hat{\psi}^{\alpha_2}_{\text{in}}(x_2)\bar{\hat{\psi}}^{\beta_2}_{\text{in}}(y_2)\mathcal{L}^\mu_{\mu_4}(z_4)\big)|0> +4<0|T\big(\bar{\hat{\psi}}^{\alpha_1}_{\text{in}}(x_1)\\
&\times \hat{\psi}^{\beta_1}_{\text{in}}(y_1)\mathcal{L}^e_{\mu_4}(z_4)|0><0|T\big(\hat{\psi}^{\alpha_2}_{\text{in}}(x_2)\bar{\hat{\psi}}^{\beta_2}_{\text{in}}(y_2)\mathcal{L}^\mu_{\mu_1}(z_1)\mathcal{L}^\mu_{\mu_2}(z_2)\\
&\times \mathcal{L}^\mu_{\mu_3}(z_3)\big)|0> +6<0|T\big(\bar{\hat{\psi}}^{\alpha_1}_{\text{in}}(x_1)\hat{\psi}^{\beta_1}_{\text{in}}(y_1)\mathcal{L}^e_{\mu_1}(z_1)\mathcal{L}^e_{\mu_2}(z_2)|0>\\
&\times <0|T\big(\hat{\psi}^{\alpha_2}_{\text{in}}(x_2)\bar{\hat{\psi}}^{\beta_2}_{\text{in}}(y_2)\mathcal{L}^\mu_{\mu_3}(z_3)\mathcal{L}^\mu_{\mu_4}(z_4)\big)|0>\bigg]. \qquad (6.146)
\end{aligned}
$$

In the above equation we have defined $\mathcal{L}_\mu(z)=\bar{\hat{\psi}}_{\rm in}(z)\gamma_\mu\hat{\psi}_{\rm in}(z)$. We will use the result

$$
\begin{aligned}
<0|T(\hat{A}^{\mu_1}(z_1)...\hat{A}^{\mu_4}(z_4))|0> &= iD^{\mu_1\mu_2}(z_1-z_2).iD^{\mu_3\mu_4}(z_3-z_4)+iD^{\mu_1\mu_3}(z_1-z_3).iD^{\mu_2\mu_4}(z_2-z_4)\\
&+ iD^{\mu_1\mu_4}(z_1-z_4).iD^{\mu_2\mu_3}(z_2-z_3).
\end{aligned}
\tag{6.147}
$$

1st term: By using Wick's theorem we compute next the expression

$$
<0|T\big(\bar{\hat{\psi}}_{\rm in}^{\alpha_1}(x_1)\hat{\psi}_{\rm in}^{\beta_1}(y_1)\mathcal{L}^e_{\mu_1}(z_1)\mathcal{L}^e_{\mu_2}(z_2)\mathcal{L}^e_{\mu_3}(z_3)|0> . \tag{6.148}
$$

There are in total 24 contractions. By dropping those disconnected contractions which contain $S_F(0)$ we will only have 11 contractions left. Using then the symmetry between the points z_1, z_2 and z_3 (under the integral and the trace) the expression is reduced further to 3 terms. These are

$$
\begin{aligned}
<0|T\big(\bar{\hat{\psi}}_{\rm in}^{\alpha_1}(x_1)\hat{\psi}_{\rm in}^{\beta_1}(y_1)\mathcal{L}^e_{\mu_1}(z_1)\mathcal{L}^e_{\mu_2}(z_2)\mathcal{L}^e_{\mu_3}(z_3)|0> &= -6\Big[S_F(y_1-z_1)\gamma_{\mu_1}S_F(z_1-z_2)\gamma_{\mu_2}S_F(z_2-z_3)\gamma_{\mu_3}\\
&\times S_F(z_3-x_1)\Big]^{\beta_1\alpha_1}+3\Big[S_F(y_1-z_1)\gamma_{\mu_1}S_F(z_1-x_1)\Big]^{\beta_1\alpha_1}\\
&\times tr\Big[S_F(z_2-z_3)\gamma_{\mu_3}S_F(z_3-z_2)\gamma_{\mu_2}\Big]\\
&+ 2S_F(y_1-x_1)^{\beta_1\alpha_1}tr\Big[\gamma_{\mu_1}S_F(z_1-z_2)\gamma_{\mu_2}S_F(z_2-z_3)\gamma_{\mu_3}\\
&\times S_F(z_3-z_1)\Big].
\end{aligned}
\tag{6.149}
$$

The last term corresponds to a disconnected contribution.

We also need the expression

$$
<0|T\big(\hat{\psi}_{\rm in}^{\alpha_2}(x_2)\bar{\hat{\psi}}_{\rm in}^{\beta_2}(y_2)\mathcal{L}^\mu_{\mu_4}(z_4)\big)|0> \tag{6.150}
$$

Again by dropping the disconnected contraction we obtain

$$
<0|T\big(\hat{\psi}_{\rm in}^{\alpha_2}(x_2)\bar{\hat{\psi}}_{\rm in}^{\beta_2}(y_2)\mathcal{L}^\mu_{\mu_4}(z_4)\big)|0>=\Big[S_F(x_2-z_4)\gamma_{\mu_4}S_F(z_4-y_2)\Big]^{\alpha_2\beta_2}. \tag{6.151}
$$

We have then the following two contributions to the first term. The first contribution consists of the three terms

$$
\begin{aligned}
{\rm con}_1 &= \frac{i^4}{4!}(-e)^4(4)(-6)\int d^4z_1...\int d^4z_4<0|T(\hat{A}^{\mu_1}(z_1)...\hat{A}^{\mu_4}(z_4))|0>\Big[S_F(y_1-z_1)\gamma_{\mu_1}S_F(z_1-z_2)\gamma_{\mu_2}\\
&\times S_F(z_2-z_3)\gamma_{\mu_3}S_F(z_3-x_1)\Big]^{\beta_1\alpha_1}\Big[S_F(x_2-z_4)\gamma_{\mu_4}S_F(z_4-y_2)\Big]^{\alpha_2\beta_2}
\end{aligned}
\tag{6.152}
$$

The second contribution consists of the term

$$
\begin{aligned}
{\rm con}_2 &= \frac{i^4}{4!}(-e)^4(4)(3.2)\int d^4z_1...\int d^4z_4 iD^{\mu_1\mu_2}(z_1-z_2).iD^{\mu_3\mu_4}(z_3-z_4)\Big[S_F(y_1-z_1)\gamma_{\mu_1}S_F(z_1-x_1)\Big]^{\beta_1\alpha_1}\\
&\times tr\Big[S_F(z_2-z_3)\gamma_{\mu_3}S_F(z_3-z_2)\gamma_{\mu_2}\Big]\Big[S_F(x_2-z_4)\gamma_{\mu_4}S_F(z_4-y_2)\Big]^{\alpha_2\beta_2}.
\end{aligned}
\tag{6.153}
$$

In this term we have used the fact that the two terms $iD^{\mu_1\mu_2}(z_1-z_2).iD^{\mu_3\mu_4}(z_3-z_4)$ and $iD^{\mu_1\mu_3}(z_1-z_3).iD^{\mu_2\mu_4}(z_2-z_4)$ in the photon 4−point function lead to identical contributions whereas the term $iD^{\mu_1\mu_4}(z_1-z_4).iD^{\mu_2\mu_3}(z_2-z_3)$ leads to a disconnected contribution and so it is neglected. In momentum space we have

$$\begin{aligned}
\text{con}_2 &= \frac{i^4}{4!}(-e)^4(4)(3.2)\int\frac{d^4q_1}{(2\pi)^4}\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4q_2}{(2\pi)^4}\int\frac{d^4q_3}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\Big[S(q_1)\gamma_{\mu_1}S(-p_1)\Big]^{\beta_1\alpha_1}\Big[S(p_2)\\
&\times\ \gamma_{\mu_4}S(-q_2)\Big]^{\alpha_2\beta_2} trS(q_3)\gamma_{\mu_3}S(p_3)\gamma_{\mu_2}\frac{-i\eta^{\mu_1\mu_2}}{(q_1+p_1)^2}\frac{-i\eta^{\mu_3\mu_4}}{(p_2+q_2)^2}(2\pi)^4\delta(q_1+p_1-q_3+p_3)(2\pi)^4\delta^4(-p_2\\
&-\ q_2-q_3+p_3)\ e^{-ip_1x_1-iq_1y_1-ip_2x_2-iq_2y_2}.
\end{aligned}\tag{6.154}$$

In above we have defined

$$S(p)=\frac{i(\gamma.p+m)}{p^2-m^2}.\tag{6.155}$$

The corresponding Fourier transform and probability amplitude are

$$\begin{aligned}
G^{\text{con}_2}_{\alpha_1,\beta_1,\alpha_2,\beta_2}(p_1,q_1,p_2,q_2) &= \frac{i^4}{4!}(-e)^4(4)(3.2)\int\frac{d^4q_3}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\Big[S(-q_1)\gamma_{\mu_1}S(p_1)\Big]^{\beta_1\alpha_1}\Big[S(-p_2)\gamma_{\mu_4}S(q_2)\Big]^{\alpha_2\beta_2}\\
&\times\ trS(q_3)\gamma_{\mu_3}S(p_3)\gamma_{\mu_2}\frac{-i\eta^{\mu_1\mu_2}}{(q_1+p_1)^2}\frac{-i\eta^{\mu_3\mu_4}}{(p_2+q_2)^2}(2\pi)^4\delta(-q_1-p_1-q_3+p_3)(2\pi)^4\delta^4(p_2\\
&+\ q_2-q_3+p_3).
\end{aligned}\tag{6.156}$$

$$\begin{aligned}
<\vec{p}_2s_2,\vec{q}_2r_2\ \text{out}|\vec{p}_1s_1,\vec{q}_1r_1\ \text{in}>_{\text{con}_2} &= \Big(\bar{v}^{r_1}(q_1)(-ie\gamma_{\mu_1})u^{s_1}(p_1)\Big).\Big(\frac{-i\eta^{\mu_1\mu_2}}{(q_1+p_1)^2}\Big).(-1).\int\frac{d^4q_3}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\\
&\times\ (2\pi)^4\delta(p_1+q_1+q_3-p_3).tr(-ie\gamma_{\mu_2})S(q_3)(-ie\gamma_{\mu_3})S(p_3)\\
&\times\ (2\pi)^4\delta^4(p_2+q_2+q_3-p_3).\Big(\frac{-i\eta^{\mu_3\mu_4}}{(p_2+q_2)^2}\Big).\Big(\bar{u}^{s_2}(p_2)(-ie\gamma_{\mu_4})v^{r_2}(q_2)\Big).
\end{aligned}\tag{6.157}$$

This can be represented by the diagram RAD0.

1st term, Continued: The three terms in the first contribution of the first term are

$$\begin{aligned}
\text{con}_1^1 &= \frac{i^4}{4!}(-e)^4(4)(-6)\int d^4z_1...\int d^4z_4 iD^{\mu_1\mu_2}(z_1-z_2).iD^{\mu_3\mu_4}(z_3-z_4)\Big[S_F(y_1-z_1)\gamma_{\mu_1}S_F(z_1-z_2)\gamma_{\mu_2}\\
&\times\ S_F(z_2-z_3)\gamma_{\mu_3}S_F(z_3-x_1)\Big]^{\beta_1\alpha_1}\Big[S_F(x_2-z_4)\gamma_{\mu_4}S_F(z_4-y_2)\Big]^{\alpha_2\beta_2}\\
&= -(-ie)^4\int\frac{d^4q_1}{(2\pi)^4}\int\frac{d^4q_2}{(2\pi)^4}\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4q_3}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\Big[S(q_1)\gamma_{\mu_1}S(p_3)\gamma_{\mu_2}S(q_3)\gamma_{\mu_3}S(-p_1)\\
&\times\ \Big]^{\beta_1\alpha_1}\Big[S(p_2)\gamma_{\mu_4}S(-q_2)\Big]^{\alpha_2\beta_2}\frac{-i\eta^{\mu_1\mu_2}}{(q_1-p_3)^2}\frac{-i\eta^{\mu_3\mu_4}}{(p_2+q_2)^2}(2\pi)^4\delta^4(q_1-q_3)(2\pi)^4\delta^4(-p_2-q_2-q_3-p_1)\\
&\times\ e^{-ip_1x_1-iq_1y_1-ip_2x_2-iq_2y_2}.
\end{aligned}\tag{6.158}$$

$$\begin{aligned}
\mathrm{con}_1^2 &= \frac{i^4}{4!}(-e)^4(4)(-6)\int d^4z_1...\int d^4z_4 iD^{\mu_1\mu_3}(z_1-z_3).iD^{\mu_2\mu_4}(z_2-z_4)\bigg[S_F(y_1-z_1)\gamma_{\mu_1}S_F(z_1-z_2)\gamma_{\mu_2}\\
&\times S_F(z_2-z_3)\gamma_{\mu_3}S_F(z_3-x_1)\bigg]^{\beta_1\alpha_1}\bigg[S_F(x_2-z_4)\gamma_{\mu_4}S_F(z_4-y_2)\bigg]^{\alpha_2\beta_2}\\
&= -(-ie)^4\int\frac{d^4q_1}{(2\pi)^4}\int\frac{d^4q_2}{(2\pi)^4}\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4q_3}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\bigg[S(q_1)\gamma_{\mu_1}S(p_3)\gamma_{\mu_2}S(q_3)\gamma_{\mu_3}S(-p_1)\\
&\times \bigg]^{\beta_1\alpha_1}\bigg[S(p_2)\gamma_{\mu_4}S(-q_2)\bigg]^{\alpha_2\beta_2}\frac{-i\eta^{\mu_1\mu_3}}{(q_1-p_3)^2}\frac{-i\eta^{\mu_2\mu_4}}{(p_2+q_2)^2}(2\pi)^4\delta^4(-q_1+p_3-q_3-p_1)(2\pi)^4\delta^4(p_2+q_2+p_3\\
&- q_3)e^{-ip_1x_1-iq_1y_1-ip_2x_2-iq_2y_2}.
\end{aligned}\tag{6.159}$$

$$\begin{aligned}
\mathrm{con}_1^3 &= \frac{i^4}{4!}(-e)^4(4)(-6)\int d^4z_1...\int d^4z_4 iD^{\mu_1\mu_4}(z_1-z_4).iD^{\mu_2\mu_3}(z_2-z_3)\bigg[S_F(y_1-z_1)\gamma_{\mu_1}S_F(z_1-z_2)\gamma_{\mu_2}\\
&\times S_F(z_2-z_3)\gamma_{\mu_3}S_F(z_3-x_1)\bigg]^{\beta_1\alpha_1}\bigg[S_F(x_2-z_4)\gamma_{\mu_4}S_F(z_4-y_2)\bigg]^{\alpha_2\beta_2}\\
&= -(-ie)^4\int\frac{d^4q_1}{(2\pi)^4}\int\frac{d^4q_2}{(2\pi)^4}\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4q_3}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\bigg[S(q_1)\gamma_{\mu_1}S(p_3)\gamma_{\mu_2}S(q_3)\gamma_{\mu_3}S(-p_1)\\
&\times \bigg]^{\beta_1\alpha_1}\bigg[S(p_2)\gamma_{\mu_4}S(-q_2)\bigg]^{\alpha_2\beta_2}\frac{-i\eta^{\mu_1\mu_4}}{(q_1-p_3)^2}\frac{-i\eta^{\mu_2\mu_3}}{(p_3-q_3)^2}(2\pi)^4\delta^4(p_3+p_1)(2\pi)^4\delta^4(q_1-p_3+p_2+q_2)\\
&\times e^{-ip_1x_1-iq_1y_1-ip_2x_2-iq_2y_2}.
\end{aligned}\tag{6.160}$$

The corresponding Fourier transforms are

$$\begin{aligned}
G^{\mathrm{con}_1^1}_{\alpha_1,\beta_1,\alpha_2,\beta_2}(p_1,q_1,p_2,q_2) &= -(-ie)^4\int\frac{d^4q_3}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\bigg[S(-q_1)\gamma_{\mu_1}S(p_3)\gamma_{\mu_2}S(q_3)\gamma_{\mu_3}S(p_1)\bigg]^{\beta_1\alpha_1}\bigg[S(-p_2)\gamma_{\mu_4}\\
&\times S(q_2)\bigg]^{\alpha_2\beta_2}\frac{-i\eta^{\mu_1\mu_2}}{(q_1+p_3)^2}\frac{-i\eta^{\mu_3\mu_4}}{(p_2+q_2)^2}(2\pi)^4\delta^4(q_1+q_3)(2\pi)^4\delta^4(p_2+q_2+p_1+q_1).
\end{aligned}\tag{6.161}$$

$$\begin{aligned}
G^{\mathrm{con}_1^2}_{\alpha_1,\beta_1,\alpha_2,\beta_2}(p_1,q_1,p_2,q_2) &= -(-ie)^4\int\frac{d^4q_3}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\bigg[S(-q_1)\gamma_{\mu_1}S(p_3)\gamma_{\mu_2}S(q_3)\gamma_{\mu_3}S(p_1)\bigg]^{\beta_1\alpha_1}\bigg[S(-p_2)\gamma_{\mu_4}\\
&\times S(q_2)\bigg]^{\alpha_2\beta_2}\frac{-i\eta^{\mu_1\mu_3}}{(q_1+p_3)^2}\frac{-i\eta^{\mu_2\mu_4}}{(p_2+q_2)^2}(2\pi)^4\delta^4(q_1+p_1+p_3-q_3)(2\pi)^4\delta^4(-q_2-p_2\\
&+ p_3-q_3).
\end{aligned}\tag{6.162}$$

$$\begin{aligned}
G^{\mathrm{con}_1^3}_{\alpha_1,\beta_1,\alpha_2,\beta_2}(p_1,q_1,p_2,q_2) &= -(-ie)^4\int\frac{d^4q_3}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\bigg[S(-q_1)\gamma_{\mu_1}S(p_3)\gamma_{\mu_2}S(q_3)\gamma_{\mu_3}S(p_1)\bigg]^{\beta_1\alpha_1}\bigg[S(-p_2)\gamma_{\mu_4}\\
&\times S(q_2)\bigg]^{\alpha_2\beta_2}\frac{-i\eta^{\mu_1\mu_4}}{(q_1+p_3)^2}\frac{-i\eta^{\mu_2\mu_3}}{(p_3-q_3)^2}(2\pi)^4\delta^4(p_3-p_1)(2\pi)^4\delta^4(q_1+p_1+q_2+p_2).
\end{aligned}\tag{6.163}$$

The corresponding probability amplitudes are

$$
\begin{aligned}
< \vec{p}_2 s_2, \vec{q}_2 r_2 \text{ out}|\vec{p}_1 s_1, \vec{q}_1 r_1 \text{ in} >_{\text{con}_1^1} &= \int \frac{d^4 q_3}{(2\pi)^4} \int \frac{d^4 p_3}{(2\pi)^4} \bigg(\bar{v}^{r_1}(q_1)(-ie\gamma_{\mu_1}) S(p_3)(-ie\gamma_{\mu_2}) S(q_3)(-ie\gamma_{\mu_3}) \\
&\times \; u^{s_1}(p_1)\bigg) . \bigg(\frac{-i\eta^{\mu_1\mu_2}}{(q_1+p_3)^2} \bigg) . (2\pi)^4 \delta^4(q_1+q_3) . (2\pi)^4 \delta^4(p_2+q_2-p_1-q_1) \\
&\times \; \bigg(\frac{-i\eta^{\mu_3\mu_4}}{(p_2+q_2)^2} \bigg) . \bigg(\bar{u}^{s_2}(p_2)(-ie\gamma_{\mu_4}) v^{r_2}(q_2) \bigg). \qquad (6.164)
\end{aligned}
$$

$$
\begin{aligned}
< \vec{p}_2 s_2, \vec{q}_2 r_2 \text{ out}|\vec{p}_1 s_1, \vec{q}_1 r_1 \text{ in} >_{\text{con}_1^2} &= \int \frac{d^4 q_3}{(2\pi)^4} \int \frac{d^4 p_3}{(2\pi)^4} \bigg(\bar{v}^{r_1}(q_1)(-ie\gamma_{\mu_1}) S(p_3)(-ie\gamma_{\mu_2}) S(q_3)(-ie\gamma_{\mu_3}) \\
&\times \; u^{s_1}(p_1)\bigg) . \bigg(\frac{-i\eta^{\mu_1\mu_3}}{(p_1-q_3)^2} \bigg) . (2\pi)^4 \delta^4(q_1+p_1+p_3-q_3) . (2\pi)^4 \delta^4(p_2+q_2 \\
&+ \; p_3-q_3) \bigg(\frac{-i\eta^{\mu_2\mu_4}}{(p_2+q_2)^2} \bigg) . \bigg(\bar{u}^{s_2}(p_2)(-ie\gamma_{\mu_4}) v^{r_2}(q_2) \bigg). \qquad (6.165)
\end{aligned}
$$

$$
\begin{aligned}
< \vec{p}_2 s_2, \vec{q}_2 r_2 \text{ out}|\vec{p}_1 s_1, \vec{q}_1 r_1 \text{ in} >_{\text{con}_1^3} &= \int \frac{d^4 q_3}{(2\pi)^4} \int \frac{d^4 p_3}{(2\pi)^4} \bigg(\bar{v}^{r_1}(q_1)(-ie\gamma_{\mu_1}) S(p_3)(-ie\gamma_{\mu_2}) S(q_3)(-ie\gamma_{\mu_3}) \\
&\times \; u^{s_1}(p_1)\bigg) . \bigg(\frac{-i\eta^{\mu_1\mu_4}}{(p_2+q_2)^2} \bigg) . (2\pi)^4 \delta^4(p_3-p_1) . (2\pi)^4 \delta^4(q_1+p_1-q_2-p_2) \\
&\times \; \bigg(\frac{-i\eta^{\mu_2\mu_3}}{(p_1-q_3)^2} \bigg) . \bigg(\bar{u}^{s_2}(p_2)(-ie\gamma_{\mu_4}) v^{r_2}(q_2) \bigg). \qquad (6.166)
\end{aligned}
$$

They are represented by the diagrams RAD1, RAD2 and RAD3 respectively.

2nd term: The calculation of the second term is identical to the calculation of the first term except that the role of the electron and the positron is interchanged with the role of the muon and antimuon. The result is represented by the sum of diagrams RAD4. This term contains two contributions which are proportional to one virtual muon propagator and two contributions which are proportional to two virtual muon propagators. Thus in the limit in which the muon is much heavier than the electron (which is actually the case here since $m_e = 0.5$ Mev and $m_\mu = 105.7$ Mev) we can neglect the second term compared to the first term. Indeed the second term is proportional to $1/m_\mu$ whereas the first term is of order 0 in $1/m_\mu$ in the limit $m_\mu \longrightarrow \infty$.

3rd term: By using Wick's theorem we compute the expression

$$< 0|T\big(\bar{\hat{\psi}}_{\text{in}}^{\alpha_1}(x_1)\hat{\psi}_{\text{in}}^{\beta_1}(y_1)\mathcal{L}^e_{\mu_1}(z_1)\mathcal{L}^e_{\mu_2}(z_2)|0> . \qquad (6.167)$$

There are in total 6 contractions. By dropping disconnected contractions which contain $S_F(0)$ we will only have 3 contractions left. Using then the symmetry between the points z_1 and z_2 (under the integral and the trace) the expression is reduced further to 2 terms. These are

$$
\begin{aligned}
< 0|T\big(\bar{\hat{\psi}}_{\text{in}}^{\alpha_1}(x_1)\hat{\psi}_{\text{in}}^{\beta_1}(y_1)\mathcal{L}^e_{\mu_1}(z_1)\mathcal{L}^e_{\mu_2}(z_2)|0> &= -2\bigg[S_F(y_1-z_1)\gamma_{\mu_1} S_F(z_1-z_2)\gamma_{\mu_2} S_F(z_2-x_1)\bigg]^{\beta_1\alpha_1} \\
&+ \; S_F^{\beta_1\alpha_1}(y_1-x_1) tr\bigg[\gamma_{\mu_1} S_F(z_1-z_2)\gamma_{\mu_2} S_F(z_2-z_1)\bigg]. \qquad (6.168)
\end{aligned}
$$

Similarly

$$< 0|T\big(\hat{\psi}_{\rm in}^{\alpha_2}(x_2)\bar{\hat{\psi}}_{\rm in}^{\beta_2}(y_2)\mathcal{L}^{\mu}_{\mu_3}(z_3)\mathcal{L}^{\mu}_{\mu_4}(z_4)\big)|0> \;=\; 2\Big[S_F(x_2-z_3)\gamma_{\mu_3}S_F(z_3-z_4)\gamma_{\mu_4}S_F(z_4-y_2)\Big]^{\alpha_2\beta_2} - S_F^{\alpha_2\beta_2}(x_2-y_2)tr\Big[\gamma_{\mu_3}S_F(z_3-z_4)\gamma_{\mu_4}S_F(z_4-z_3)\Big]. \quad (6.169)$$

The second terms in the above two equations correspond to disconnected contributions. Also the first term $iD^{\mu_1\mu_2}(z_1-z_2).iD^{\mu_3\mu_4}(z_3-z_4)$ in the photon 4−point function leads to a disconnected contribution. The second term $iD^{\mu_1\mu_3}(z_1-z_3).iD^{\mu_2\mu_4}(z_2-z_4)$ gives on the other hand the contribution

$$< 0|T\big(\bar{\hat{\psi}}^{\alpha_1}(x_1)\hat{\psi}^{\beta_1}(y_1)\hat{\psi}^{\alpha_2}(x_2)\bar{\hat{\psi}}^{\beta_2}(y_2)\big)|0> \;=\; -e^4\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\int\frac{d^4q_1}{(2\pi)^4}\int\frac{d^4q_2}{(2\pi)^4}\int\frac{d^4q_3}{(2\pi)^4} \times \Big[S(q_1)\gamma_{\mu_1}S(p_3)\gamma_{\mu_2}S(-p_1)\Big]^{\beta_1\alpha_1}\Big[S(p_2)\gamma_{\mu_3}S(-q_3)\gamma_{\mu_4}S(-q_2) \times \Big]^{\alpha_2\beta_2}\frac{-i\eta^{\mu_1\mu_3}}{(q_1-p_3)^2}\frac{-i\eta^{\mu_2\mu_4}}{(-q_3+q_2)^2}(2\pi)^4\delta(q_3-q_2-p_3-p_1) \times (2\pi)^4\delta^4(p_1+q_1+p_2+q_2)\; e^{-ip_1x_1-iq_1y_1-ip_2x_2-iq_2y_2}. \quad (6.170)$$

Thus

$$G_{\alpha_1,\beta_1,\alpha_2,\beta_2}(p_1,q_1,p_2,q_2) \;=\; -e^4\int\frac{d^4p_3}{(2\pi)^4}\int\frac{d^4q_3}{(2\pi)^4}\Big[S(-q_1)\gamma_{\mu_1}S(p_3)\gamma_{\mu_2}S(p_1)\Big]^{\beta_1\alpha_1}\Big[S(-p_2)\gamma_{\mu_3}S(-q_3)\gamma_{\mu_4} \times S(q_2)\Big]^{\alpha_2\beta_2}\frac{-i\eta^{\mu_1\mu_3}}{(-q_1-p_3)^2}\frac{-i\eta^{\mu_2\mu_4}}{(-q_3-q_2)^2}(2\pi)^4\delta(q_3+q_2-p_3+p_1).(2\pi)^4\delta^4(p_1+q_1 + p_2+q_2). \quad (6.171)$$

The third term $iD^{\mu_1\mu_4}(z_1-z_4).iD^{\mu_2\mu_3}(z_2-z_3)$ in the photon 4−point function gives the contribution

$$< 0|T\big(\bar{\hat{\psi}}^{\alpha_1}(x_1)\hat{\psi}^{\beta_1}(y_1)\hat{\psi}^{\alpha_2}(x_2)\bar{\hat{\psi}}^{\beta_2}(y_2)\big)|0> \;=\; -e^4\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\int\frac{d^4p_3}{(2\pi)^4}\int\frac{d^4q_1}{(2\pi)^4}\int\frac{d^4q_2}{(2\pi)^4}\int\frac{d^4q_3}{(2\pi)^4} \times \Big[S(q_1)\gamma_{\mu_1}S(p_3)\gamma_{\mu_2}S(-p_1)\Big]^{\beta_1\alpha_1}\Big[S(p_2)\gamma_{\mu_3}S(-q_3)\gamma_{\mu_4}S(-q_2) \times \Big]^{\alpha_2\beta_2}\frac{-i\eta^{\mu_1\mu_4}}{(q_1-p_3)^2}\frac{-i\eta^{\mu_2\mu_3}}{(p_2+q_3)^2}(2\pi)^4\delta(p_1+p_2+q_3+p_3) \times (2\pi)^4\delta^4(q_1+q_2-q_3-p_3)\; e^{-ip_1x_1-iq_1y_1-ip_2x_2-iq_2y_2}. \quad (6.172)$$

Thus

$$G_{\alpha_1,\beta_1,\alpha_2,\beta_2}(p_1,q_1,p_2,q_2) = -e^4\int\frac{d^4p_3}{(2\pi)^4}\int\frac{d^4q_3}{(2\pi)^4}\bigg[S(-q_1)\gamma_{\mu_1}S(p_3)\gamma_{\mu_2}S(p_1)\bigg]^{\beta_1\alpha_1}\bigg[S(-p_2)\gamma_{\mu_3}S(-q_3)\gamma_{\mu_4} \times S(q_2)\bigg]^{\alpha_2\beta_2}\frac{-i\eta^{\mu_1\mu_4}}{(q_1+p_3)^2}\frac{-i\eta^{\mu_2\mu_3}}{(p_2-q_3)^2}(2\pi)^4\delta(p_1+p_2-q_3-p_3)(2\pi)^4\delta^4(q_1+q_2 + q_3+p_3). \quad (6.173)$$

The two contributions (6.171) and (6.173) correspond to the diagrams RAD5 and RAD6 respectively. They will be neglected under the assumption that the muon mass is very large compared to the electron mass. These two diagrams in the limit of infinite muon mass go as $1/m_\mu$ which corresponds to the single internal muon propagator.

6.6 LSZ Reduction formulas for Photons

6.6.1 Example II: $e^- + \gamma \longrightarrow e^- + \gamma$

Let us consider now the process

$$e^-(p_1)+\gamma(k_1)\longrightarrow e^-(p_2)+\gamma(k_2). \quad (6.174)$$

The initial and final states are given by

$$\text{initial state} = |\vec{p}_1,s_1>|\vec{k}_1,\lambda_1> = \sqrt{2E_{\vec{p}_1}}\ \hat{b}(\vec{p}_1,s_1)^+\hat{a}(\vec{k}_1,\lambda_1)^+|0>. \quad (6.175)$$

$$\text{final state} = |\vec{p}_2,s_2>|\vec{k}_2,\lambda_2> = \sqrt{2E_{\vec{p}_2}}\ \hat{b}(\vec{p}_2,s_2)^+\hat{a}(\vec{k}_2,\lambda_2)^+|0>. \quad (6.176)$$

The probability amplitude of interest in this case is

$$<\vec{p}_2s_2,\vec{k}_2\lambda_2\ \text{out}|\vec{p}_1s_1,\vec{k}_1\lambda_1\ \text{in}> = \sqrt{2E_{\vec{p}_2}}<\vec{k}_2\lambda_2\ \text{out}|\hat{b}_{\text{out}}(\vec{p}_2,s_2)|\vec{p}_1s_1,\vec{k}_1\lambda_1\ \text{in}>(6.177)$$

By assuming that $p_1\neq p_2$ and $s_1\neq s_2$ and then using the appropriate LSZ reduction formulas we get

$$\begin{aligned}<\vec{p}_2s_2,\vec{k}_2\lambda_2\ \text{out}|\vec{p}_1s_1,\vec{k}_1\lambda_1\ \text{in}> &= \sqrt{2E_{\vec{p}_2}}<\vec{k}_2\lambda_2\ \text{out}|\hat{b}_{\text{out}}(\vec{p}_2,s_2)|\vec{p}_1s_1,\vec{k}_1\lambda_1\ \text{in}>\\ &= \frac{1}{i}\int d^4x_2e^{ip_2x_2}\bar{u}^{s_2}(p_2)(i\gamma^\mu\partial_{\mu,x_2}-m)<\vec{k}_2\lambda_2\ \text{out}|\hat{\psi}(x_2)|\vec{p}_1s_1,\vec{k}_1\lambda_1\ \text{in}>\\ &= \frac{1}{i}\sqrt{2E_{\vec{p}_1}}\int d^4x_2e^{ip_2x_2}\bar{u}^{s_2}(p_2)(i\gamma^\mu\partial_{\mu,x_2}-m)<\vec{k}_2\lambda_2\ \text{out}|\hat{\psi}(x_2)\\ &\times\ \hat{b}_{\text{in}}(\vec{p}_1,s_1)^+|\vec{k}_1\lambda_1\ \text{in}>\\ &= \frac{1}{i^2}\int d^4x_2e^{ip_2x_2}\int d^4x_1e^{-ip_1x_1}\bar{u}^{s_2}(p_2)(i\gamma^\mu\partial_{\mu,x_2}-m)<\vec{k}_2\lambda_2\ \text{out}|T(\\ &\times\ \hat{\psi}(x_2)\hat{\bar{\psi}}(x_1))|\vec{k}_1\lambda_1\ \text{in}>(-i\gamma^\mu\overleftarrow{\partial}_{\mu,x_1}-m)u^{s_1}(p_1). \end{aligned} \quad (6.178)$$

We need now to reduce the photon states. We need reduction formulas for photons. By analogy with the scalar field case the reduction formulas for the electromagnetic field read

$$\hat{a}_{\text{out}}(k,\lambda)T(...)-T(...)\hat{a}_{\text{in}}(k,\lambda) = -\int d^4xe^{ikx}\ \epsilon^\mu_\lambda(\vec{k})i\partial^2T(\hat{A}_\mu(x)...). \quad (6.179)$$

$$\hat{a}^+_{\rm out}(k,\lambda)T(...)-T(...)\hat{a}^+_{\rm in}(k,\lambda)=\int d^4x e^{-ikx}\ \epsilon^{\mu}_{\lambda}(\vec{k})i\partial^2 T(\hat{A}_{\mu}(x)...). \tag{6.180}$$

We have then

$$\begin{aligned} <\vec{p}_2 s_2,\vec{k}_2\lambda_2\ {\rm out}|\vec{p}_1 s_1,\vec{k}_1\lambda_1\ {\rm in}> &= \frac{1}{i^2}\int d^4x_2 e^{ip_2x_2}\int d^4x_1 e^{-ip_1x_1}\bar{u}^{s_2}(p_2)(i\gamma^{\mu}\partial_{\mu,x_2}-m)<0\ {\rm out}|\hat{a}_{\rm out}(k_2,\\ &\times\ \lambda_2)T(\hat{\psi}(x_2)\bar{\hat{\psi}}(x_1))|\vec{k}_1\lambda_1\ {\rm in}>(-i\gamma^{\mu}\overleftarrow{\partial}_{\mu,x_1}-m)u^{s_1}(p_1). \end{aligned} \tag{6.181}$$

Again by assuming that $k_1\neq k_2$ and $\lambda_1\neq\lambda_2$ we get

$$\begin{aligned} <\vec{p}_2 s_2,\vec{k}_2\lambda_2\ {\rm out}|\vec{p}_1 s_1,\vec{k}_1\lambda_1\ {\rm in}> &= -\frac{1}{i^2}i\int d^4x_2 e^{ip_2x_2}\int d^4x_1 e^{-ip_1x_1}\bar{u}^{s_2}(p_2)(i\gamma^{\mu}\partial_{\mu,x_2}-m)\int d^4y_2 e^{ik_2y_2}\\ &\times\ \epsilon^{\mu_2}_{\lambda_2}(\vec{k_2})\partial^2_{y_2}<0\ {\rm out}|T(\hat{A}_{\mu_2}(y_2)\hat{\psi}(x_2)\bar{\hat{\psi}}(x_1))\hat{a}_{\rm in}(\vec{k}_1,\lambda_1)^+|0\ {\rm in}>\\ &\times\ (-i\gamma^{\mu}\overleftarrow{\partial}_{\mu,x_1}-m)u^{s_1}(p_1)\\ &= \frac{1}{i^2}i^2\int d^4x_2 e^{ip_2x_2}\int d^4x_1 e^{-ip_1x_1}\bar{u}^{s_2}(p_2)(i\gamma^{\mu}\partial_{\mu,x_2}-m)\int d^4y_2 e^{ik_2y_2}\\ &\times\ \int d^4y_1 e^{-ik_1y_1}\ \epsilon^{\mu_2}_{\lambda_2}(\vec{k_2})\partial^2_{y_2}<0\ {\rm out}|T(\hat{A}_{\mu_1}(y_1)\hat{A}_{\mu_2}(y_2)\hat{\psi}(x_2)\bar{\hat{\psi}}(x_1))|0\ {\rm in}>\\ &\times\ \epsilon^{\mu_1}_{\lambda_1}(\vec{k_1})\overleftarrow{\partial}^2_{y_1}(-i\gamma^{\mu}\overleftarrow{\partial}_{\mu,x_1}-m)u^{s_1}(p_1). \end{aligned} \tag{6.182}$$

This depends on the Green's function

$$\begin{aligned} G_{\alpha_1,\mu_1,\alpha_2,\mu_2}(x_1,y_1,x_2,y_2) &= <0\ {\rm out}|T\bigg(\hat{A}_{\mu_1}(y_1)\hat{A}_{\mu_2}(y_2)\hat{\psi}_{\alpha_2}(x_2)\bar{\hat{\psi}}_{\alpha_1}(x_1)\bigg)|0\ {\rm in}>\\ &= \int\frac{d^4p_1}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}G_{\alpha_1,\mu_1,\alpha_2,\mu_2}(p_1,k_1,p_2,k_2)\ e^{ip_1x_1+ik_1y_1+ip_2x_2+ik_2y_2}. \end{aligned} \tag{6.183}$$

Thus we get

$$\begin{aligned} <\vec{p}_2 s_2,\vec{k}_2\lambda_2\ {\rm out}|\vec{p}_1 s_1,\vec{k}_1\lambda_1\ {\rm in}> &= k_1^2k_2^2\epsilon^{\mu_1}_{\lambda_1}(k_1)\epsilon^{\mu_2}_{\lambda_2}(k_2)\bigg(\bar{u}^{s_2}(p_2)(\gamma.p_2-m)\bigg)_{\alpha_2}G_{\alpha_1,\mu_1,\alpha_2,\mu_2}(p_1,k_1,-p_2,-k_2)\\ &\times\ \bigg((\gamma.p_1-m)u^{s_1}(p_1)\bigg)_{\alpha_1}. \end{aligned} \tag{6.184}$$

6.6.2 Perturbation Theory

We need now to compute the Green's function

$$<0\ {\rm out}|T\bigg(\hat{A}_{\mu_1}(y_1)\hat{A}_{\mu_2}(y_2)\hat{\psi}_{\alpha_2}(x_2)\bar{\hat{\psi}}_{\alpha_1}(x_1)\bigg)|0\ {\rm in}>. \tag{6.185}$$

By using the Gell-Mann Low formula we have

$$\begin{aligned} <0\ {\rm out}|T\bigg(\hat{A}^{\mu_1}(y_1)\hat{A}^{\mu_2}(y_2)\hat{\psi}^{\alpha_2}(x_2)\bar{\hat{\psi}}^{\alpha_1}(x_1)\bigg)|0\ {\rm in}> &= <0\ {\rm in}|T\bigg(\hat{A}^{\mu_1}_{\rm in}(y_1)\hat{A}^{\mu_2}_{\rm in}(y_2)\hat{\psi}^{\alpha_2}_{\rm in}(x_2)\bar{\hat{\psi}}^{\alpha_1}_{\rm in}(x_1)\\ &\times\ S\bigg)|0\ {\rm in}>. \end{aligned} \tag{6.186}$$

As before we will set $|0\text{ out}>=|0\text{ in}>=|0>$ for simplicity. The first non-zero contribution (tree level) is

$$
\begin{aligned}
<0|T\Big(\hat{A}^{\mu_1}(y_1)\hat{A}^{\mu_2}(y_2)\hat{\psi}^{\alpha_2}(x_2)\bar{\hat{\psi}}^{\alpha_1}(x_1)\Big)|0> &= \frac{i^2}{2!}\int d^4z_1\int d^4z_2 <0|T\Big(\hat{A}^{\mu_1}_{\rm in}(y_1)\hat{A}^{\mu_2}_{\rm in}(y_2)\hat{\psi}^{\alpha_2}_{\rm in}(x_2)\bar{\hat{\psi}}^{\alpha_1}_{\rm in}(x_1)\\
&\times \mathcal{L}_{\rm int}(z_1)\mathcal{L}_{\rm int}(z_2)\Big)|0>\\
&= \frac{(-ie)^2}{2!}\int d^4z_1\int d^4z_2 <0|T\Big(\hat{A}^{\mu_1}_{\rm in}(y_1)\hat{A}^{\mu_2}_{\rm in}(y_2)\hat{A}^{\nu_1}_{\rm in}(z_1)\\
&\times \hat{A}^{\nu_2}_{\rm in}(z_2)\Big)|0><0|T\Big(\hat{\psi}^{\alpha_2}_{\rm in}(x_2)\bar{\hat{\psi}}^{\alpha_1}_{\rm in}(x_1)\mathcal{L}_{\nu_1}(z_1)\mathcal{L}_{\nu_2}(z_2)\Big)|0>
\end{aligned}
\tag{6.187}
$$

The only contribution in the fermion Green's function $<0|T\big(\hat{\psi}^{\alpha_2}_{\rm in}(x_2)\bar{\hat{\psi}}^{\alpha_1}_{\rm in}(x_1)\mathcal{L}_{\nu_1}(z_1)\mathcal{L}_{\nu_2}(z_2)|0>$ which will lead to connected diagrams is

$$
<0|T\big(\hat{\psi}^{\alpha_2}_{\rm in}(x_2)\bar{\hat{\psi}}^{\alpha_1}_{\rm in}(x_1)\mathcal{L}_{\nu_1}(z_1)\mathcal{L}_{\nu_2}(z_2)|0> = 2\Big[S_F(x_2-z_1)\gamma_{\nu_1}S_F(z_1-z_2)\gamma_{\nu_2}S_F(z_2-x_1)\Big]^{\alpha_2\alpha_1}.
\tag{6.188}
$$

The only contributions in the gauge field Green's function $<0|T\Big(\hat{A}^{\mu_1}_{\rm in}(y_1)\hat{A}^{\mu_2}_{\rm in}(y_2)\hat{A}^{\nu_1}_{\rm in}(z_1)\hat{A}^{\nu_2}_{\rm in}(z_2)\Big)|0>$ which will lead to connected diagrams are

$$
\begin{aligned}
<0|T\Big(\hat{A}^{\mu_1}_{\rm in}(y_1)\hat{A}^{\mu_2}_{\rm in}(y_2)\hat{A}^{\nu_1}_{\rm in}(z_1)\hat{A}^{\nu_2}_{\rm in}(z_2)\Big)|0> &= iD^{\mu_1\nu_1}(y_1-z_1).iD^{\mu_2\nu_2}(y_2-z_2)\\
&+ iD^{\mu_1\nu_2}(y_1-z_2).iD^{\mu_2\nu_1}(y_2-z_1).
\end{aligned}
\tag{6.189}
$$

Hence

$$
\begin{aligned}
<0|T\left(\hat{A}^{\mu_1}(y_1)\hat{A}^{\mu_2}(y_2)\hat{\psi}^{\alpha_2}(x_2)\hat{\bar{\psi}}^{\alpha_1}(x_1)\right)|0> &= (-ie)^2\int d^4z_1\int d^4z_2 iD^{\mu_1\nu_1}(y_1-z_1).iD^{\mu_2\nu_2}(y_2-z_2)\Big[\\
&\times S_F(x_2-z_1)\gamma_{\nu_1}S_F(z_1-z_2)\gamma_{\nu_2}S_F(z_2-x_1)\Big]^{\alpha_2\alpha_1}\\
&+ (-ie)^2\int d^4z_1\int d^4z_2 iD^{\mu_1\nu_2}(y_1-z_2).iD^{\mu_2\nu_1}(y_2-z_1)\Big[\\
&\times S_F(x_2-z_1)\gamma_{\nu_1}S_F(z_1-z_2)\gamma_{\nu_2}S_F(z_2-x_1)\Big]^{\alpha_2\alpha_1}\\
&= (-ie)^2\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\Big[S(p_2)\gamma_{\nu_1}\\
&\times S(k_1+p_2)\gamma_{\nu_2}S(-p_1)\Big]^{\alpha_2\alpha_1}\frac{-i\eta^{\mu_1\nu_1}}{k_1^2}\frac{-i\eta^{\mu_2\nu_2}}{k_2^2}(2\pi)^4\delta^4(k_2+k_1\\
&+ p_2+p_1)e^{-ip_1x_1-ik_1y_1-ip_2x_2-ik_2y_2}\\
&+ (-ie)^2\int\frac{d^4k_1}{(2\pi)^4}\int\frac{d^4k_2}{(2\pi)^4}\int\frac{d^4p_1}{(2\pi)^4}\int\frac{d^4p_2}{(2\pi)^4}\Big[S(p_2)\gamma_{\nu_1}\\
&\times S(k_2+p_2)\gamma_{\nu_2}S(-p_1)\Big]^{\alpha_2\alpha_1}\frac{-i\eta^{\mu_1\nu_2}}{k_1^2}\frac{-i\eta^{\mu_2\nu_1}}{k_2^2}(2\pi)^4\delta^4(k_2+k_1\\
&+ p_2+p_1)e^{-ip_1x_1-ik_1y_1-ip_2x_2-ik_2y_2}.
\end{aligned}
\tag{6.190}
$$

We deduce therefore the Fourier expansion

$$
\begin{aligned}
G^{\alpha_1,\mu_1,\alpha_2,\mu_2}(-p_1,-k_1,-p_2,-k_2) &= (-ie)^2\Big[S(p_2)\gamma_{\nu_1}S(k_1+p_2)\gamma_{\nu_2}S(-p_1)\Big]^{\alpha_2\alpha_1}\frac{-i\eta^{\mu_1\nu_1}}{k_1^2}\frac{-i\eta^{\mu_2\nu_2}}{k_2^2}(2\pi)^4\delta^4(k_2\\
&+ k_1+p_2+p_1)\\
&+ (-ie)^2\Big[S(p_2)\gamma_{\nu_1}S(k_2+p_2)\gamma_{\nu_2}S(-p_1)\Big]^{\alpha_2\alpha_1}\frac{-i\eta^{\mu_1\nu_2}}{k_1^2}\frac{-i\eta^{\mu_2\nu_1}}{k_2^2}(2\pi)^4\delta^4(k_2\\
&+ k_1+p_2+p_1).
\end{aligned}
\tag{6.191}
$$

The probability amplitude of the process $\gamma+e^-\longrightarrow\gamma+e^-$ becomes

$$
\begin{aligned}
<\vec{p}_2s_2,\vec{k}_2\lambda_2\ \mathrm{out}|\vec{p}_1s_1,\vec{k}_1\lambda_1\ \mathrm{in}> &= (-ie)^2\epsilon_{\lambda_1}^{\mu_1}(k_1)\Big[\bar{u}^{s_2}(p_2)\gamma_{\mu_1}S(-k_1+p_2)\gamma_{\mu_2}u^{s_1}(p_1)\Big]\epsilon_{\lambda_2}^{\mu_2}(k_2)(2\pi)^4\delta^4(k_2\\
&+ p_2-k_1-p_1)\\
&+ (-ie)^2\epsilon_{\lambda_1}^{\mu_1}(k_1)\Big[\bar{u}^{s_2}(p_2)\gamma_{\mu_2}S(k_2+p_2)\gamma_{\mu_1}u^{s_1}(p_1)\Big]\epsilon_{\lambda_2}^{\mu_2}(k_2)(2\pi)^4\delta^4(k_2\\
&+ p_2-k_1-p_1).
\end{aligned}
\tag{6.192}
$$

These two terms are represented by the two diagrams COMP1 and COMP2 respectively.

6.7 Feynman Rules for QED

From the above two examples we can summarize Feynman rules for QED in momentum space as follows. First we draw all connected Feynman graphs which will contribute to a given process then we associate an expression for every diagram in the perturbative expansion by applying the following rules:

- **Energy Conservation**:
  - We assign a 4−momentum vector to each line.
  - We impose energy conservation at each vertex.
  - We will integrate (at the end) over all undetermined, i.e. internal, momenta.

- **External Legs**:
  - We attach a spinor $u^s(p)$ to any initial fermion state with incoming momentum p and spin s.
  - We attach a spinor $\bar{u}^s(p)$ to any final fermion state with outgoing momentum p and spin s.
  - We attach a spinor $\bar{v}^s(p)$ to any initial antifermion fermion state with incoming momentum p and spin s.
  - We attach a spinor $v^s(p)$ to any final antifermion fermion state with outgoing momentum p and spin s.
  - We attach a photon polarization 4−vector $\epsilon^\mu_\lambda(k)$ to any photon state with momentum k and polarization λ.
  - We will put arrows on fermion and antifermion lines. For fermions the arrow is in the same direction as the momentum carried by the line. For antifermions the arrow is opposite to the momentum carried by the line.

- **Propagators**:
  - We attach a propagator $S^{\alpha\beta}(p) = i(\gamma.p + m)^{\alpha\beta}/(p^2 - m^2 + i\epsilon)$ to any fermion line carrying a momentum p in the same direction as the arrow on the line. We will attach a propagator $S^{\alpha\beta}(-p)$ if the momentum p of the fermion is opposite to the arrow on the line. Remark that antifermions are included in this rule automatically since any antifermion line which will carry a momentum p opposite to the arrow on the line will be associated with a propagator $S^{\alpha\beta}(-p)$.
  - We attach a propagator $-i\eta^{\mu\nu}/(p^2 + i\epsilon)$ to any photon line.
  - External fermion and photon lines will not be associated with propagators. We say that external lines are amputated.

- **Vertex**:
  - The vector indices of photon propagators and photon polarization 4−vectors will be connected together via interaction vertices. The value of QED vertex is $-ie(\gamma^\mu)_{\alpha\beta}$. The spinor indices of the vertex will connect together spinor indices of fermion propagators and fermion external legs.
  - All spinor and vector indices coming from vertices, propagators and external legs must be contracted appropriately.

- **Fermion Loops**:
  - A fermion loop is always associated with an overall minus sign.

6.8 Cross Sections

Transition Probability: In real experiments we measure cross sections and decay rates and not probability amplitudes, $S-$matrix elements and correlation functions. The main point of this section will be therefore to establish a relation between the cross section of the process

$$1(k_1) + 2(k_2) \longrightarrow 1^{'}(k_1^{'}) + ... + N^{'}(k_N^{'}), \quad (6.193)$$

and the $S-$matrix element (probability amplitude) of this process given by

$$< \beta \text{ out}|\alpha \text{ in} > . \quad (6.194)$$

The "in" state consists of two particles 1 and 2 with momenta k_1 and k_2 respectively while the "out" state consists of N particles $1^{'}$, ...,$N^{'}$ with momenta $k_1^{'}$,...,$k_N^{'}$ respectively. We will assume that all these particles are scalar and thus we have

$$< \beta \text{ out}|\alpha \text{ in} >= \sqrt{2E_{k_1}}\sqrt{2E_{k_2}}\sqrt{2E_{k_1^{'}}}...\sqrt{2E_{k_N^{'}}} < 0 \text{ out}|\hat{a}_{\text{out}}(k_N^{'})...\hat{a}_{\text{out}}(k_1^{'})\hat{a}_{\text{in}}^{+}(k_1)\hat{a}_{\text{in}}^{+}(k_2)|0 \text{ in} > . \quad (6.195)$$

The $S-$matrix is given by

$$S = T\bigg(e^{-i\int dt V_I(t)}\bigg) \ , \ V_I(t) = -\int d^3x \mathcal{L}_{\text{int}}(\hat{\phi}_{\text{in}}(\vec{x}, t)). \quad (6.196)$$

We will introduce the $T-$matrix by

$$S = 1 + i\int d^4x T(x). \quad (6.197)$$

In other words

$$T(x) = \mathcal{L}_{\text{int}}(\hat{\phi}_{\text{in}}(x)) + \frac{i}{2}\int d^4x_1 T\bigg(\mathcal{L}_{\text{int}}(\hat{\phi}_{\text{in}}(x))\mathcal{L}_{\text{int}}(\hat{\phi}_{\text{in}}(x_1))\bigg) + ... \quad (6.198)$$

Let P_μ be the 4$-$momentum operator. We have

$$[P_\mu, \hat{\phi}_{\text{in}}(x)] = -i\partial_\mu \hat{\phi}_{\text{in}}. \quad (6.199)$$

It is straightforward to show that

$$[P_\mu, \int d^4x \mathcal{L}_{\text{int}}(\hat{\phi}_{\text{in}}(x))] = 0. \quad (6.200)$$

Hence

$$[P_\mu, S] = 0. \quad (6.201)$$

We know that P_μ generates spacetime translations. This expression then means that the $S-$matrix operator is invariant under spacetime translations. We expect therefore that $S-$matrix elements conserve energy-momentum. To show this we start from

$$[P_\mu, T(x)] = -i\partial_\mu T(x). \quad (6.202)$$

By integrating both sides of this equation we get (use the fact that $[P_\mu, P_\nu] = 0$)

$$T(x) = e^{iPx}T(0)e^{-iPx}. \tag{6.203}$$

Now we can compute

$$\begin{aligned} <\beta\ \text{out}|\alpha\ \text{in}> &= <\beta\ \text{in}|S|\alpha\ \text{in}> \\ &= <\beta\ \text{in}|\alpha\ \text{in}> + i(2\pi)^4\delta^4(P_\alpha - P_\beta) <\beta\ \text{in}|T(0)|\alpha\ \text{in}>. \end{aligned} \tag{6.204}$$

By assuming that the "in" and "out" states are different we get

$$<\beta\ \text{out}|\alpha\ \text{in}> = i(2\pi)^4\delta^4(P_\alpha - P_\beta) <\beta\ \text{in}|T(0)|\alpha\ \text{in}>. \tag{6.205}$$

Thus the process conserve energy-momentum as it should. The invariance of the vacuum under translations is expressed by the fact that the energy-momentum operator annihilates the vacuum, namely

$$P_\mu|0\ \text{in}>= 0. \tag{6.206}$$

Let us now recall that when we go to the box normalization, i.e. when we impose periodic boundary conditions in the spatial directions, the commutator $[\hat{a}(p), \hat{a}^+(q)] = (2\pi)^3\delta^3(\vec{p} - \vec{q})$ becomes $[\hat{a}(p), \hat{a}^+(q)] = V\delta_{\vec{p},\vec{q}}$. In other words when we go to the box normalization we make the replacement

$$(2\pi)^3\delta^3(\vec{p} - \vec{q}) \longrightarrow V\delta_{\vec{p},\vec{q}}. \tag{6.207}$$

By imposing periodic boundary condition in the time direction with a period T we can similarly replace the energy conserving delta function $(2\pi)\delta(p^0 - q^0)$ with $T\delta_{p^0,q^0}$, ,viz

$$(2\pi)\delta^3(p^0 - q^0) \longrightarrow T\delta_{p^0,q^0}. \tag{6.208}$$

It is understood that in the above two equations p^i, q^i, p^0 and q^0 are discrete variables. By making these two replacements in the $S-$matrix element $<\beta\ \text{out}|\alpha\ \text{in}>$ we obtain

$$<\beta\ \text{out}|\alpha\ \text{in}> = iTV\delta_{p^0,q^0}\delta_{\vec{p},\vec{q}} <k_1^{'}...k_N^{'}\ \text{in}|T(0)|k_1k_2\ \text{in}>. \tag{6.209}$$

Let us recall that the normalization of the one-particle states is given by

$$<\vec{p}|\vec{q}> = 2E_pV\delta_{\vec{p},\vec{q}}. \tag{6.210}$$

Taking this normalization into account, i.e. by working only with normalized states, we get the probability amplitude

$$<\beta\ \text{out}|\alpha\ \text{in}> = iTV\delta_{p^0,q^0}\delta_{\vec{p},\vec{q}}\frac{1}{\sqrt{2E_{k_1^{'}}V}}...\frac{1}{\sqrt{2E_{k_N^{'}}V}}\frac{1}{\sqrt{2E_{k_1}V}}\frac{1}{\sqrt{2E_{k_2}V}} <k_1^{'}...k_N^{'}\ \text{in}|T(0)|k_1k_2\ \text{in}>. \tag{6.211}$$

The probability is then given by

$$\begin{aligned} |<\beta\ \text{out}|\alpha\ \text{in}>|^2 &= T^2V^2\delta_{p^0,q^0}\delta_{\vec{p},\vec{q}}\frac{1}{2E_{k_1^{'}}V}...\frac{1}{2E_{k_N^{'}}V}\frac{1}{2E_{k_1}V}\frac{1}{2E_{k_2}V}|<k_1^{'}...k_N^{'}\ \text{in}|T(0)|k_1k_2\ \text{in}>|^2 \\ &= TV(2\pi)^4\delta^4(P_\alpha - P_\beta)\frac{1}{2E_{k_1^{'}}V}...\frac{1}{2E_{k_N^{'}}V}\frac{1}{2E_{k_1}V}\frac{1}{2E_{k_2}V} \\ &\times\ |<k_1^{'}...k_N^{'}\ \text{in}|T(0)|k_1k_2\ \text{in}>|^2. \end{aligned} \tag{6.212}$$

The transition probability per unit volume and per unit time is defined by

$$\frac{1}{TV}|<\beta \text{ out}|\alpha \text{ in}>|^2 = (2\pi)^4\delta^4(P_\alpha - P_\beta)\frac{1}{2E_{k_1^{'}}V}...\frac{1}{2E_{k_N^{'}}V}\frac{1}{2E_{k_1}V}\frac{1}{2E_{k_2}V} \times |<k_1^{'}...k_N^{'} \text{ in}|T(0)|k_1k_2 \text{ in}>|^2. \tag{6.213}$$

In real experiments we are interested in transitions to final states where the 4−momentum $k_i^{'}$ of the ith particle is not well determined but it is only known that it lies in a volume $d^3k_i^{'}$. From the correspondence $\sum_{\vec{k}}/V \longrightarrow \int d^3k/(2\pi)^3$ we see that we have $(Vd^3k)/(2\pi)^3$ states in the volume d^3k. Hence the transition probability per unit volume and per unit time of interest to real experiments is

$$\begin{aligned} d\nu &= \frac{1}{TV}|<\beta \text{ out}|\alpha \text{ in}>|^2\frac{Vd^3k_1^{'}}{(2\pi)^3}...\frac{Vd^3k_N^{'}}{(2\pi)^3} \\ &= (2\pi)^4\delta^4(P_\alpha - P_\beta)\frac{d^3k_1^{'}}{(2\pi)^3}\frac{1}{2E_{k_1^{'}}}...\frac{d^3k_N^{'}}{(2\pi)^3}\frac{1}{2E_{k_N^{'}}}|<k_1^{'}...k_N^{'} \text{ in}|T(0)|k_1k_2 \text{ in}>|^2\frac{1}{2E_{k_1}V}\frac{1}{2E_{k_2}V}. \end{aligned} \tag{6.214}$$

Remark that $d^3k/((2\pi)^3\sqrt{2E_k})$ is the Lorentz-invariant 3−dimensional measure. We also remark that in the limit $V \longrightarrow \infty$ this transition probability vanishes. In large volumes the interaction between the two initial particles has a less chance of happening at all. In order to increase the transition probability we increase the number of initial particles.

Reaction Rate and Cross Section: Let N_1 and N_2 be the number of initial particles of types 1 and 2 respectively. Clearly the number of transitions (collisions) per unit volume per unit time dN divided by the total number of pairs N_1N_2 is the transition probability per unit volume and per unit time $d\nu$. In other words

$$dN = N_1N_2d\nu. \tag{6.215}$$

This is also called the reaction rate.

The 4−vector density is defined by $J^\mu = \rho u^\mu$ where ρ is the density in the rest frame and u^μ is the 4−vector velocity, viz $u^0 = 1/\sqrt{1-v^2}$ and $u^i = v^i/\sqrt{1-v^2}$. Thus $J^0dx^1dx^2dx^3$ is the number of particles in the volume $dx^1dx^2dx^3$ while $J^1dx^2dx^3dx^0$ is the number of particles which cross the area dx^2dx^3 during a time dx^0. Clearly $J^i = J^0v^i$ with $v^i = k^i/E_k$. Using these definitions we have

$$N_1 = VJ_0^{(1)} \ , \ N_2 = VJ_0^{(2)}. \tag{6.216}$$

Thus

$$dN = V^2J_0^{(1)}J_0^{(2)}d\nu. \tag{6.217}$$

We introduce now the differential cross section by

$$dN = J_0^{(1)}J_0^{(2)}\frac{I}{E_{k_1}E_{k_2}}d\sigma. \tag{6.218}$$

The Lorentz-invariant factor I is defined by

$$I = \sqrt{(k_1k_2)^2 - k_1^2k_2^2}. \tag{6.219}$$

We compute

$$I = E_{k_1}E_{k_2}\sqrt{(\vec{v}_1-\vec{v}_2)^2+(\vec{v}_1\vec{v}_2)^2-\vec{v}_1^2\vec{v}_2^2}$$
$$= E_{k_1}E_{k_2}\sqrt{(\vec{v}_1-\vec{v}_2)^2-(\vec{v}_1\times\vec{v}_2)^2}. \quad (6.220)$$

The motivation behind this definition of the differential cross section goes as follows. Let us go to the lab reference frame. This is the reference frame in which $\vec{v_2}=0$. In other words it is the frame in which particles of type 1 play the role of incident beam while particles of type 2 play the role of target. In this case we get

$$dN = J_0^{(2)}|\vec{J}^{(1)}|d\sigma. \quad (6.221)$$

The number of incident particles per unit area normal to the beam per unit time is $|\vec{J}^{(1)}|$. Thus $|\vec{J}^{(1)}|d\sigma$ is the number of particles which cross $d\sigma$ per unit time. Since we have $J_0^{(2)}$ target particles per unit volume, the total number of transitions (collisions or scattering events) per unit volume per unit time is $J_0^{(2)}\times|\vec{J}^{(1)}|d\sigma$. We will usually write $d\sigma=(d\sigma/d\Omega)d\Omega$. Thus

$$dN = J_0^{(2)}|\vec{J}^{(1)}|\frac{d\sigma}{d\Omega}d\Omega. \quad (6.222)$$

Hence dN is the number of particles per unit volume per unit time scattered into the solid angle $d\Omega$. The differential cross section $d\sigma=(d\sigma/d\Omega)d\Omega$ is therefore the number of particles per unit volume per unit time scattered into the solid angle $d\Omega$ divided by the product of the incident flux density $|\vec{J}^{(1)}|$ and the target density $J_0^{(2)}$. From equations (6.217) and (6.218) we get

$$d\nu = Id\sigma\frac{1}{VE_{k_1}}\frac{1}{VE_{k_2}}. \quad (6.223)$$

By combining this last equation with (6.214) we obtain the result

$$d\sigma = (2\pi)^4\delta^4(P_\alpha-P_\beta)\frac{d^3k_1^{'}}{(2\pi)^3}\frac{1}{2E_{k_1^{'}}}...\frac{d^3k_N^{'}}{(2\pi)^3}\frac{1}{2E_{k_N^{'}}}|<k_1^{'}...k_N^{'}\ \text{in}|T(0)|k_1k_2\ \text{in}>|^2\frac{1}{4I}. \quad (6.224)$$

Fermi's Golden Rule: Let us consider the case $N=2$ (two particles in the final state) in the center of mass frame ($\vec{k}_1+\vec{k}_2=0$). We have

$$d\sigma = (2\pi)^4\delta^4(k_1+k_2-k_1^{'}-k_2^{'})\frac{d^3k_1^{'}}{(2\pi)^3}\frac{1}{2E_{k_1^{'}}}\frac{d^3k_2^{'}}{(2\pi)^3}\frac{1}{2E_{k_2^{'}}}|<k_1^{'}k_2^{'}\ \text{in}|T(0)|k_1k_2\ \text{in}>|^2\frac{1}{4I}$$
$$= (2\pi)^4\delta^3(\vec{k}_1^{'}+\vec{k}_2^{'})\delta(E_{k_1}+E_{k_2}-E_{k_1^{'}}-E_{k_2^{'}})\frac{d^3k_1^{'}}{(2\pi)^3}\frac{1}{2E_{k_1^{'}}}\frac{d^3k_2^{'}}{(2\pi)^3}\frac{1}{2E_{k_2^{'}}}|<k_1^{'}k_2^{'}\ \text{in}|T(0)|k_1k_2\ \text{in}>|^2\frac{1}{4I}. \quad (6.225)$$

The integral over $\vec{k}_2^{'}$ can be done. We obtain

$$d\sigma = \left[(2\pi)\delta(E_{k_1}+E_{k_2}-E_{k_1^{'}}-E_{k_2^{'}})\frac{d^3k_1^{'}}{(2\pi)^3}\frac{1}{2E_{k_1^{'}}}\frac{1}{2E_{k_2^{'}}}|<k_1^{'}k_2^{'}\ \text{in}|T(0)|k_1k_2\ \text{in}>|^2\frac{1}{4I}\right]_{\vec{k}_2=-\vec{k}_1}. \quad (6.226)$$

Since $E_{k_1^{'}}=\sqrt{\vec{k}_1^{'2}+m_1^{'2}}$ and $E_{k_2^{'}}=\sqrt{\vec{k}_1^{'2}+m_2^{'2}}$ we compute $E_{k_1^{'}}dE_{k_1^{'}}=E_{k_2^{'}}dE_{k_2^{'}}=k^{'}dk^{'}$ where $k^{'}=|\vec{k}_1^{'}|=|\vec{k}_2^{'}|$. Thus $E_{k_1^{'}}E_{k_2^{'}}d(E_{k_1^{'}}+E_{k_2^{'}})=(E_{k_1^{'}}+E_{k_2^{'}})k^{'}dk^{'}$. We have then

$$d^3k_1^{'}=k_1^{'2}dk_1^{'}d\Omega^{'}=k^{'}\frac{E_{k_1^{'}}E_{k_2^{'}}}{E_{k_1^{'}}+E_{k_2^{'}}}d(E_{k_1^{'}}+E_{k_2^{'}})d\Omega^{'}. \tag{6.227}$$

We get then the result

$$d\sigma = \frac{1}{64\pi^2}\frac{k^{'}}{I(E_{k_1}+E_{k_2})}d\Omega^{'}\bigg[|<k_1^{'}k_2^{'}\ \mathrm{in}|T(0)|k_1k_2\ \mathrm{in}>|^2\bigg]_{\vec{k_2}=-\vec{k}_1}. \tag{6.228}$$

In this equation $k^{'}$ should be thought of as a function of $E_{k_1}+E_{k_2}$ obtained by solving the equation $\sqrt{k^{'2}+m_1^{'2}}+\sqrt{k^{'2}+m_2^{'2}}=E_{k_1}+E_{k_2}$. In the center of mass system we have $I=E_{k_1}E_{k_2}|\vec{v}_1-\vec{v}_2|=E_{k_1}E_{k_2}(|\vec{v}_1|+|\vec{v}_2|)=(E_{k_1}+E_{k_2})k$ where $k=|\vec{k}_1|=|\vec{k}_2|$. Hence we get the final result (with $s=(E_{k_1}+E_{k_2})^2$ is the square of the center of mass energy)

$$d\sigma = \frac{1}{64\pi^2 s}\frac{k^{'}}{k}d\Omega^{'}\bigg[|<k_1^{'}k_2^{'}\ \mathrm{in}|T(0)|k_1k_2\ \mathrm{in}>|^2\bigg]_{\vec{k_2}=-\vec{k}_1}. \tag{6.229}$$

6.9 Tree Level Cross Sections: An Example

The tree level probability amplitude for the process $e^-+e^+\longrightarrow\mu^-+\mu^+$ was found to be given by

$$\begin{aligned}<\vec{p}_2s_2,\vec{q}_2r_2\ \mathrm{out}|\vec{p}_1s_1,\vec{q}_1r_1\ \mathrm{in}> &= \bigg(\bar{v}^{r_1}(q_1)(-ie\gamma_\mu)u^{s_1}(p_1)\bigg)\frac{-i\eta^{\mu\nu}}{(p_1+q_1)^2}\bigg(\bar{u}^{s_2}(p_2)(-ie\gamma_\nu)v^{r_2}(q_2)\bigg)\\ &\times\ (2\pi)^4\delta^4(q_1+p_1-p_2-q_2).\end{aligned} \tag{6.230}$$

From the definition (6.205) we deduce the $T-$matrix element (with $q=p_1+q_1$)

$$\begin{aligned}i<\vec{p}_2s_2,\vec{q}_2r_2\ \mathrm{in}|T(0)|\vec{p}_1s_1,\vec{q}_1r_1\ \mathrm{in}> &= \bigg(\bar{v}^{r_1}(q_1)(-ie\gamma_\mu)u^{s_1}(p_1)\bigg)\frac{-i\eta^{\mu\nu}}{q^2}\bigg(\bar{u}^{s_2}(p_2)(-ie\gamma_\nu)v^{r_2}(q_2)\bigg)\\ &= \frac{ie^2}{q^2}\bigg(\bar{v}^{r_1}(q_1)\gamma_\mu u^{s_1}(p_1)\bigg)\bigg(\bar{u}^{s_2}(p_2)\gamma^\mu v^{r_2}(q_2)\bigg).\end{aligned} \tag{6.231}$$

In the formula of the cross section we need the square of this matrix element. Recalling that $(\gamma^0)^2=1$, $(\gamma^i)^2=-1$, $(\gamma^0)^+=\gamma^0$, $(\gamma^i)^+=-\gamma^i$ we get $\bar{\psi}\gamma^\mu\chi=\bar{\chi}\gamma^\mu\psi$. Thus

$$\begin{aligned}|<\vec{p}_2s_2,\vec{q}_2r_2\ \mathrm{in}|T(0)|\vec{p}_1s_1,\vec{q}_1r_1\ \mathrm{in}>|^2 &= \frac{e^4}{(q^2)^2}\bigg(\bar{v}^{r_1}(q_1)\gamma_\mu u^{s_1}(p_1)\bigg)\bigg(\bar{u}^{s_2}(p_2)\gamma^\mu v^{r_2}(q_2)\bigg)\bigg(\bar{u}^{s_1}(p_1)\gamma_\nu v^{r_1}(q_1)\bigg)\\ &\times\ \bigg(\bar{v}^{r_2}(q_2)\gamma^\nu u^{s_2}(p_2)\bigg).\end{aligned} \tag{6.232}$$

Unpolarized Cross Section: The first possibility which is motivated by experimental considerations is to compute the cross section of the process $e^-+e^+\longrightarrow\mu^-+\mu^+$ for unpolarized initial and final spin states. In a real experiment initial spin states are prepared and so unpolarized initial spin states means taking an average over the initial spins s_1 and r_1 of the electron and positron beams. The final spin states are the output in any real experiment and thus unpolarized final spin states means summing over all possible final spin states s_2 and r_2 of the muon and antimuon. This is equivalent to saying that the detectors do not care to measure the spins of the final particles. So we really want to compute

$$\frac{1}{2}\sum_{s_1}\frac{1}{2}\sum_{r_1}\sum_{s_2}\sum_{r_2}|<\vec{p}_2s_2,\vec{q}_2r_2\text{ in}|T(0)|\vec{p}_1s_1,\vec{q}_1r_1\text{ in}>|^2. \tag{6.233}$$

We have explicitly

$$\begin{aligned}\frac{1}{2}\sum_{s_1}\frac{1}{2}\sum_{r_1}\sum_{s_2}\sum_{r_2}|<\vec{p}_2s_2,\vec{q}_2r_2\text{ in}|T(0)|\vec{p}_1s_1,\vec{q}_1r_1\text{ in}>|^2 &= \frac{e^4}{4(q^2)^2}(\gamma_\mu)_{\alpha_1\beta_1}(\gamma^\mu)_{\alpha_2\beta_2}(\gamma_\nu)_{\rho_1\gamma_1}(\gamma^\nu)_{\rho_2\gamma_2}\\ &\times \sum_{s_1}u^{s_1}_{\beta_1}(p_1)\bar{u}^{s_1}_{\rho_1}(p_1)\sum_{r_1}v^{r_1}_{\gamma_1}(q_1)\bar{v}^{r_1}_{\alpha_1}(q_1)\\ &\times \sum_{s_2}u^{s_2}_{\gamma_2}(p_2)\bar{u}^{s_2}_{\alpha_2}(p_2)\sum_{r_2}v^{r_2}_{\beta_2}(q_2)\bar{v}^{r_2}_{\rho_2}(q_2).\end{aligned} \tag{6.234}$$

We recall the identities $\sum_s u^s_\alpha(p)\bar{u}^s_\beta(p)=(\gamma.p+m)_{\alpha\beta}$ and $\sum_s v^s_\alpha(p)\bar{v}^s_\beta(p)=(\gamma.p-m)_{\alpha\beta}$. We get then

$$\begin{aligned}\frac{1}{2}\sum_{s_1}\frac{1}{2}\sum_{r_1}\sum_{s_2}\sum_{r_2}|<\vec{p}_2s_2,\vec{q}_2r_2\text{ in}|T(0)|\vec{p}_1s_1,\vec{q}_1r_1\text{ in}>|^2 &= \frac{e^4}{4(q^2)^2}\text{tr}\gamma_\mu(\gamma.p_1+m_e)\gamma_\nu(\gamma.q_1-m_e)\\ &\times \text{tr}\gamma^\mu(\gamma.q_2-m_\mu)\gamma^\nu(\gamma.p_2+m_\mu).\end{aligned} \tag{6.235}$$

We can easily compute

$$\begin{aligned}&\text{tr}\gamma^\mu=0\\ &\text{tr}\gamma^\mu\gamma^\nu=4\eta^{\mu\nu}\\ &\text{tr}\gamma^\mu\gamma^\nu\gamma^\rho=0\\ &\text{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma=4\bigg(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}\bigg).\end{aligned} \tag{6.236}$$

Using these identities we calculate

$$\text{tr}\gamma_\mu(\gamma.p_1+m_e)\gamma_\nu(\gamma.q_1-m_e)=4p_{1\mu}q_{1\nu}+4p_{1\nu}q_{1\mu}-4\eta_{\mu\nu}p_1.q_1-4\eta_{\mu\nu}m_e^2. \tag{6.237}$$

$$\text{tr}\gamma^\mu(\gamma.q_2-m_\mu)\gamma^\nu(\gamma.p_2+m_\mu)=4p_2^\mu q_2^\nu+4p_2^\nu q_2^\mu-4\eta^{\mu\nu}p_2.q_2-4\eta^{\mu\nu}m_\mu^2. \tag{6.238}$$

We get then

$$\begin{aligned}\frac{1}{2}\sum_{s_1}\frac{1}{2}\sum_{r_1}\sum_{s_2}\sum_{r_2}|<\vec{p}_2s_2,\vec{q}_2r_2\text{ in}|T(0)|\vec{p}_1s_1,\vec{q}_1r_1\text{ in}>|^2 &= \frac{8e^4}{(q^2)^2}\bigg((p_1p_2)(q_1q_2)+(p_1q_2)(q_1p_2)+m_\mu^2p_1q_1\\ &+ m_e^2p_2q_2+2m_\mu^2m_e^2\bigg).\end{aligned} \tag{6.239}$$

Since we are assuming that $m_e<<m_\mu$ we obtain

$$\frac{1}{2}\sum_{s_1}\frac{1}{2}\sum_{r_1}\sum_{s_2}\sum_{r_2}|<\vec{p}_2s_2,\vec{q}_2r_2\text{ in}|T(0)|\vec{p}_1s_1,\vec{q}_1r_1\text{ in}>|^2 = \frac{8e^4}{(q^2)^2}\bigg((p_1p_2)(q_1q_2)+(p_1q_2)(q_1p_2)+m_\mu^2p_1q_1\bigg). \tag{6.240}$$

In the center of mass system we have $\vec{p}_1=-\vec{q}_1=\vec{k}$ and $\vec{p}_2=-\vec{q}_2=\vec{k}^{'}$. We compute $p_1p_2=q_1q_2=\sqrt{m_e^2+\vec{k}^2}\sqrt{m_\mu^2+\vec{k}^{'2}}-\vec{k}\vec{k}^{'}$ and $p_1q_2=q_1p_2=\sqrt{m_e^2+\vec{k}^2}\sqrt{m_\mu^2+\vec{k}^{'2}}+\vec{k}\vec{k}^{'}$. Thus by dropping terms proportional to m_e^2 we obtain

$$\begin{aligned}(p_1p_2)(q_1q_2)+(p_1q_2)(q_1p_2)+m_\mu^2p_1q_1 &= 2(\vec{k}\vec{k}^{'})^2+2\vec{k}^2\vec{k}^{'2}+4m_\mu^2\vec{k}^2\\ &= 2\vec{k}^2\vec{k}^{'2}\cos^2\theta+2\vec{k}^2\vec{k}^{'2}+4m_\mu^2\vec{k}^2.\end{aligned} \quad (6.241)$$

Conservation of energy reads in this case $2\sqrt{\vec{k}^{'2}+m_\mu^2}=2\sqrt{\vec{k}^2+m_e^2}$. Hence we must have $\vec{k}^{'2}=\vec{k}^2-m_\mu^2$ and as a consequence we get

$$(p_1p_2)(q_1q_2)+(p_1q_2)(q_1p_2)+m_\mu^2p_1q_1 = 2(\vec{k}^2)^2\bigg(1+\frac{m_\mu^2}{\vec{k}^2}+(1-\frac{m_\mu^2}{\vec{k}^2})\cos^2\theta\bigg). \quad (6.242)$$

Since $q^2=4\vec{k}^2$ we get the result

$$\frac{1}{2}\sum_{s_1}\frac{1}{2}\sum_{r_1}\sum_{s_2}\sum_{r_2}|<\vec{p}_2s_2,\vec{q}_2r_2\text{ in}|T(0)|\vec{p}_1s_1,\vec{q}_1r_1\text{ in}>|^2 = e^4\bigg(1+\frac{m_\mu^2}{\vec{k}^2}+(1-\frac{m_\mu^2}{\vec{k}^2})\cos^2\theta\bigg). \quad (6.243)$$

The differential cross section (6.229) becomes (with $\alpha=e^2/4\pi$)

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{4s}\sqrt{1-\frac{m_\mu^2}{\vec{k}^2}}\bigg(1+\frac{m_\mu^2}{\vec{k}^2}+(1-\frac{m_\mu^2}{\vec{k}^2})\cos^2\theta\bigg). \quad (6.244)$$

The high energy limit of this equation ($m_\mu<<|\vec{k}|$) reads

$$\frac{d\sigma}{d\Omega}=\frac{\alpha^2}{4s}\bigg(1+\cos^2\theta\bigg). \quad (6.245)$$

Polarized Cross Section: We can also compute the polarized cross section of the process $e^-+e^+\longrightarrow\mu^-+\mu^+$ as follows. It is customary to quantize the spin along the direction of motion of the particle. In this case the spin states are referred to as helicity states. Since we are assuming that $m_e<<m_\mu$ which is equivalent to treating the electron and positron as massless particles the left-handed and right-handed helicity states of the electron and the positron will be completely independent. They provide independent representations of the Lorentz group. In the high energy limit where we can assume that $m_\mu<<|\vec{k}|$ the muon and antimuon too behave as if they are massless particles and as a consequence the corresponding left-handed and right-handed helicity states will also be independent.

We recall the definition of the spinors u and v given by

$$u^s=\begin{pmatrix}\sqrt{\sigma_\mu p^\mu}\xi^s\\ \sqrt{\bar{\sigma}_\mu p^\mu}\xi^s\end{pmatrix}\ ,\ v^s=\begin{pmatrix}\sqrt{\sigma_\mu p^\mu}\eta^s\\ -\sqrt{\bar{\sigma}_\mu p^\mu}\eta^s\end{pmatrix}. \quad (6.246)$$

In the limit of high energy we have $\sigma_\mu p^\mu=E-\vec{\sigma}\vec{p}\simeq 2E\sigma$ where σ is the two-dimensional projection operator $\sigma=(1-\vec{\sigma}\hat{p})/2$ with $\hat{p}=\vec{p}/|\vec{p}|$. Indeed we can check that σ is an idempotent, viz $\sigma^2=\sigma$. Similarly we have in the high energy limit $\bar{\sigma}_\mu p^\mu=E+\vec{\sigma}\vec{p}\simeq 2E\bar{\sigma}$ where $\bar{\sigma}$ is the two-dimensional projection operator $\bar{\sigma}=(1+\vec{\sigma}\hat{p})/2$. Thus we find that

$$u^s=\sqrt{2E}\begin{pmatrix}\sigma\xi^s\\ \bar{\sigma}\xi^s\end{pmatrix}\ ,\ v^s=\sqrt{2E}\begin{pmatrix}\sigma\eta^s\\ -\bar{\sigma}\eta^s\end{pmatrix}. \quad (6.247)$$

The spinors $\xi_R=\bar{\sigma}\xi^s$ and $\eta_R=\bar{\sigma}\eta^s$ are right-handed spinors in the sense that $\vec{\sigma}\hat{p}\xi_R=\xi_R$ and $\vec{\sigma}\hat{p}\eta_R=\eta_R$ whereas $\xi_L=\sigma\xi^s$ and $\eta_L=\sigma\eta^s$ are left-handed spinors in the sense that $\vec{\sigma}\hat{p}\xi_L=-\xi_L$ and $\vec{\sigma}\hat{p}\eta_L=-\eta_L$. We introduce the four-dimensional projection operators onto the right-handed and left-handed sectors respectively by

$$P_R=\frac{1+\gamma^5}{2}=\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix}\;,\;P_L=\frac{1-\gamma^5}{2}=\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}. \tag{6.248}$$

Indeed we compute

$$u_R=P_Ru^s=\sqrt{2E}\begin{pmatrix} 0\\ \xi_R\end{pmatrix}\;,\;v_R=P_Rv^s=\sqrt{2E}\begin{pmatrix} 0\\ -\eta_R\end{pmatrix}. \tag{6.249}$$

$$u_L=P_Lu^s=\sqrt{2E}\begin{pmatrix} \xi_L\\ 0\end{pmatrix}\;,\;v_L=P_Lv^s=\sqrt{2E}\begin{pmatrix} \eta_L\\ 0\end{pmatrix}. \tag{6.250}$$

Now we go back to the probability amplitude

$$i<\vec{p}_2s_2,\vec{q}_2r_2\;\text{in}|T(0)|\vec{p}_1s_1,\vec{q}_1r_1\;\text{in}> \;=\; \frac{ie^2}{q^2}\bigg(\bar{v}^{r_1}(q_1)\gamma_\mu u^{s_1}(p_1)\bigg)\bigg(\bar{u}^{s_2}(p_2)\gamma^\mu v^{r_2}(q_2)\bigg). \tag{6.251}$$

We compute using $u^s=u_L+u_R$ and $v^r=v_L+v_R$ for any s and r that

$$\begin{aligned}\bar{v}^{r_1}(q_1)\gamma_\mu u^{s_1}(p_1) &= v_L^+(q_1)\gamma^0\gamma_\mu u_L(p_1)+v_R^+(q_1)\gamma^0\gamma_\mu u_R(p_1)\\ &= \bar{v}_R(q_1)\gamma_\mu u_L(p_1)+\bar{v}_L(q_1)\gamma_\mu u_R(p_1).\end{aligned} \tag{6.252}$$

In the above equation we have used the fact that $v_L^+\gamma^0=\bar{v}_R$ and $v_R^+\gamma^0=\bar{v}_L$. In other words left-handed spinor v corresponds to a right-handed positron while right-handed spinor v corresponds to left-handed positron. This is related to the general result that particles and antiparticles have opposite handedness. The probability amplitude becomes then

$$\begin{aligned}i<\vec{p}_2s_2,\vec{q}_2r_2\;\text{in}|T(0)|\vec{p}_1s_1,\vec{q}_1r_1\;\text{in}> &= \frac{ie^2}{q^2}\bigg(\bar{v}_R(q_1)\gamma_\mu u_L(p_1)\bigg)\bigg(\bar{u}_R(p_2)\gamma^\mu v_L(q_2)\bigg)\\ &+ \frac{ie^2}{q^2}\bigg(\bar{v}_R(q_1)\gamma_\mu u_L(p_1)\bigg)\bigg(\bar{u}_L(p_2)\gamma^\mu v_R(q_2)\bigg)\\ &+ \frac{ie^2}{q^2}\bigg(\bar{v}_L(q_1)\gamma_\mu u_R(p_1)\bigg)\bigg(\bar{u}_R(p_2)\gamma^\mu v_L(q_2)\bigg)\\ &+ \frac{ie^2}{q^2}\bigg(\bar{v}_L(q_1)\gamma_\mu u_R(p_1)\bigg)\bigg(\bar{u}_L(p_2)\gamma^\mu v_R(q_2)\bigg)\end{aligned} \tag{6.253}$$

The four terms correspond to the four processes

$$\begin{aligned}&e_L^-+e_R^+\longrightarrow\mu_L^-+\mu_R^+\\ &e_L^-+e_R^+\longrightarrow\mu_R^-+\mu_L^+\\ &e_R^-+e_L^+\longrightarrow\mu_L^-+\mu_R^+\\ &e_R^-+e_L^+\longrightarrow\mu_R^-+\mu_L^+.\end{aligned} \tag{6.254}$$

In the square of the above $T-$matrix element there will be 16 terms. Since left-handed and right-handed spinors are orthogonal to each other most of these 16 terms will be zero except the

4 terms corresponding to the above 4 processes. In a sense the above 4 processes are mutually exclusive and so there is no interference between them. We have then

$$\begin{aligned} |<\vec{p}_2 s_2, \vec{q}_2 r_2 \text{ in}|T(0)|\vec{p}_1 s_1, \vec{q}_1 r_1 \text{ in}>|^2 &= \frac{e^4}{(q^2)^2}|\bar{v}_R(q_1)\gamma_\mu u_L(p_1).\bar{u}_R(p_2)\gamma^\mu v_L(q_2)|^2 \\ &+ \frac{e^4}{(q^2)^2}|\bar{v}_R(q_1)\gamma_\mu u_L(p_1).\bar{u}_L(p_2)\gamma^\mu v_R(q_2)|^2 \\ &+ \frac{e^4}{(q^2)^2}|\bar{v}_L(q_1)\gamma_\mu u_R(p_1).\bar{u}_R(p_2)\gamma^\mu v_L(q_2)|^2 \\ &+ \frac{e^4}{(q^2)^2}|\bar{v}_L(q_1)\gamma_\mu u_R(p_1).\bar{u}_L(p_2)\gamma^\mu v_R(q_2)|^2 \quad (6.255) \end{aligned}$$

From now on we will concentrate only on the first term since the others are similar. We have

$$\begin{aligned} \sum_{\text{spins}}(\bar{v}_R(q_1)\gamma_\mu u_L(p_1))(\bar{v}_R(q_1)\gamma_\nu u_L(p_1))^* &= \sum_{\text{spins}}(\bar{v}(q_1)\gamma_\mu\frac{1-\gamma_5}{2}u(p_1)).(\bar{v}(q_1)\gamma_\nu\frac{1-\gamma_5}{2}u(p_1))^* \\ &= \sum_{s_1,r_1}\bar{v}^{r_1}_{\alpha_1}(q_1)(\gamma_\mu\frac{1-\gamma_5}{2})_{\alpha_1\beta_1}u^{s_1}_{\beta_1}(p_1).\bar{u}^{s_1}_{\gamma_1}(p_1)(\gamma_\nu\frac{1-\gamma_5}{2})_{\gamma_1\delta_1}v^{r_1}_{\delta_1}(q_1) \\ &= \text{tr}\gamma_\mu\frac{1-\gamma_5}{2}(\gamma.p_1)\gamma_\nu\frac{1-\gamma_5}{2}(\gamma q_1). \quad (6.256) \end{aligned}$$

$$\begin{aligned} \sum_{\text{spins}}(\bar{u}_R(p_2)\gamma^\mu v_L(q_2))(\bar{u}_R(p_2)\gamma^\nu v_L(q_2))^* &= \sum_{\text{spins}}(\bar{u}(p_2)\gamma^\mu\frac{1-\gamma_5}{2}v(q_2)).(\bar{u}(p_2)\gamma^\nu\frac{1-\gamma_5}{2}v(q_2))^* \\ &= \sum_{s_2,r_2}\bar{u}^{s_2}_{\alpha_2}(p_2)(\gamma^\mu\frac{1-\gamma_5}{2})_{\alpha_2\beta_2}v^{r_2}_{\beta_2}(q_2).\bar{v}^{r_2}_{\gamma_2}(q_2)(\gamma^\nu\frac{1-\gamma_5}{2})_{\gamma_2\delta_2}u^{s_2}_{\delta_2}(p_2) \\ &= \text{tr}\gamma^\mu\frac{1-\gamma_5}{2}(\gamma.q_2)\gamma^\nu\frac{1-\gamma_5}{2}(\gamma p_2). \quad (6.257) \end{aligned}$$

From the above two results it is obvious that all 12 interference terms in the square of the $T-$matrix element $<\vec{p}_2 s_2, \vec{q}_2 r_2 \text{ in}|T(0)|\vec{p}_1 s_1, \vec{q}_1 r_1 \text{ in}>$ will indeed vanish because they will involve traces of products of gamma matrices with one factor equal $(1+\gamma_5)/2$ and one factor equal $(1-\gamma_5)/2$.

Next we will use the results

$$\text{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5 = -4i\epsilon^{\mu\nu\rho\sigma}. \quad (6.258)$$

$$\epsilon_{\mu\rho\nu\sigma}\epsilon^{\mu\rho^{'}\nu\sigma^{'}} = -2(\eta_\rho^{\rho^{'}}\eta_\sigma^{\sigma^{'}} - \eta_\rho^{\sigma^{'}}\eta_\sigma^{\rho^{'}}). \quad (6.259)$$

We compute

$$\sum_{\text{spins}}(\bar{v}_R(q_1)\gamma_\mu u_L(p_1))(\bar{v}_R(q_1)\gamma_\nu u_L(p_1))^* = 2\bigg(p_{1\mu}q_{1\nu} + p_{1\nu}q_{1\mu} - \eta_{\mu\nu}p_1q_1 - i\epsilon_{\mu\rho\nu\sigma}p_1^\rho q_1^\sigma\bigg) \quad (6.260)$$

$$\sum_{\text{spins}}(\bar{u}_R(p_2)\gamma^\mu v_L(q_2))(\bar{u}_R(p_2)\gamma^\nu v_L(q_2))^* = 2\bigg(q_2^\mu p_2^\nu + q_2^\nu p_2^\mu - \eta^{\mu\nu}q_2p_2 - i\epsilon^{\mu\rho\nu\sigma}q_{2\rho}p_{2\sigma}\bigg) \quad (6.261)$$

Hence

$$\frac{e^4}{(q^2)^2}|\bar{v}_R(q_1)\gamma_\mu u_L(p_1).\bar{u}_R(p_2)\gamma^\mu v_L(q_2)|^2 = \frac{16e^4}{(q^2)^2}(p_1q_2)(q_1p_2) = e^4(1+\cos\theta)^2. \quad (6.262)$$

The last line is in the center of mass system. The corresponding cross section of the process $e_L^- + e_R^+ \longrightarrow \mu_L^- + \mu_R^+$ is

$$\frac{d\sigma}{d\Omega}(e_L^- + e_R^+ \longrightarrow \mu_L^- + \mu_R^+) = \frac{\alpha^2}{4s}\bigg(1+\cos\theta\bigg)^2. \quad (6.263)$$

The other polarized cross sections are

$$\frac{d\sigma}{d\Omega}(e_L^- + e_R^+ \longrightarrow \mu_R^- + \mu_L^+) = \frac{\alpha^2}{4s}\bigg(1-\cos\theta\bigg)^2. \quad (6.264)$$

$$\frac{d\sigma}{d\Omega}(e_R^- + e_L^+ \longrightarrow \mu_L^- + \mu_R^+) = \frac{\alpha^2}{4s}\bigg(1-\cos\theta\bigg)^2. \quad (6.265)$$

$$\frac{d\sigma}{d\Omega}(e_R^- + e_L^+ \longrightarrow \mu_R^- + \mu_L^+) = \frac{\alpha^2}{4s}\bigg(1+\cos\theta\bigg)^2. \quad (6.266)$$

The average of these four polarized cross sections obtained by taking their sum and then dividing by the number of initial polarization states (2×2) gives precisely the unpolarized cross section calculated previously.

6.10 Exercises and Problems

The LSZ Reduction Formulas for Fermions

- Verify equations (6.75) and (6.76).
- Verify equations (6.81) and (6.82).
- Prove the LSZ reduction formulas (6.85)-(6.88) for one fermion operator.

The LSZ Reduction Formulas for Photons

- Write down the electromagnetic field operator in the limits $t \longrightarrow \pm\infty$ where it is assumed that the QED interaction vanishes.
- Express the creation and annihilation operators $\hat{a}^+_{\rm in}(k,\lambda)$, $\hat{a}^+_{\rm out}(k,\lambda)$ and $\hat{a}_{\rm in}(k,\lambda)$, $\hat{a}_{\rm out}(k,\lambda)$ in terms of the field operators $\hat{A}_{\mu,{\rm in}}(t,\vec{p})$ and $\hat{A}_{\mu,{\rm out}}(t,p)$ defined by

$$\hat{A}_\mu(t,\vec{k}) = \int d^3x \hat{A}_\mu(x)\ e^{-i\vec{k}\vec{x}}. \quad (6.267)$$

- Prove the LSZ reduction formulas (6.179) and (6.180) for zero photon operator.

Wick's Theorem

- Verify equation (6.36).
- Check equations (6.46),(6.47) and (6.52) .
- Verify explicitly that

$$\frac{i^6}{6!}\int d^4x_1\int d^4x_1^{'}...\int d^4x_3\int d^4x_3^{'} < 0|T(L(x_1)L(x_1^{'})...L(x_3)L(x_3^{'}))|0 >= (\frac{i^3}{3!})^2\int d^4x_1\int d^4x_1^{'}... \int d^4x_3\int d^4x_3^{'} < 0|T(\bar{\eta}(x_1)\hat{\psi}_{\rm in}(x_1)...\bar{\eta}(x_3)\hat{\psi}_{\rm in}(x_3).\bar{\hat{\psi}}_{\rm in}(x_1^{'})\eta(x_1^{'})...\bar{\hat{\psi}}_{\rm in}(x_3^{'})\eta(x_3^{'}))|0 > . \tag{6.268}$$

In this expression $L(x)$ is given by the expression $L(x) = \bar{\eta}(x)\hat{\psi}_{\rm in}(x) + \bar{\hat{\psi}}_{\rm in}(x)\eta(x)$.

- Use Wick's theorem (6.62) to derive the $2-$, $4-$ and $6-$point free fermion correlators.
- Verify equation (6.149).
- Verify equation (6.168).
- Verify that equation (6.133) leads to equation (6.134).

Interaction Picture

- Write down the equation relating the Schrödinger and interaction fields.
- Write down the equation relating the Heisenberg and interaction fields.
- Show that the interaction fields ψ_I and A_I^{μ} are free fields.

Gell-Mann Low Formula

- Show the Gell-Mann Low formula

$$\hat{\psi}(x) = S^{-1}T\bigg(\hat{\psi}_{\rm in}(x)S\bigg). \tag{6.269}$$

- Express $\hat{\psi}(x)\hat{\psi}(y)$ in terms of $\hat{\psi}_{\rm in}(x)\hat{\psi}_{\rm in}(y)$.

Energy-Momentum Conservation

- Solve the equation

$$[P_{\mu}, T(x)] = -i\partial_{\mu}T(x). \tag{6.270}$$

- Show that

$$< \beta \text{ out}|\alpha \text{ in} > = i(2\pi)^4\delta^4(P_{\alpha} - P_{\beta}) < \beta \text{ in}|T(0)|\alpha \text{ in} > . \tag{6.271}$$

- Show that

$$\begin{aligned} I &= \sqrt{(k_1k_2)^2 - k_1^2k_2^2} \\ &= E_{k_1}E_{k_2}\sqrt{(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2}. \end{aligned} \tag{6.272}$$

Trace Technology:

- Show that

$$
\begin{aligned}
&\mathrm{tr}\gamma^\mu = 0\\
&\mathrm{tr}\gamma^\mu\gamma^\nu = 4\eta^{\mu\nu}\\
&\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho = 0\\
&\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4\bigg(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\bigg).
\end{aligned} \tag{6.273}
$$

- Show that

$$\gamma^5 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma. \tag{6.274}$$

$$\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = -4!. \tag{6.275}$$

- Show that

$$
\begin{aligned}
&\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^5 = 0\\
&\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5 = -4i\epsilon^{\mu\nu\rho\sigma}.
\end{aligned} \tag{6.276}
$$

Compton Scattering:

- The probability amplitude of the process $\gamma + e^- \longrightarrow \gamma + e^-$ is given by

$$
\begin{aligned}
< \vec{p}_2 s_2, \vec{k}_2\lambda_2 \text{ out}|\vec{p}_1 s_1, \vec{k}_1\lambda_1 \text{ in} > \;&=\; (-ie)^2\epsilon_{\lambda_1}^{\mu_1}(k_1)\bigg[\bar{u}^{s_2}(p_2)\gamma_{\mu_1}S(-k_1+p_2)\gamma_{\mu_2}u^{s_1}(p_1)\bigg]\epsilon_{\lambda_2}^{\mu_2}(k_2)\\
&\times\; (2\pi)^4\delta^4(k_2+p_2-k_1-p_1)\\
&+\; (-ie)^2\epsilon_{\lambda_1}^{\mu_1}(k_1)\bigg[\bar{u}^{s_2}(p_2)\gamma_{\mu_2}S(k_2+p_2)\gamma_{\mu_1}u^{s_1}(p_1)\bigg]\epsilon_{\lambda_2}^{\mu_2}(k_2)\\
&\times\; (2\pi)^4\delta^4(k_2+p_2-k_1-p_1).
\end{aligned} \tag{6.277}
$$

Derive the corresponding unpolarized cross section (Klein-Nishina formula).

7

Renormalization of QED

7.1 Example III: $e^- + \mu^- \longrightarrow e^- + \mu^-$

The most important one-loop correction to the probability amplitude of the process $e^- + e^+ \longrightarrow \mu^- + \mu^+$ is given by the Feynamn diagram RAD2. This is known as the vertex correction as it gives quantum correction to the QED interaction vertex $-ie\gamma^\mu$. It has profound observable measurable physical consequences. For example it will lead among other things to the infamous anomalous magnetic moment of the electron. This is a generic effect. Indeed vertex correction should appear in all electromagnetic processes.

Let us consider here as an example the different process

$$e^-(p) + \mu^-(k) \longrightarrow e^-(p^{'}) + \mu^-(k^{'}). \tag{7.1}$$

This is related to the process $e^- + e^+ \longrightarrow \mu^- + \mu^+$ by the so-called crossing symmetry or substitution law. Remark that the incoming positron became the outgoing electron and the outgoing antimuon became the incoming muon. The substitution law is essentially the statement that the probability amplitudes of these two processes can be obtained from the same Green's function. Instead of following this route we will simply use Feynman rules to write down the probability amplitude of the above process of electron scattering from a heavy particle which is here the muon.

For vertex correction we will need to add the probability amplitudes of the three Feynamn diagrams VERTEX. The tree level contribution (first graph) is (with $q = p - p^{'}$ and $l^{'} = l - q$)

$$(2\pi)^4\delta^4(k + p - k^{'} - p^{'})\frac{ie^2}{q^2}(\bar{u}^{s^{'}}(p^{'})\gamma^\mu u^s(p))(\bar{u}^{r^{'}}(k^{'})\gamma_\mu u^r(k)). \tag{7.2}$$

The electron vertex correction (the second graph) is

$$(2\pi)^4\delta^4(k + p - k^{'} - p^{'})\frac{-e^4}{q^2}\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l-p)^2 + i\epsilon}\bigg(\bar{u}^{s^{'}}(p^{'})\gamma^\lambda\frac{i(\gamma.l^{'} + m_e)}{l'^2 - m_e^2 + i\epsilon}\gamma^\mu\frac{i(\gamma.l + m_e)}{l^2 - m_e^2 + i\epsilon}\gamma_\lambda u^s(p)\bigg)$$
$$\times \quad (\bar{u}^{r^{'}}(k^{'})\gamma_\mu u^r(k)). \tag{7.3}$$

The muon vertex correction (the third graph) is similar to the electron vertex correction but since it will be neglected in the limit $m_\mu \longrightarrow \infty$ we will not write down here.

Adding the three diagrams together we obtain

$$(2\pi)^4\delta^4(k+p-k^{'}-p^{'})\frac{ie^2}{q^2}(\bar{u}^{s^{'}}(p^{'})\Gamma^\mu(p^{'},p)u^s(p))(\bar{u}^{r^{'}}(k^{'})\gamma_\mu u^r(k)). \quad (7.4)$$

This is the same as the tree level term with an effective vertex $-ie\Gamma^\mu(p^{'},p)$ where $\Gamma^\mu(p^{'},p)$ is given by

$$\Gamma^\mu(p^{'},p) = \gamma^\mu + ie^2\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l-p)^2+i\epsilon}\bigg(\gamma^\lambda\frac{i(\gamma.l^{'}+m_e)}{l'^2-m_e^2+i\epsilon}\gamma^\mu\frac{i(\gamma.l+m_e)}{l^2-m_e^2+i\epsilon}\gamma_\lambda\bigg). \quad (7.5)$$

If we did not take the limit $m_\mu \longrightarrow \infty$ the muon vertex would have also been corrected in the same fashion.

The corrections to external legs are given by the four diagrams WAVEFUNCTION. We only write explicitly the first of these diagrams. This is given by

$$(2\pi)^4\delta^4(k+p-k^{'}-p^{'})\frac{e^4}{q^2}\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l-p)^2+i\epsilon}(\bar{u}^{s^{'}}(p^{'})\gamma^\mu\frac{\gamma.p+m_e}{p^2-m_e^2}\gamma^\lambda\frac{\gamma.l+m_e}{l^2-m_e^2}\gamma_\lambda u^s(p))(\bar{u}^{r^{'}}(k^{'})\gamma_\mu u^r(k)). \quad (7.6)$$

The last diagram contributing to the one-loop radiative corrections is the vacuum polarization diagram shown on figure PHOTONVACUUM. It is given by

$$(2\pi)^4\delta^4(k+p-k^{'}-p^{'})\frac{ie^2}{(q^2)^2}(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\Pi_2^{\mu\nu}(q)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)). \quad (7.7)$$

$$i\Pi_2^{\mu\nu}(q) = (-1)\int\frac{d^4k}{(2\pi)^4}\mathrm{tr}(-ie\gamma^\mu)\frac{i(\gamma.k+m_e)}{k^2-m_e^2+i\epsilon}(-ie\gamma^\nu)\frac{i(\gamma.(k+q)+m_e)}{(k+q)^2-m_e^2+i\epsilon}. \quad (7.8)$$

7.2 Example IV : Scattering From External Electromagnetic Fields

We will now consider the problem of scattering of electrons from a fixed external electromagnetic field A_μ^{backgr}, viz

$$e^-(p) \longrightarrow e^-(p^{'}). \quad (7.9)$$

The transfer momentum which is here $q = p^{'} - p$ is taken by the background electromagnetic field A_μ^{backgr}. Besides this background field there will also be a fluctuating quantum electromagnetic field A_μ as usual. This means in particular that the interaction Lagrangian is of the form

$$\mathcal{L}_{\mathrm{in}} = -e\bar{\hat{\psi}}_{\mathrm{in}}\gamma_\mu\hat{\psi}_{\mathrm{in}}(\hat{A}^\mu + A^{\mu,\mathrm{backgr}}). \quad (7.10)$$

The initial and final states in this case are given by

$$|\vec{p},s\ \mathrm{in}>=\sqrt{2E_{\vec{p}}}\ \hat{b}_{\mathrm{in}}(\vec{p},s)^+|0\ \mathrm{in}> . \quad (7.11)$$

$$|\vec{p}^{'},s^{'}\ \mathrm{out}>=\sqrt{2E_{\vec{p}^{'}}}\ \hat{b}_{\mathrm{out}}(\vec{p}^{'},s^{'})^+|0\ \mathrm{out}> . \quad (7.12)$$

The probability amplitude after reducing the initial and final electron states using the appropriate reduction formulas is given by

$$< \vec{p}^{'} s^{'} \text{ out}|\vec{p}s \text{ in} > \;=\; -\Big[\bar{u}^{s^{'}}(p^{'})(\gamma.p^{'} - m_e)\Big]_{\alpha^{'}} G_{\alpha^{'}\alpha}(-p^{'},p)\Big[(\gamma.p - m_e)u^s(p)\Big]_{\alpha}. \quad (7.13)$$

Here $G_{\alpha^{'}\alpha}(p^{'},p)$ is the Fourier transform of the 2−point Green's function $< 0 \text{ out}|T(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_{\alpha}(x))|0 \text{ in} >$, viz

$$< 0 \text{ out}|T\bigg(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_{\alpha}(x)\bigg)|0 \text{ in} > \;=\; \int \frac{d^4p^{'}}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}G_{\alpha^{'},\alpha}(p^{'},p)\; e^{ipx+ip^{'}x^{'}}. \quad (7.14)$$

By using the Gell-Mann Low formula we get

$$< 0 \text{ out}|T\bigg(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_{\alpha}(x)\bigg)|0 \text{ in} > \;=\; < 0 \text{ in}|T\bigg(\hat{\psi}_{\alpha^{'},\text{in}}(x^{'})\bar{\hat{\psi}}_{\alpha,\text{in}}(x)S\bigg)|0 \text{ in} > . \quad (7.15)$$

Now we use Wick's theorem. The first term in S leads 0. The second term in S leads to the contribution

$$\begin{aligned}
i\int d^4z < 0 \text{ in}|T\bigg(\hat{\psi}_{\alpha^{'},\text{in}}(x^{'})\bar{\hat{\psi}}_{\alpha,\text{in}}(x)\mathcal{L}_{\text{in}}(z)\bigg)|0 \text{ in} > \;&=\; (-ie)\int d^4z < 0 \text{ in}|T\bigg(\hat{\psi}_{\alpha^{'},\text{in}}(x^{'})\bar{\hat{\psi}}_{\alpha,\text{in}}(x).\bar{\hat{\psi}}_{\text{in}}(z)\gamma_{\mu}\\
&\times\;\; \hat{\psi}_{\text{in}}(z)\bigg)|0 \text{ in} > A^{\mu,\text{backgr}}(z)\\
&=\; (-ie)\int d^4z\bigg(S_F(x^{'}-z)\gamma_{\mu}S_F(z-x)\bigg)^{\alpha^{'}\alpha}A^{\mu,\text{backgr}}(z)\\
&=\; (-ie)\int\frac{d^4p^{'}}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\bigg(S(p^{'})\gamma_{\mu}S(p)\bigg)^{\alpha^{'}\alpha}A^{\mu,\text{backgr}}(q)\\
&\times\;\; e^{ipx-ip^{'}x^{'}}. \qquad (7.16)
\end{aligned}$$

We read from this equation the Fourier transform

$$G_{\alpha^{'}\alpha}(-p^{'},p) \;=\; (-ie)\bigg(S(p^{'})\gamma_{\mu}S(p)\bigg)^{\alpha^{'}\alpha}A^{\mu,\text{backgr}}(q). \quad (7.17)$$

The tree level probability amplitude is therefore given by

$$< \vec{p}^{'} s^{'} \text{ out}|\vec{p}s \text{ in} > \;=\; -ie\bigg(\bar{u}^{s^{'}}(p^{'})\gamma_{\mu}u^s(p)\bigg)A^{\mu,\text{backgr}}(q). \quad (7.18)$$

The Fourier transform $A^{\mu,\text{backgr}}(q)$ is defined by

$$A^{\mu,\text{backgr}}(x) = \int\frac{d^4q}{(2\pi)^4}A^{\mu,\text{backgr}}(q)\; e^{-iqx}. \quad (7.19)$$

This tree level process corresponds to the Feynman diagram EXT-TREE.

The background field is usually assumed to be small. So we will only keep linear terms in $A^{\mu,\text{backgr}}(x)$. The third term in S does not lead to any correction which is linear in $A^{\mu,\text{backgr}}(x)$.

The fourth term in S leads to a linear term in $A^{\mu,\mathrm{backgr}}(x)$ given by

$$\frac{(-ie)^3}{3!}(3)\int d^4z_1\int d^4z_2\int d^4z_3 <0\ \mathrm{in}|T\bigg(\hat{\psi}_{\alpha',\mathrm{in}}(x')\bar{\hat{\psi}}_{\alpha,\mathrm{in}}(x).\bar{\hat{\psi}}_{\mathrm{in}}(z_1)\gamma_\mu\hat{\psi}_{\mathrm{in}}(z_1).\bar{\hat{\psi}}_{\mathrm{in}}(z_2)\gamma_\nu\hat{\psi}_{\mathrm{in}}(z_2).\bar{\hat{\psi}}_{\mathrm{in}}(z_3)$$
$$\times\quad \gamma_\lambda\hat{\psi}_{\mathrm{in}}(z_3)\bigg)|0\ \mathrm{in}><0\ \mathrm{out}|T(\hat{A}^\mu(z_1)\hat{A}^\nu(z_2))|0\ \mathrm{in}>A^{\lambda,\mathrm{backgr}}(z_3). \tag{7.20}$$

We use Wick's theorem. For the gauge fields the result is trivial. It is simply given by the photon propagator. For the fermion fields the result is quite complicated. As before there are in total 24 contractions. By dropping those disconnected contractions which contain $S_F(0)$ we will only have 11 contractions left. By further inspection we see that only 8 are really disconnected. By using then the symmetry between the internal points z_1 and z_2 we obtain the four terms

$$<0\ \mathrm{in}|T\bigg(\hat{\psi}_{\alpha',\mathrm{in}}(x')\bar{\hat{\psi}}_{\alpha,\mathrm{in}}(x).\bar{\hat{\psi}}_{\mathrm{in}}(z_1)\gamma_\mu\hat{\psi}_{\mathrm{in}}(z_1).\bar{\hat{\psi}}_{\mathrm{in}}(z_2)\gamma_\nu\hat{\psi}_{\mathrm{in}}(z_2).\bar{\hat{\psi}}_{\mathrm{in}}(z_3)\gamma_\lambda\hat{\psi}_{\mathrm{in}}(z_3)\bigg)|0\ \mathrm{in}>$$
$$= -2\bigg[S_F(x'-z_1)\gamma_\mu S_F(z_1-x)\bigg]^{\alpha'\alpha}\mathrm{tr}\gamma_\nu S_F(z_2-z_3)\gamma_\lambda S_F(z_3-z_2)$$
$$+\ 2\bigg[S_F(x'-z_1)\gamma_\mu S_F(z_1-z_2)\gamma_\nu S_F(z_2-z_3)\gamma_\lambda S_F(z_3-x)\bigg]^{\alpha'\alpha}$$
$$+\ 2\bigg[S_F(x'-z_3)\gamma_\lambda S_F(z_3-z_2)\gamma_\nu S_F(z_2-z_1)\gamma_\mu S_F(z_1-x)\bigg]^{\alpha'\alpha}$$
$$+\ 2\bigg[S_F(x'-z_1)\gamma_\mu S_F(z_1-z_3)\gamma_\lambda S_F(z_3-z_2)\gamma_\nu S_F(z_2-x)\bigg]^{\alpha'\alpha}. \tag{7.21}$$

These four terms correspond to the four Feynman diagrams on figure EXT-RAD. Clearly only the last diagram will contribute to the vertex correction so we will only focus on it in the rest of this discussion. The fourth term in S leads therefore to a linear term in the background field $A^{\mu,\mathrm{backgr}}(x)$ given by

$$(-ie)^3\int d^4z_1\int d^4z_2\int d^4z_3\bigg[S_F(x'-z_1)\gamma_\mu S_F(z_1-z_3)\gamma_\lambda S_F(z_3-z_2)\gamma_\nu S_F(z_2-x)\bigg]^{\alpha'\alpha} iD_F^{\mu\nu}(z_1-z_2)$$
$$\times\quad A^{\lambda,\mathrm{backgr}}(z_3)=e^3\int\frac{d^4p'}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k'}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(p'-k)^2+i\epsilon}\bigg(S(p')\gamma_\mu S(k)\gamma_\lambda S(k')\gamma^\mu S(p)\bigg)^{\alpha'\alpha}$$
$$\times\quad A^{\lambda,\mathrm{backgr}}(q)\ (2\pi)^4\delta^4(q-k+k')\ e^{ipx-ip'x'}. \tag{7.22}$$

The corresponding Fourier transform is

$$G_{\alpha',\alpha}(-p',p) = e^3\int\frac{d^4k}{(2\pi)^4}\frac{1}{(p'-k)^2+i\epsilon}\bigg(S(p')\gamma_\mu S(k)\gamma_\lambda S(k-q)\gamma^\mu S(p)\bigg)^{\alpha'\alpha}A^{\lambda,\mathrm{backgr}}(q). \tag{7.23}$$

The probability amplitude (including also the tree level contribution) is therefore given by

$$
\begin{aligned}
< \vec{p}^{'} s^{'} \text{ out}|\vec{p}s \text{ in} > &= -ie\bigg(\bar{u}^{s^{'}}(p^{'})\gamma_{\lambda}u^s(p)\bigg)A^{\lambda,\text{backgr}}(q)\\
&+ e^3\int\frac{d^4k}{(2\pi)^4}\frac{1}{(p-k)^2+i\epsilon}\bigg(\bar{u}^{s^{'}}(p^{'})\gamma_{\mu}S(k+q)\gamma_{\lambda}S(k)\gamma^{\mu}u^s(p)\bigg)A^{\lambda,\text{backgr}}(q)\\
&= -ie\bigg(\bar{u}^{s^{'}}(p^{'})\Gamma_{\lambda}(p^{'},p)u^s(p)\bigg)A^{\lambda,\text{backgr}}(q). \qquad (7.24)
\end{aligned}
$$

The effective vertex $\Gamma_{\lambda}(p^{'},p)$ is given by the same formula as before. This is a general result. The quantum electron vertex at one-loop is always given by the function $\Gamma_{\lambda}(p^{'},p)$.

7.3 One-loop Calculation I: Vertex Correction

7.3.1 Feynman Parameters and Wick Rotation

We will calculate $\delta\Gamma^{\mu}(p^{'},p)=\Gamma^{\mu}(p^{'},p)-\gamma^{\mu}$. First we use the identities $\gamma^{\nu}\gamma^{\mu}\gamma_{\nu}=-2\gamma^{\mu}$, $\gamma^{\lambda}\gamma^{\rho}\gamma^{\mu}\gamma_{\lambda}=4\eta^{\rho\mu}$ and

$$
\begin{aligned}
\gamma^{\lambda}\gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\gamma_{\lambda} &= 2\gamma^{\sigma}\gamma^{\rho}\gamma^{\mu}-2\gamma^{\mu}\gamma^{\rho}\gamma^{\sigma}-2\gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\\
&= -2\gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}. \qquad (7.25)
\end{aligned}
$$

We have

$$
\begin{aligned}
\bar{u}^{s^{'}}(p^{'})\delta\Gamma^{\mu}(p^{'},p)u^s(p) &= 2ie^2\int\frac{d^4l}{(2\pi)^4}\frac{1}{((l-p)^2+i\epsilon)(l^{'2}-m_e^2+i\epsilon)(l^2-m_e^2+i\epsilon)}\bar{u}^{s^{'}}(p^{'})\bigg((\gamma.l)\gamma^{\mu}(\gamma.l^{'})\\
&+ m_e^2\gamma^{\mu}-2m_e(l+l^{'})^{\mu}\bigg)u^s(p). \qquad (7.26)
\end{aligned}
$$

Feynman Parameters: Now we note the identity

$$
\frac{1}{A_1A_2...A_n}=\int_0^1 dx_1dx_2...dx_n\delta(x_1+x_2+...+x_n-1)\frac{(n-1)!}{(x_1A_1+x_2A_2+...+x_nA_n)^n}. \qquad (7.27)
$$

For $n=2$ this is obvious since

$$
\begin{aligned}
\frac{1}{A_1A_2} &= \int_0^1 dx_1dx_2\delta(x_1+x_2-1)\frac{1}{(x_1A_1+x_2A_2)^2}\\
&= \int_0^1 dx_1\frac{1}{(x_1A_1+(1-x_1)A_2)^2}\\
&= \frac{1}{(A_1-A_2)^2}\int_{A_2/(A_1-A_2)}^{A_1/(A_1-A_2)}\frac{dx_1}{x_1^2}. \qquad (7.28)
\end{aligned}
$$

In general the identity can be proven as follows. Let ϵ be a small positive real number. We start from the identity

$$
\frac{1}{A}=\int_0^{\infty}dt\ e^{-t(A+\epsilon)}. \qquad (7.29)
$$

Hence

$$\frac{1}{A_1A_2...A_n}=\int_0^\infty dt_1dt_2...dt_n\ e^{-\sum_{i=1}^n t_i(A_i+\epsilon)}. \tag{7.30}$$

Since $t_i\geq 0$ we have also the identity

$$\int_0^\infty \frac{d\lambda}{\lambda}\ \delta(1-\frac{1}{\lambda}\sum_{i=1}^n t_i)=1. \tag{7.31}$$

Inserting (7.31) into (7.30) we obtain

$$\frac{1}{A_1A_2...A_n}=\int_0^\infty dt_1dt_2...dt_n\int_0^\infty \frac{d\lambda}{\lambda}\ \delta(1-\frac{1}{\lambda}\sum_{i=1}^n t_i)\ e^{-\sum_{i=1}^n t_i(A_i+\epsilon)}. \tag{7.32}$$

We change variables from t_i to $x_i=t_i/\lambda$. We obtain

$$\frac{1}{A_1A_2...A_n}=\int_0^\infty dx_1dx_2...dx_n\int_0^\infty d\lambda\lambda^{n-1}\ \delta(1-\sum_{i=1}^n x_i)\ e^{-\lambda\sum_{i=1}^n x_i(A_i+\epsilon)}. \tag{7.33}$$

We use now the integral representation of the gamma function given by (with $\mathrm{Re}(X)>0$)

$$\Gamma(n)=(n-1)!=X^n\int_0^\infty d\lambda\lambda^{n-1}\ e^{-\lambda X}. \tag{7.34}$$

We obtain

$$\frac{1}{A_1A_2...A_n}=\int_0^\infty dx_1dx_2...dx_n\ \delta(1-\sum_{i=1}^n x_i)\ \frac{(n-1)!}{\bigg(\sum_{i=1}^n x_i(A_i+\epsilon)\bigg)^n}. \tag{7.35}$$

Since $x_i\geq 0$ and $\sum_{i=1}^n x_i=1$ we must have $0\leq x_i\leq 1$. Thus

$$\frac{1}{A_1A_2...A_n}=\int_0^1 dx_1dx_2...dx_n\ \delta(1-\sum_{i=1}^n x_i)\ \frac{(n-1)!}{\bigg(A_1x_1+A_2x_2+...+A_nx_n\bigg)^n}. \tag{7.36}$$

The variables x_i are called Feynman parameters.

This identity will allow us to convert a product of propagators into a single fraction. Let us see how this works in our current case. We have

$$\frac{1}{((l-p)^2+i\epsilon)(l'^2-m_e^2+i\epsilon)(l^2-m_e^2+i\epsilon)}=2\int_0^1 dxdydz\ \delta(x+y+z-1)\frac{1}{D^3}. \tag{7.37}$$

$$D=x((l-p)^2+i\epsilon)+y(l^{'2}-m_e^2+i\epsilon)+z(l^2-m_e^2+i\epsilon). \tag{7.38}$$

Let us recall that the variable of integration is the four-momentum l. Clearly we must try to complete the square. By using $x+y+z=1$ we have

$$\begin{aligned}D &= l^2-2(xp+yq)l+xp^2+yq^2-(y+z)m_e^2+i\epsilon\\ &= \bigg(l-xp-yq\bigg)^2-x^2p^2-y^2q^2-2xypq+xp^2+yq^2-(y+z)m_e^2+i\epsilon\\ &= \bigg(l-xp-yq\bigg)^2+xzp^2+xyp^{'2}+yzq^2-(y+z)m_e^2+i\epsilon.\end{aligned} \tag{7.39}$$

Since this will act on $u^s(p)$ and $\bar{u}^{s^{'}}(p^{'})$ and since $p^2u^s(p)=m_e^2u^s(p)$ and $p^{'2}\bar{u}^{s^{'}}(p^{'})=m_e^2\bar{u}^{s^{'}}(p^{'})$ we can replace both p^2 and $p^{'2}$ in D with their on-shell value m_e^2. We get then

$$D = \bigg(l-xp-yq\bigg)^2+yzq^2-(1-x)^2m_e^2+i\epsilon. \tag{7.40}$$

We will define

$$\Delta = -yzq^2+(1-x)^2m_e^2. \tag{7.41}$$

This is always positive since $q^2<0$ for scattering processes. We shift the variable l as $l\longrightarrow L=l-xp-yq$. We get

$$D = L^2-\Delta+i\epsilon. \tag{7.42}$$

Plugging this result into our original integral we get

$$\begin{aligned}\bar{u}^{s^{'}}(p^{'})\delta\Gamma^{\mu}(p^{'},p)u^s(p) &= 4ie^2\int_0^1 dxdydz\ \delta(x+y+z-1)\int\frac{d^4L}{(2\pi)^4}\frac{1}{(L^2-\Delta+i\epsilon)^3}\bar{u}^{s^{'}}(p^{'})\bigg((\gamma.l)\gamma^{\mu}(\gamma.l^{'})\\ &+ m_e^2\gamma^{\mu}-2m_e(l+l^{'})^{\mu}\bigg)u^s(p).\end{aligned} \tag{7.43}$$

In this equation $l=L+xp+yq$ and $l^{'}=L+xp+(y-1)q$. By dropping odd terms in L which must vanish by summetry we get

$$\begin{aligned}\bar{u}^{s^{'}}(p^{'})\delta\Gamma^{\mu}(p^{'},p)u^s(p) &= 4ie^2\int_0^1 dxdydz\ \delta(x+y+z-1)\int\frac{d^4L}{(2\pi)^4}\frac{1}{(L^2-\Delta+i\epsilon)^3}\bar{u}^{s^{'}}(p^{'})\bigg((\gamma.L)\gamma^{\mu}(\gamma.L)\\ &+ m_e^2\gamma^{\mu}+(x\gamma.p+y\gamma.q)\gamma^{\mu}(x\gamma.p+(y-1)\gamma.q)-2m_e(2xp+(2y-1)q)^{\mu}\bigg)u^s(p).\end{aligned} \tag{7.44}$$

Again by using symmetry considerations quadratic terms in L must be given by

$$\int\frac{d^4L}{(2\pi)^4}\frac{L^{\mu}L^{\nu}}{(L^2-\Delta+i\epsilon)^3}=\int\frac{d^4L}{(2\pi)^4}\frac{\frac{1}{4}\eta^{\mu\nu}L^2}{(L^2-\Delta+i\epsilon)^3} \tag{7.45}$$

Thus

$$\begin{aligned}\bar{u}^{s^{'}}(p^{'})\delta\Gamma^{\mu}(p^{'},p)u^s(p) &= 4ie^2\int_0^1 dxdydz\ \delta(x+y+z-1)\int\frac{d^4L}{(2\pi)^4}\frac{1}{(L^2-\Delta+i\epsilon)^3}\bar{u}^{s^{'}}(p^{'})\bigg(-\frac{1}{2}\gamma^{\mu}L^2\\ &+ m_e^2\gamma^{\mu}+(x\gamma.p+y\gamma.q)\gamma^{\mu}(x\gamma.p+(y-1)\gamma.q)-2m_e(2xp+(2y-1)q)^{\mu}\bigg)u^s(p).\end{aligned} \tag{7.46}$$

By using $\gamma.pu^s(p)=m_eu^s(p)$, $\bar{u}^{s^{'}}(p^{'})\gamma.p^{'}=m_e\bar{u}^{s^{'}}(p^{'})$ and $\gamma.p\gamma^{\mu}=2p^{\mu}-\gamma^{\mu}\gamma.p$, $\gamma^{\mu}\gamma.p^{'}=$

$2p'^{\mu} - \gamma.p'\gamma^{\mu}$ we can make the replacement

$$\begin{aligned}
\bar{u}^{s'}(p')\Big[(x\gamma.p + y\gamma.q)\gamma^{\mu}(x\gamma.p + (y-1)\gamma.q)\Big]u^s(p) &\longrightarrow \bar{u}^{s'}(p')\Big[\Big((x+y)\gamma.p - ym_e\Big)\gamma^{\mu}\Big((x+y-1)m_e \\
&- (y-1)\gamma.p'\Big)\Big]u^s(p) \\
&\longrightarrow \bar{u}^{s'}(p')\Big[m_e(x+y)(x+y-1)(2p^{\mu} - m_e\gamma^{\mu}) \\
&- (x+y)(y-1)\Big(2m_e(p+p')^{\mu} + q^2\gamma^{\mu} - 3m_e^2\gamma^{\mu}\Big) \\
&- m_e^2y(x+y-1)\gamma^{\mu} + m_ey(y-1)(2p'^{\mu} - m_e\gamma^{\mu})\Big] \\
&\times u^s(p). \qquad (7.47)
\end{aligned}$$

After some more algebra we obtain the result

$$\begin{aligned}
\bar{u}^{s'}(p')\delta\Gamma^{\mu}(p',p)u^s(p) &= 4ie^2\int_0^1 dxdydz\ \delta(x+y+z-1)\int\frac{d^4L}{(2\pi)^4}\frac{1}{(L^2-\Delta+i\epsilon)^3}\bar{u}^{s'}(p')\Big[\gamma^{\mu}\Big(-\frac{1}{2}L^2 \\
&+ (1-z)(1-y)q^2 + (1-x^2-2x)m_e^2\Big) + m_ex(x-1)(p+p')^{\mu} \\
&+ m_e(x-2)(x+2y-1)m_eq^{\mu}\Big]u^s(p). \qquad (7.48)
\end{aligned}$$

The term proportional to $q^{\mu} = p^{\mu} - p'^{\mu}$ is zero because it is odd under the exchange $y \leftrightarrow z$ since $x + 2y - 1 = y - z$. This is our first manifestation of the so-called Ward identity. In other words we have

$$\begin{aligned}
\bar{u}^{s'}(p')\delta\Gamma^{\mu}(p',p)u^s(p) &= 4ie^2\int_0^1 dxdydz\ \delta(x+y+z-1)\int\frac{d^4L}{(2\pi)^4}\frac{1}{(L^2-\Delta+i\epsilon)^3}\bar{u}^{s'}(p')\Big[\gamma^{\mu}\Big(-\frac{1}{2}L^2 \\
&+ (1-z)(1-y)q^2 + (1-x^2-2x)m_e^2\Big) + m_ex(x-1)(p+p')^{\mu}\Big]u^s(p). \qquad (7.49)
\end{aligned}$$

Now we use the so-called Gordon's identity given by (with the spin matrices $\sigma^{\mu\nu} = 2\Gamma^{\mu\nu} = i[\gamma^{\mu},\gamma^{\nu}]/2$)

$$\bar{u}^{s'}(p')\gamma^{\mu}u^s(p) = \frac{1}{2m_e}\bar{u}^{s'}(p')\Big[(p+p')^{\mu} - i\sigma^{\mu\nu}q_{\nu}\Big]u^s(p). \qquad (7.50)$$

This means that we can make the replacement

$$\bar{u}^{s'}(p')(p+p')^{\mu}u^s(p) \longrightarrow \bar{u}^{s'}(p')\Big[2m_e\gamma^{\mu} + i\sigma^{\mu\nu}q_{\nu}\Big]u^s(p). \qquad (7.51)$$

Hence we get

$$\begin{aligned}
\bar{u}^{s'}(p')\delta\Gamma^{\mu}(p',p)u^s(p) &= 4ie^2\int_0^1 dxdydz\ \delta(x+y+z-1)\int\frac{d^4L}{(2\pi)^4}\frac{1}{(L^2-\Delta+i\epsilon)^3}\bar{u}^{s'}(p')\Big[\gamma^{\mu}\Big(-\frac{1}{2}L^2 \\
&+ (1-z)(1-y)q^2 + (1+x^2-4x)m_e^2\Big) + im_ex(x-1)\sigma^{\mu\nu}q_{\nu}\Big]u^s(p). \qquad (7.52)
\end{aligned}$$

Wick Rotation: The natural step at this stage is to actually do the 4−dimensional integral over L. Towards this end we will perform the so-called Wick rotation of the real integration variable L^0 to a pure imaginary variable $L^4=-iL^0$ which will allow us to convert the Minkowskian signature of the metric into an Euclidean signature. Indeed the Minkowski line element $dL^2=(dL^0)^2-(dL^i)^2$ becomes under Wick rotation the Euclid line element $dL^2=-(dL^4)^2-(dL^i)^2$. In a very profound sense the quantum field theory integral becomes under Wick rotation a statistical mechanics integral. This is of course possible because of the location of the poles $\sqrt{\vec{L}^2+\Delta}-i\epsilon^{'}$ and $-\sqrt{\vec{L}^2+\Delta}+i\epsilon^{'}$ of the L^0 integration and because the integral over L^0 goes to 0 rapidly enough for large positive L^0. Note that the prescription $L^4=-iL^0$ corresponds to a rotation by $\pi/2$ counterclockwise of the L^0 axis.

Let us now compute

$$\int\frac{d^4L}{(2\pi)^4}\frac{(L^2)^n}{(L^2-\Delta+i\epsilon)^m} = \frac{i}{(2\pi)^4}\frac{(-1)^n}{(-1)^m}\int d^4L_E\frac{(L_E^2)^n}{(L_E^2+\Delta)^m}. \tag{7.53}$$

In this equation $\vec{L}_E=(L^1,L^2,L^3,L^4)$. Since we are dealing with Euclidean coordinates in four dimensions we can go to spherical coordinates in four dimensions defined by (with $0\leq r\leq\infty$, $0\leq\theta\leq\pi$, $0\leq\phi\leq 2\pi$ and $0\leq\omega\leq\pi$)

$$\begin{aligned}L^1 &= r\sin\omega\sin\theta\cos\phi\\ L^2 &= r\sin\omega\sin\theta\sin\phi\\ L^3 &= r\sin\omega\cos\theta\\ L^4 &= r\cos\omega.\end{aligned} \tag{7.54}$$

We also know that

$$d^4L_E=r^3\sin^2\omega\sin\theta dr d\theta d\phi d\omega. \tag{7.55}$$

We calculate then

$$\begin{aligned}\int\frac{d^4L}{(2\pi)^4}\frac{(L^2)^n}{(L^2-\Delta+i\epsilon)^m} &= \frac{i}{(2\pi)^4}\frac{(-1)^n}{(-1)^m}\int\frac{r^{2n+3}dr}{(r^2+\Delta)^m}\int\sin^2\omega\sin\theta d\theta d\phi d\omega\\ &= \frac{2i\pi^2}{(2\pi)^4}\frac{(-1)^n}{(-1)^m}\int\frac{r^{2n+3}dr}{(r^2+\Delta)^m}.\end{aligned} \tag{7.56}$$

The case $n=0$ is easy. We have

$$\begin{aligned}\int\frac{d^4L}{(2\pi)^4}\frac{1}{(L^2-\Delta+i\epsilon)^m} &= \frac{2i\pi^2}{(2\pi)^4}\frac{1}{(-1)^m}\int\frac{r^3dr}{(r^2+\Delta)^m}\\ &= \frac{i\pi^2}{(2\pi)^4}\frac{1}{(-1)^m}\int_{\Delta}^{\infty}\frac{(x-\Delta)dx}{x^m}\\ &= \frac{i}{(4\pi)^2}\frac{(-1)^m}{(m-2)(m-1)}\frac{1}{\Delta^{m-2}}.\end{aligned} \tag{7.57}$$

The case $n=1$ turns out to be divergent

$$\begin{aligned}\int \frac{d^4L}{(2\pi)^4}\frac{L^2}{(L^2-\Delta+i\epsilon)^m} &= \frac{2i\pi^2}{(2\pi)^4}\frac{-1}{(-1)^m}\int \frac{r^5dr}{(r^2+\Delta)^m}\\ &= \frac{i\pi^2}{(2\pi)^4}\frac{-1}{(-1)^m}\int_{\Delta}^{\infty}\frac{(x-\Delta)^2dx}{x^m}\\ &= \frac{i\pi^2}{(2\pi)^4}\frac{-1}{(-1)^m}\bigg(\frac{x^{3-m}}{3-m}-2\Delta\frac{x^{2-m}}{2-m}+\Delta^2\frac{x^{1-m}}{1-m}\bigg)_{\Delta}^{\infty}\\ &= \frac{i}{(4\pi)^2}\frac{(-1)^{m+1}}{(m-3)(m-2)(m-1)}\frac{2}{\Delta^{m-3}}. \end{aligned} \tag{7.58}$$

This does not make sense for $m=3$ which is the case of interest.

7.3.2 Pauli-Villars Regularization

We will now show that this divergence is ultraviolet in the sense that it is coming from integrating arbitrarily high momenta in the loop integral. We will also show the existence of an infrared divergence coming from integrating arbitrarily small momenta in the loop integral. In order to control these infinities we need to regularize the loop integral in one way or another. We adopt here the so-called Pauli-Villars regularization. This is given by making the following replacement

$$\frac{1}{(l-p)^2+i\epsilon}\longrightarrow\frac{1}{(l-p)^2-\mu^2+i\epsilon}-\frac{1}{(l-p)^2-\Lambda+i\epsilon}. \tag{7.59}$$

The infrared cutoff μ will be taken to zero at the end and thus it should be thought of as a small mass for the physical photon. The ultraviolet cutoff Λ will be taken to ∞ at the end. The UV cutoff Λ does also look like a a very large mass for a fictitious photon which becomes infinitely heavy and thus unobservable in the limit $\Lambda\longrightarrow\infty$.

Now it is not difficult to see that

$$\frac{1}{((l-p)^2-\mu^2+i\epsilon)(l'^2-m_e^2+i\epsilon)(l^2-m_e^2+i\epsilon)}=2\int_0^1 dxdydz\ \delta(x+y+z-1)\frac{1}{D_{\mu}^3}. \tag{7.60}$$

$$D_{\mu}=D-\mu^2x=L^2-\Delta_{\mu}+i\epsilon\ ,\ \Delta_{\mu}=\Delta+\mu^2x. \tag{7.61}$$

$$\frac{1}{((l-p)^2-\Lambda^2+i\epsilon)(l'^2-m_e^2+i\epsilon)(l^2-m_e^2+i\epsilon)}=2\int_0^1 dxdydz\ \delta(x+y+z-1)\frac{1}{D_{\Lambda}^3}. \tag{7.62}$$

$$D_{\Lambda}=D-\Lambda^2x=L^2-\Delta_{\Lambda}+i\epsilon\ ,\ \Delta_{\Lambda}=\Delta+\Lambda^2x. \tag{7.63}$$

The result (7.52) becomes

$$\begin{aligned}\bar{u}^{s'}(p')\delta\Gamma^{\mu}(p',p)u^s(p) &= 4ie^2\int_0^1 dxdydz\ \delta(x+y+z-1)\int\frac{d^4L}{(2\pi)^4}\bigg[\frac{1}{(L^2-\Delta_{\mu}+i\epsilon)^3}-\frac{1}{(L^2-\Delta_{\Lambda}+i\epsilon)^3}\bigg]\\ &\times\ \bar{u}^{s'}(p')\bigg[\gamma^{\mu}\bigg(-\frac{1}{2}L^2+(1-z)(1-y)q^2+(1+x^2-4x)m_e^2\bigg)+im_ex(x-1)\sigma^{\mu\nu}q_{\nu}\bigg]\\ &\times\ u^s(p). \end{aligned} \tag{7.64}$$

We compute now (after Wick rotation)

$$\begin{aligned}\int\frac{d^4L}{(2\pi)^4}\bigg[\frac{L^2}{(L^2-\Delta_\mu+i\epsilon)^3}-\frac{L^2}{(L^2-\Delta_\Lambda+i\epsilon)^3}\bigg] &= \frac{2i}{(4\pi)^2}\bigg[\int\frac{r^5dr}{(r^2+\Delta_\mu)^3}-\int\frac{r^5dr}{(r^2+\Delta_\Lambda)^3}\bigg]\\ &= \frac{i}{(4\pi)^2}\bigg[\int_{\Delta_\mu}^{\infty}\frac{(x-\Delta_\mu)^2dx}{x^3}-\int_{\Delta_\Lambda}^{\infty}\frac{(x-\Delta_\Lambda)^2dx}{x^3}\bigg]\\ &= \frac{i}{(4\pi)^2}\ln\frac{\Delta_\Lambda}{\Delta_\mu}. \qquad (7.65)\end{aligned}$$

Clearly in the limit $\Lambda\longrightarrow\infty$ this goes as $\ln\Lambda^2$. This shows explicitly that the divergence problem seen earlier is a UV one,i.e. coming from high momenta. Also we compute

$$\begin{aligned}\int\frac{d^4L}{(2\pi)^4}\bigg[\frac{1}{(L^2-\Delta_\mu+i\epsilon)^3}-\frac{1}{(L^2-\Delta_\Lambda+i\epsilon)^3}\bigg] &= -\frac{2i}{(4\pi)^2}\bigg[\int\frac{r^3dr}{(r^2+\Delta_\mu)^3}-\int\frac{r^3dr}{(r^2+\Delta_\Lambda)^3}\bigg]\\ &= -\frac{i}{(4\pi)^2}\bigg[\int_{\Delta_\mu}^{\infty}\frac{(x-\Delta_\mu)dx}{x^3}-\int_{\Delta_\Lambda}^{\infty}\frac{(x-\Delta_\Lambda)dx}{x^3}\bigg]\\ &= -\frac{i}{2(4\pi)^2}\bigg(\frac{1}{\Delta_\mu}-\frac{1}{\Delta_\Lambda}\bigg). \qquad (7.66)\end{aligned}$$

The second term vanishes in the limit $\Lambda\longrightarrow\infty$. We get then the result

$$\begin{aligned}\bar{u}^{s^{'}}(p^{'})\delta\Gamma^\mu(p^{'},p)u^s(p) &= (4ie^2)(-\frac{i}{2(4\pi)^2})\int_0^1 dxdydz\ \delta(x+y+z-1)\bar{u}^{s^{'}}(p^{'})\bigg[\gamma^\mu\bigg(\ln\frac{\Delta_\Lambda}{\Delta_\mu}\\ &+ \frac{(1-z)(1-y)q^2+(1+x^2-4x)m_e^2}{\Delta_\mu}\bigg)+\frac{i}{\Delta_\mu}m_e x(x-1)\sigma^{\mu\nu}q_\nu\bigg]u^s(p)\\ &= \bar{u}^{s^{'}}(p^{'})\bigg(\gamma^\mu(F_1(q^2)-1)-\frac{i\sigma^{\mu\nu}q_\nu}{2m_e}F_2(q^2)\bigg)u^s(p). \qquad (7.67)\end{aligned}$$

$$F_1(q^2) = 1+\frac{\alpha}{2\pi}\int_0^1 dxdydz\ \delta(x+y+z-1)\bigg(\ln\frac{\Lambda^2x}{\Delta_\mu}+\frac{(1-z)(1-y)q^2+(1+x^2-4x)m_e^2}{\Delta_\mu}\bigg). \qquad (7.68)$$

$$F_2(q^2) = \frac{\alpha}{2\pi}\int_0^1 dxdydz\ \delta(x+y+z-1)\frac{2m_e^2x(1-x)}{\Delta_\mu}. \qquad (7.69)$$

The functions $F_1(q^2)$ and $F_2(q^2)$ are known as the form factors of the electron. The form factor $F_1(q^2)$ is logarithmically UV divergent and requires a redefinition which is termed a renormalization. This will be done in the next section. This form factor is also IR divergent. To see this recall that $\Delta_\mu=-yzq^2+(1-x)^2m_e^2+\mu^2x$. Now set $q^2=0$ and $\mu^2=0$. The term proportional to $1/\Delta_\mu$ is

$$\begin{aligned} F_1(0) &= ... + \frac{\alpha}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^1 dz\ \delta(x+y+z-1)\frac{1+x^2-4x}{(1-x)^2} \\ &= ... + \frac{\alpha}{2\pi}\int_0^1 dx \int_0^1 dy \int_0^{1-y} dt\ \delta(x-t)\frac{1+x^2-4x}{(1-x)^2} \\ &= ... + \frac{\alpha}{2\pi}\int_0^1 dy \int_0^{1-y} dt\ \frac{1+t^2-4t}{(1-t)^2} \\ &= ... - \frac{\alpha}{2\pi}\int_0^1 dy \int_1^{y} dt\ (1+\frac{2}{t}-\frac{2}{t^2}) \\ &= ... - \frac{\alpha}{2\pi}\int_0^1 dy\ (y+2\ln y+\frac{2}{y}-3). \end{aligned} \quad (7.70)$$

As it turns out this infrared divergence will cancel exactly the infrared divergence coming from bremsstrahlung diagrams. Bremsstrahlung is scattering with radiation, i.e. scattering with emission of very low energy photons which can not be detected.

7.3.3 Renormalization (Minimal Subtraction) and Anomalous Magnetic Moment

Electric Charge and Magnetic Moment of the Electron: The form factors $F_1(q^2)$ and $F_2(q^2)$ define the charge and the magnetic moment of the electron. To see this we go to the problem of scattering of electrons from an external electromagnetic field. The probability amplitude is given by equation (7.24) with $q = p^{'} - p$. Thus

$$\begin{aligned} <\vec{p}^{'} s^{'}\ \text{out}|\vec{p}s\ \text{in}> &= -ie\bar{u}^{s^{'}}(p^{'})\Gamma_\lambda(p^{'},p)u^s(p).A^{\lambda,\text{backgr}}(q) \\ &= -ie\bar{u}^{s^{'}}(p^{'})\bigg[\gamma_\lambda F_1(q^2) + \frac{i\sigma_{\lambda\gamma}q^\gamma}{2m_e}F_2(q^2)\bigg]u^s(p).A^{\lambda,\text{backgr}}(q). \end{aligned} \quad (7.71)$$

Firstly we will consider an electrostatic potential $\phi(\vec{x})$, viz $A^{\lambda,\text{backgr}}(q) = (2\pi\delta(q^0)\phi(\vec{q}),0)$. We have then

$$<\vec{p}^{'} s^{'}\ \text{out}|\vec{p}s\ \text{in}> = -ieu^{s^{'}+}(p^{'})\bigg[F_1(-\vec{q}^2) + \frac{F_2(-\vec{q}^2)}{2m_e}\gamma^i q^i\bigg]u^s(p).2\pi\delta(q^0)\phi(\vec{q}). \quad (7.72)$$

We will assume that the electrostatic potential $\phi(\vec{x})$ is slowly varying over a large region so that $\phi(\vec{q})$ is concentrated around $\vec{q}=0$. In other words the momentum $\vec{q}$ can be treated as small and as a consequence the momenta $\vec{p}$ and $\vec{p}^{'}$ are also small.

In the nonrelativistic limit the spinor $u^s(p)$ behaves as (recall that $\sigma_\mu p^\mu = E - \vec{\sigma}\vec{p}$ and $\bar{\sigma}_\mu p^\mu = E + \vec{\sigma}\vec{p}$)

$$u^s(p) = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\xi^s \\ \sqrt{\bar{\sigma}_\mu p^\mu}\xi^s \end{pmatrix} = \sqrt{m_e}\begin{pmatrix} (1-\frac{\vec{\sigma}\vec{p}}{2m_e}+O(\frac{\vec{p}^2}{m_e^2}))\xi^s \\ (1+\frac{\vec{\sigma}\vec{p}}{2m_e}+O(\frac{\vec{p}^2}{m_e^2}))\xi^s \end{pmatrix}. \quad (7.73)$$

We remark that the nonrelativistic limit is equivalent to the limit of small momenta. Thus by dropping all terms which are at least linear in the momenta we get

$$
\begin{aligned}
<\vec{p}^{'}s^{'}\ \text{out}|\vec{p}s\ \text{in}> &= -ieu^{s^{'}+}(p^{'})F_1(0)u^s(p).2\pi\delta(q^0)\phi(\vec{q})\\
&= -ieF_1(0).2m_e\xi^{s^{'}+}\xi^s.2\pi\delta(q^0)\phi(\vec{q})\\
&= -ieF_1(0)\phi(\vec{q}).2m_e\delta^{s^{'}s}.2\pi\delta(q^0). \qquad (7.74)
\end{aligned}
$$

The corresponding $T-$matrix element is thus

$$
<\vec{p}^{'}s^{'}\ \text{in}|iT|\vec{p}s\ \text{in}> = -ieF_1(0)\phi(\vec{q}).2m_e\delta^{s^{'}s}. \qquad (7.75)
$$

This should be compared with the Born approximation of the probability amplitude of scattering from a potential $V(\vec{x})$ (with $V(\vec{q})=\int d^3xV(\vec{x})e^{-i\vec{q}\vec{x}}$)

$$
<\vec{p}^{'}\ \text{in}|iT|\vec{p}\ \text{in}> = iV(\vec{q}). \qquad (7.76)
$$

The factor $2m_e$ should not bother us because it is only due to our normalization of spinors and so it should be omitted in the comparison. The Kronecker's delta $\delta^{s^{'}s}$ coincides with the prediction of nonrelativistic quantum mechanics. Thus the problem is equivalent to scattering from the potential

$$
V(\vec{x})=-eF_1(0)\phi(\vec{x}). \qquad (7.77)
$$

The charge of the electron in units of $-e$ is precisely $F_1(0)$.

Next we will consider a vector potential $\vec{A}(\vec{x})$, viz $A^{\lambda,\text{backgr}}(q)=(0,2\pi\delta(q^0)\vec{A}(\vec{q}))$. We have

$$
<\vec{p}^{'}s^{'}\ \text{in}|iT|\vec{p}s\ \text{in}> = -ie\bar{u}^{s^{'}}(p^{'})\bigg[\gamma_iF_1(-\vec{q}^2)+\frac{i\sigma_{ij}q^j}{2m_e}F_2(-\vec{q}^2)\bigg]u^s(p).A^{i,\text{backgr}}(\vec{q}). \qquad (7.78)
$$

We will keep up to the linear term in the momenta. Thus

$$
<\vec{p}^{'}s^{'}\ \text{in}|iT|\vec{p}s\ \text{in}> = -ieu^{s^{'}+}(p^{'})\gamma^0\bigg[\gamma_iF_1(0)-\frac{[\gamma_i,\gamma_j]q^j}{4m_e}F_2(0)\bigg]u^s(p).A^{i,\text{backgr}}(\vec{q}). \qquad (7.79)
$$

We compute

$$
\begin{aligned}
u^{s^{'}+}(p^{'})\gamma^0\gamma_iu^s(p) &= m_e\xi^{s^{'}+}\bigg((1-\frac{\vec{\sigma}\vec{p}^{'}}{2m_e})\sigma^i(1-\frac{\vec{\sigma}\vec{p}}{2m_e})-(1+\frac{\vec{\sigma}\vec{p}^{'}}{2m_e})\sigma^i(1+\frac{\vec{\sigma}\vec{p}}{2m_e})\bigg)\xi^s\\
&= \xi^{s^{'}+}\bigg(-(p+p^{'})^i+i\epsilon^{ijk}q^j\sigma^k\bigg)\xi^s. \qquad (7.80)
\end{aligned}
$$

$$
u^{s^{'}+}(p^{'})\gamma^0[\gamma_i,\gamma_j]q^ju^s(p) = 2m_e\xi^{s^{'}+}\bigg(-2i\epsilon^{ijk}q^j\sigma^k\bigg)\xi^s. \qquad (7.81)
$$

We get then

$$
\begin{aligned}
<\vec{p}^{'}s^{'}\ \text{in}|iT|\vec{p}s\ \text{in}> &= -ie\xi^{s^{'}+}\bigg[-(p^i+p^{'i})F_1(0)\bigg]\xi^s.A^{i,\text{backgr}}(\vec{q})\\
&- ie\xi^{s^{'}+}\bigg[i\epsilon^{ijk}q^j\sigma^k(F_1(0)+F_2(0))\bigg]\xi^s.A^{i,\text{backgr}}(\vec{q}). \qquad (7.82)
\end{aligned}
$$

The first term corresponds to the interaction term $\vec{\hat{p}}\vec{\hat{A}}+\vec{\hat{A}}\vec{\hat{p}}$ in the Schrödinger equation. The second term is the magnetic moment interaction. Thus

$$\begin{aligned}
<\vec{p}^{\,'}s^{'}\ \mathrm{in}|iT|\vec{p}s\ \mathrm{in}>_{\mathrm{magn\ moment}} &= -ie\xi^{s^{'}+}\bigg[i\epsilon^{ijk}q^j\sigma^k(F_1(0)+F_2(0))\bigg]\xi^s.A^{i,\mathrm{backgr}}(\vec{q}) \\
&= -ie\xi^{s^{'}+}\bigg[\sigma^k(F_1(0)+F_2(0))\bigg]\xi^s.B^{k,\mathrm{backgr}}(\vec{q}) \\
&= -i<\mu^k>.B^{k,\mathrm{backgr}}(\vec{q}).2m_e \\
&= iV(\vec{q}).2m_e. \qquad (7.83)
\end{aligned}$$

The magnetic field is defined by $\vec{B}^{\mathrm{backgr}}(\vec{x})=\vec{\nabla}\times\vec{A}^{\mathrm{backgr}}(\vec{x})$ and thus $B^k(\vec{q})=i\epsilon^{ijk}q^jA^{i,\mathrm{backgr}}(\vec{q})$. The magnetic moment is defined by

$$<\mu^k>=\frac{e}{m_e}\xi^{s^{'}+}\bigg[\frac{\sigma^k}{2}(F_1(0)+F_2(0))\bigg]\xi^s\Leftrightarrow\mu^k=g\frac{e}{2m_e}\frac{\sigma^k}{2}. \qquad (7.84)$$

The gyromagnetic ratio (Landé g-factor) is then given by

$$g=2(F_1(0)+F_2(0)). \qquad (7.85)$$

Renormalization: We have found that the charge of the electron is $-eF_1(0)$ and not $-e$. This is a tree level result. Thus one must have $F_1(0)=1$. Substituting $q^2=0$ in (7.68) we get

$$F_1(0) = 1+\frac{\alpha}{2\pi}\int_0^1 dxdydz\ \delta(x+y+z-1)\bigg(\ln\frac{\Lambda^2x}{\Delta_\mu(0)}+\frac{(1+x^2-4x)m_e^2}{\Delta_\mu(0)}\bigg). \qquad (7.86)$$

This is clearly not equal 1. In fact $F_1(0)\longrightarrow\infty$ logarithmically when $\Lambda\longrightarrow\infty$. We need to redefine (renormalize) the value of $F_1(q^2)$ in such a way that $F_1(0)=1$. We adopt here a prescription termed minimal subtraction which consists in subtracting from $\delta F_1(q^2)=F_1(q^2)-1$ (which is the actual one-loop correction to the vertex) the divergence $\delta F_1(0)$. We define

$$\begin{aligned}
F_1^{\mathrm{ren}}(q^2) &= F_1(q^2)-\delta F_1(0) \\
&= 1+\frac{\alpha}{2\pi}\int_0^1 dxdydz\ \delta(x+y+z-1)\bigg(\ln\frac{\Delta_\mu(0)}{\Delta_\mu(q^2)}+\frac{(1-z)(1-y)q^2}{\Delta_\mu(q^2)}+\frac{(1+x^2-4x)m_e^2}{\Delta_\mu(q^2)} \\
&- \frac{(1+x^2-4x)m_e^2}{\Delta_\mu(0)}\bigg). \qquad (7.87)
\end{aligned}$$

This formula satisfies automatically $F_1^{\mathrm{ren}}(0)=1$.

The form factor $F_2(0)$ is UV finite since it does not depend on Λ. It is also as point out earlier IR finite and thus one can simply set $\mu=0$ in this function. The magnetic moment of the electron is proportional to the gyromagnetic ratio $g=2F_1(0)+2F_2(0)$. Since $F_1(0)$ was renormalized to $F_1^{\mathrm{ren}}(0)$ the renormalized magnetic moment of the electron will be proportional to the gyromagnetic ratio

$$\begin{aligned}
g^{\mathrm{ren}} &= 2F_1^{\mathrm{ren}}(0)+2F_2(0) \\
&= 2+2F_2(0). \qquad (7.88)
\end{aligned}$$

The first term is precisely the prediction of the Dirac theory (tree level). The second term which is due to the quantum one-loop effect will lead to the so-called anomalous magnetic moment. This is given by

$$\begin{aligned}
F_2(0) &= \frac{\alpha}{\pi}\int_0^1 dx\int_0^1 dy\int_0^1 dz\ \delta(x+y+z-1)\frac{x}{1-x}\\
&= \frac{\alpha}{\pi}\int_0^1 dx\int_0^1 dy\int_{-y}^{1-y} dt\ \delta(x-t)\frac{x}{1-x}\\
&= \frac{\alpha}{\pi}\int_0^1 dx\int_0^1 dy\int_{0}^{1-y} dt\ \delta(x-t)\frac{x}{1-x}\\
&= \frac{\alpha}{\pi}\int_0^1 dy\int_{0}^{1-y} dt\ \frac{t}{1-t}\\
&= \frac{\alpha}{\pi}\int_0^1 dy(y-1-\ln y)\\
&= \frac{\alpha}{\pi}\bigg(\frac{1}{2}(y-1)^2+y-y\ln y\bigg)_0^1\\
&= \frac{\alpha}{2\pi}.
\end{aligned}\tag{7.89}$$

7.4 Exact Fermion 2−Point Function

For simplicity we will consider in this section a scalar field theory and then we will generalize to a spinor field theory. As we have already seen the free 2−point function $<0|T(\hat{\phi}_{\rm in}(x)\hat{\phi}_{\rm in}(y))|0>$ is the probability amplitude for a free scalar particle to propagate from a spacetime point y to a spacetime x. In the interacting theory the 2−point function is $<\Omega|T(\hat{\phi}(x)\hat{\phi}(y))|\Omega>$ where $|\Omega>=|0>/\sqrt{<0|0>}$ is the ground state of the full Hamiltonian $\hat{H}$.

The full Hamiltonian $\hat{H}$ commutes with the full momentum operator $\vec{\hat{P}}$. Let $|\lambda_0>$ be an eigenstate of $\hat{H}$ with momentum $\vec{0}$. There could be many such states corresponding to one-particle states with mass m_r and 2−particle and multiparticle states which have a continuous mass spectrum starting at $2m_r$. By Lorentz invariance a generic state of $\hat{H}$ with a momentum $\vec{p}\neq 0$ can be obtained from one of the $|\lambda_0>$ by the application of a boost. Generic eigenstates of $\hat{H}$ are denoted $|\lambda_p>$ and they have energy $E_p(\lambda)=\sqrt{\vec{p}^2+m_\lambda^2}$ where m_λ is the energy of the corresponding $|\lambda_0>$. We have the completeness relation in the full Hilbert space

$$\mathbf{1}=|\Omega><\Omega|+\sum_\lambda\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)}|\lambda_p><\lambda_p|.\tag{7.90}$$

The sum over λ runs over all the 0−momentum eigenstates $|\lambda_0>$. Compare this with the completeness relation of the free one-particle states given by

$$\mathbf{1}=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}|\vec{p}><\vec{p}|\ ,\ E_p=\sqrt{\vec{p}^2+m^2}.\tag{7.91}$$

By inserting the completeness relation in the full Hilbert space, the full 2−point function becomes (for $x^0>y^0$)

$$\begin{aligned}
<\Omega|T(\hat{\phi}(x)\hat{\phi}(y))|\Omega> &= <\Omega|\hat{\phi}(x)|\Omega><\Omega|\hat{\phi}(y)|\Omega>\\
&+ \sum_\lambda\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)}<\Omega|\hat{\phi}(x)|\lambda_p><\lambda_p|\hat{\phi}(y)|\Omega>.
\end{aligned}\tag{7.92}$$

The first term vanishes by symmetry (scalar field) or by Lorentz invariance (spinor and gauge fields). By translation invariance $\hat{\phi}(x) = \exp(iPx)\hat{\phi}(0)\exp(-iPx)$. Furthermore $|\lambda_P >= U|\lambda_0 >$ where U is the unitary transformation which implements the Lorentz boost which takes the momentum $\vec{0}$ to the momentum $\vec{p}$. Also we recall that the field operator $\hat{\phi}(0)$ and the ground state $|\Omega >$ are both Lorentz invariant. By using all these facts we can verify that $< \Omega|\hat{\phi}(x)|\lambda_p >= e^{-ipx} < \Omega|\hat{\phi}(0)|\lambda_0 >$. We get then

$$< \Omega|T(\hat{\phi}(x)\hat{\phi}(y))|\Omega >= \sum_\lambda \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)} \, e^{-ip(x-y)}| < \Omega|\hat{\phi}(0)|\lambda_0 > |^2. \tag{7.93}$$

In this expression $p^0 = E_p(\lambda)$. We use the identity (the contour is closed below since $x^0 > y^0$)

$$\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m_\lambda^2 + i\epsilon} \, e^{-ip(x-y)} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p(\lambda)} \, e^{-ip(x-y)}. \tag{7.94}$$

Hence we get

$$\begin{aligned} < \Omega|T(\hat{\phi}(x)\hat{\phi}(y))|\Omega > &= \sum_\lambda \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m_\lambda^2 + i\epsilon} \, e^{-ip(x-y)}| < \Omega|\hat{\phi}(0)|\lambda_0 > |^2 \\ &= \sum_\lambda D_F(x-y; m_\lambda)| < \Omega|\hat{\phi}(0)|\lambda_0 > |^2. \end{aligned} \tag{7.95}$$

We get the same result for $x^0 < y^0$. We put this result into the suggestive form

$$< \Omega|T(\hat{\phi}(x)\hat{\phi}(y))|\Omega >= \int_0^\infty \frac{dM^2}{2\pi} D_F(x-y; M)\rho(M^2). \tag{7.96}$$

$$\rho(M^2) = \sum_\lambda (2\pi)\delta(M^2 - m_\lambda^2)| < \Omega|\hat{\phi}(0)|\lambda_0 > |^2. \tag{7.97}$$

The distribution $\rho(M^2)$ is called Källén-Lehmann spectral density. The one-particle states will contribute to the spectral density only a delta function corresponding to the pole at the exact or physical mass m_r of the scalar ϕ particle, viz

$$\rho(M^2) = (2\pi)\delta(M^2 - m_r^2)Z + ... \tag{7.98}$$

We note that the mass m appearing in the Lagrangian (the bare mass) is generally different from the physical mass. The coefficient Z is the so-called field-strength or wave function renormalization and it is equal to the corresponding probability $| < \Omega|\hat{\phi}(0)|\lambda_0 > |^2$. We have then

$$< \Omega|T(\hat{\phi}(x)\hat{\phi}(y))|\Omega >= ZD_F(x-y; m_r) + \int_{4m_r^2}^\infty \frac{dM^2}{2\pi} D_F(x-y; M)\rho(M^2). \tag{7.99}$$

The lower bound $4m_r^2$ comes from the fact that there will be essentially nothing else between the one-particle states at the simple pole $p^2 = m_r^2$ and the 2−particle and multiparticle continuum states starting at $p^2 = 4m_r^2$ which correspond to a branch cut. Indeed by taking the Fourier transform of the above equation we get

$$\int d^4x e^{ip(x-y)} \ < \Omega|T(\hat{\phi}(x)\hat{\phi}(y))|\Omega >= \frac{iZ}{p^2 - m_r^2 + i\epsilon} + \int_{4m_r^2}^\infty \frac{dM^2}{2\pi}\frac{i}{p^2 - M^2 + i\epsilon}\rho(M^2). \tag{7.100}$$

For a spinor field the same result holds. The Fourier transform of the full 2−point function $<\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega>$ is precisely given by the free Dirac propagator in momentum space with the physical mass m_r instead of the bare mass m times a field-strength normalization Z_2. In other words

$$\int d^4x e^{ip(x-y)} \quad <\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega>=\frac{iZ_2(\gamma.p+m_r)}{p^2-m_r^2+i\epsilon}+... \tag{7.101}$$

7.5 One-loop Calculation II: Electron Self-Energy

7.5.1 Electron Mass at One-Loop

From our discussion of the processes $e^-+e^+\longrightarrow\mu^-+\mu^+$, $e^-+\mu^-\longrightarrow e^-+\mu^-$ and electron scattering from an external electromagnetic field we know that there are radiative corrections to the probability amplitudes which involve correction to the external legs. From the corresponding Feynman diagrams we can immediately infer that the first two terms (tree level+one-loop) in the perturbative expansion of the fermion 2−point function $\int d^4x e^{ip(x-y)} \quad <\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega>$ is given by the two diagrams 2POINTFER. By using Feynamn rules we find the expression

$$\begin{aligned}\int d^4x e^{ip(x-y)} \quad <\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega> &= \frac{i(\gamma.p+m_e)}{p^2-m_e^2+i\epsilon}+\frac{i(\gamma.p+m_e)}{p^2-m_e^2+i\epsilon}(-ie\gamma^\mu)\\ &\times \int\frac{d^4k}{(2\pi)^4}\frac{i(\gamma.k+m_e)}{k^2-m_e^2+i\epsilon}\frac{-i\eta_{\mu\nu}}{(p-k)^2+i\epsilon}(-ie\gamma^\nu)\frac{i(\gamma.p+m_e)}{p^2-m_e^2+i\epsilon}\\ &= \frac{i(\gamma.p+m_e)}{p^2-m_e^2+i\epsilon}+\frac{i(\gamma.p+m_e)}{p^2-m_e^2+i\epsilon}(-i\Sigma_2(p))\frac{i(\gamma.p+m_e)}{p^2-m_e^2+i\epsilon}.\end{aligned} \tag{7.102}$$

The second term is the so-called self-energy of the electron. It is given in terms of the loop integral $\Sigma_2(p)$ which in turn is given by

$$-i\Sigma_2(p) \quad = \quad (-ie)^2\int\frac{d^4k}{(2\pi)^4}\gamma^\mu\frac{i(\gamma.k+m_e)}{k^2-m_e^2+i\epsilon}\gamma_\mu\frac{-i}{(p-k)^2+i\epsilon}. \tag{7.103}$$

Sometimes we will also call this quantity the electron self-energy. The two-point function $\int d^4x e^{ip(x-y)} \quad <\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega>$ is not of the form (7.101). To see this more clearly we rewrite the above equation in the form

$$\begin{aligned}\int d^4x e^{ip(x-y)} \quad <\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega> &= \frac{i}{\gamma.p-m_e}+\frac{i}{\gamma.p-m_e}(-i\Sigma_2(p))\frac{i}{\gamma.p-m_e}\\ &= \frac{i}{\gamma.p-m_e}\left[1+\Sigma_2(p)\frac{1}{\gamma.p-m_e}\right].\end{aligned} \tag{7.104}$$

By using now the fact that $\Sigma_2(p)$ commutes with $\gamma.p$ (see below) and the fact that it is supposed to be small of order e^2 we rewrite this equation in the form

$$\int d^4x e^{ip(x-y)} \quad <\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega> \quad = \quad \frac{i}{\gamma.p-m_e-\Sigma_2(p)}. \tag{7.105}$$

This is almost of the desired form (7.101). The loop-integral $\Sigma_2(p)$ is precisely the one-loop correction to the electron mass.

Physically what we have done here is to add together all the Feynman diagrams with an arbitrary number of insertions of the loop integral $\Sigma_2(p)$. These are given by the Feynman diagrams SELF. By using Feynamn rules we find the expression

$$\begin{aligned}\int d^4x e^{ip(x-y)} < \Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega > &= \frac{i}{\gamma.p-m_e}+\frac{i}{\gamma.p-m_e}(-i\Sigma_2(p))\frac{i}{\gamma.p-m_e}\\ &+ \frac{i}{\gamma.p-m_e}(-i\Sigma_2(p))\frac{i}{\gamma.p-m_e}(-i\Sigma_2(p))\frac{i}{\gamma.p-m_e}\\ &+ ...\\ &= \frac{i}{\gamma.p-m_e}\bigg[1+\Sigma_2(p)\frac{1}{\gamma.p-m_e}+(\Sigma_2(p)\frac{1}{\gamma.p-m_e})^2+...\bigg].\end{aligned} \tag{7.106}$$

This is a geometric series. The summation of this geometric series is precisely (7.105).

The loop integral $-i\Sigma_2(p)$ is an example of a one-particle irreducible (1PI) diagram. The one-particle irreducible diagrams are those diagrams which can not be split in two by cutting a single internal line. The loop integral $-i\Sigma_2(p)$ is the first 1PI diagram (order e^2) in the sum $-i\Sigma(p)$ of all 1PI diagrams with 2 fermion lines shown on ONEPARTICLE. Thus the full two-point function $\int d^4x e^{ip(x-y)} < \Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega >$ is actually of the form

$$\begin{aligned}\int d^4x e^{ip(x-y)} < \Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega > &= \frac{i}{\gamma.p-m_e}+\frac{i}{\gamma.p-m_e}(-i\Sigma(p))\frac{i}{\gamma.p-m_e}\\ &+ \frac{i}{\gamma.p-m_e}(-i\Sigma(p))\frac{i}{\gamma.p-m_e}(-i\Sigma(p))\frac{i}{\gamma.p-m_e}\\ &+ ...\\ &= \frac{i}{\gamma.p-m_e-\Sigma(p)}.\end{aligned} \tag{7.107}$$

The physical or renormalized mass m_r is defined as the pole of the two-point function $\int d^4x e^{ip(x-y)} < \Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega >$, viz

$$(\gamma.p-m_e-\Sigma(p))_{\gamma.p=m_r}=0. \tag{7.108}$$

Since $\Sigma(p)=\Sigma(\gamma.p)$ (see below) we have

$$m_r-m_e-\Sigma(m_r)=0. \tag{7.109}$$

We expand $\Sigma(p)=\Sigma(\gamma.p)$ as

$$\Sigma(p)=\Sigma(m_r)+(\gamma.p-m_r)\frac{d\Sigma}{d\gamma.p}|_{\gamma.p=m_r}+O((\gamma.p-m_r)^2). \tag{7.110}$$

Hence

$$\begin{aligned}\gamma.p-m_e-\Sigma(p) &= (\gamma.p-m_r)\frac{1}{Z_2}-O((\gamma.p-m_r)^2)\\ &= (\gamma.p-m_r)\frac{1}{Z_2}(1+O^{'}((\gamma.p-m_r))).\end{aligned} \tag{7.111}$$

$$Z_2^{-1}=1-\frac{d\Sigma}{d\gamma.p}|_{\gamma.p=m_r}. \tag{7.112}$$

Thus

$$\int d^4x e^{ip(x-y)} \quad <\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega> \quad = \quad \frac{iZ_2}{\gamma.p-m_r}. \tag{7.113}$$

This is the desired form (7.101). The correction to the mass is given by (7.109) or equivalently

$$\delta m_r = m_r - m_e = \Sigma(m_r). \tag{7.114}$$

We are interested in just the one-loop correction. Thus

$$\delta m_r = m_r - m_e = \Sigma_2(m_r). \tag{7.115}$$

We evaluate the loop integral $\Sigma_2(p)$ by the same method used for the vertex correction, i.e. we introduce Feynman parameters, we Wick rotate and then we regularize the ultraviolet divergence using the Pauli-Villars method. Clearly the integral is infrared divergent so we will also add a small photon mass. In summary we would like to compute

$$-i\Sigma_2(p) \quad = \quad (-ie)^2\int \frac{d^4k}{(2\pi)^4}\gamma^{\mu}\frac{i(\gamma.k+m_e)}{k^2-m_e^2+i\epsilon}\gamma_{\mu}\bigg[\frac{-i}{(p-k)^2-\mu^2+i\epsilon}-\frac{-i}{(p-k)^2-\Lambda^2+i\epsilon}\bigg]. \tag{7.116}$$

We have (with $L=k-(1-x_1)p$, $\Delta_{\mu}=-x_1(1-x_1)p^2+x_1m_e^2+(1-x_1)\mu^2$)

$$\begin{aligned}\frac{1}{k^2-m_e^2+i\epsilon}\frac{1}{(p-k)^2-\mu^2+i\epsilon} &= \int dx_1\frac{1}{\bigg[x_1(k^2-m_e^2+i\epsilon)+(1-x_1)((p-k)^2-\mu^2+i\epsilon)\bigg]^2}\\ &= \int dx_1\frac{1}{(L^2-\Delta_{\mu}+i\epsilon)^2}.\end{aligned} \tag{7.117}$$

Thus

$$\begin{aligned}-i\Sigma_2(p) &= -e^2\int\frac{d^4k}{(2\pi)^4}\gamma^{\mu}(\gamma.k+m_e)\gamma_{\mu}\bigg[\int dx_1\frac{1}{(L^2-\Delta_{\mu}+i\epsilon)^2}-\int dx_1\frac{1}{(L^2-\Delta_{\Lambda}+i\epsilon)^2}\bigg]\\ &= -e^2\int\frac{d^4k}{(2\pi)^4}(-2\gamma.k+4m_e)\bigg[\int dx_1\frac{1}{(L^2-\Delta_{\mu}+i\epsilon)^2}-\int dx_1\frac{1}{(L^2-\Delta_{\Lambda}+i\epsilon)^2}\bigg]\\ &= -e^2\int dx_1(-2(1-x_1)\gamma.p+4m_e)\int\frac{d^4L}{(2\pi)^4}\bigg[\frac{1}{(L^2-\Delta_{\mu}+i\epsilon)^2}-\frac{1}{(L^2-\Delta_{\Lambda}+i\epsilon)^2}\bigg]\\ &= -ie^2\int dx_1(-2(1-x_1)\gamma.p+4m_e)\int\frac{d^4L_E}{(2\pi)^4}\bigg[\frac{1}{(L_E^2+\Delta_{\mu})^2}-\frac{1}{(L_E^2+\Delta_{\Lambda})^2}\bigg]\\ &= -\frac{ie^2}{8\pi^2}\int dx_1(-2(1-x_1)\gamma.p+4m_e)\int r^3dr\bigg[\frac{1}{(r^2+\Delta_{\mu})^2}-\frac{1}{(r^2+\Delta_{\Lambda})^2}\bigg]\\ &= -\frac{ie^2}{16\pi^2}\int dx_1(-2(1-x_1)\gamma.p+4m_e)\ln\frac{\Delta_{\Lambda}}{\Delta_{\mu}}.\end{aligned} \tag{7.118}$$

The final result is

$$\Sigma_2(p) \quad = \quad \frac{\alpha}{2\pi}\int dx_1(-(1-x_1)\gamma.p+2m_e)\ln\frac{(1-x_1)\Lambda^2}{-x_1(1-x_1)p^2+x_1m_e^2+(1-x_1)\mu^2} \tag{7.119}$$

This is logarithmically divergent. Thus the mass correction or shift at one-loop is logarithmically divergent given by

$$\delta m_r = \Sigma_2(\gamma.p = m_r) = \frac{\alpha m_e}{2\pi}\int dx_1 (2-x_1)\ln\frac{x_1\Lambda^2}{(1-x_1)^2 m_e^2 + x_1\mu^2}. \quad (7.120)$$

The physical mass is therefore given by

$$m_r = m_e\bigg[1 + \frac{\alpha}{2\pi}\int dx_1(2-x_1)\ln\frac{x_1\Lambda^2}{(1-x_1)^2m_e^2 + x_1\mu^2}\bigg]. \quad (7.121)$$

Clearly the bare mass m_e must depend on the cutoff Λ in such a way that in the limit $\Lambda \longrightarrow \infty$ the physical mass m_r remains finite.

7.5.2 The Wave-Function Renormalization Z_2

At one-loop order we also need to compute the wave function renormalization. We have

$$\begin{aligned} Z_2^{-1} &= 1 - \frac{d\Sigma_2}{d\gamma.p}|_{\gamma.p=m_r} \\ &= 1 - \frac{\alpha}{2\pi}\int dx_1\bigg[-(1-x_1)\ln\frac{(1-x_1)\Lambda^2}{-x_1(1-x_1)p^2 + x_1m_e^2 + (1-x_1)\mu^2} \\ &+ (-(1-x_1)\gamma.p + 2m_e)(2\gamma.p)\frac{x_1(1-x_1)}{-x_1(1-x_1)p^2 + x_1m_e^2 + (1-x_1)\mu^2}\bigg]_{\gamma.p=m_r} \\ &= 1 - \frac{\alpha}{2\pi}\int dx_1\bigg[-(1-x_1)\ln\frac{(1-x_1)\Lambda^2}{x_1^2m_e^2 + (1-x_1)\mu^2} + \frac{2m_e^2x_1(1-x_1)(1+x_1)}{x_1^2m_e^2 + (1-x_1)\mu^2}\bigg] \end{aligned} \quad (7.122)$$

Thus

$$Z_2 = 1 + \delta Z_2. \quad (7.123)$$

$$\delta Z_2 = \frac{\alpha}{2\pi}\int_0^1 dx_1\bigg[-(1-x_1)\ln\frac{(1-x_1)\Lambda^2}{x_1^2m_e^2 + (1-x_1)\mu^2} + \frac{2m_e^2x_1(1-x_1)(1+x_1)}{x_1^2m_e^2 + (1-x_1)\mu^2}\bigg]. \quad (7.124)$$

A very deep observation is given by the identity $\delta Z_2 = \delta F_1(0) = F_1(0) - 1$ where $F_1(q^2)$ is given by (7.68). We have

$$\delta F_1(0) = \frac{\alpha}{2\pi}\int dxdydz\ \delta(x+y+z-1)\bigg[\ln\frac{x\Lambda^2}{(1-x)^2m_e^2 + x\mu^2} + \frac{m_e^2(1+x^2-4x)}{(1-x)^2m_e^2 + x\mu^2}\bigg]. \quad (7.125)$$

Clearly for $x=0$ we have $\int_0^1 dy \int_0^1 dz\ \delta(y+z-1) = 1$ whereas for $x=1$ we have $\int_0^1 dy\int_0^1 dz\ \delta(y+z) = 0$. In general

$$\int_0^1 dy\int_0^1 dz\ \delta(x+y+z-1) = 1-x. \quad (7.126)$$

The proof is simple. Since $0 \leq x \leq 1$ we have $0 \leq 1-x \leq 1$ and $1/(1-x) \geq 1$. We shift the variables as $y = (1-x)y^{'}$ and $z = (1-x)z^{'}$. We have

$$\begin{aligned} \int_0^1 dy\int_0^1 dz\ \delta(x+y+z-1) &= (1-x)^2\int_0^{1/(1-x)} dy^{'}\int_0^{1/(1-x)} dz^{'}\ \frac{1}{1-x}\delta(y^{'}+z^{'}-1) \\ &= 1-x. \end{aligned} \quad (7.127)$$

By using this identity we get

$$
\begin{aligned}
\delta F_1(0) &= \frac{\alpha}{2\pi}\int dx(1-x)\bigg[\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}+\frac{m_e^2(1+x^2-4x)}{(1-x)^2m_e^2+x\mu^2}\bigg]\\
&= \frac{\alpha}{2\pi}\int dx\bigg[x\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}+(1-2x)\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}+\frac{m_e^2(1-x)(1+x^2-4x)}{(1-x)^2m_e^2+x\mu^2}\bigg]\\
&= \frac{\alpha}{2\pi}\int dx\bigg[x\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}+\frac{d(x-x^2)}{dx}\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}+\frac{m_e^2(1-x)(1+x^2-4x)}{(1-x)^2m_e^2+x\mu^2}\bigg]\\
&= \frac{\alpha}{2\pi}\int dx\bigg[x\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}-(x-x^2)\frac{d}{dx}\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}+\frac{m_e^2(1-x)(1+x^2-4x)}{(1-x)^2m_e^2+x\mu^2}\bigg]\\
&= \frac{\alpha}{2\pi}\int dx\bigg[x\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}-\frac{m_e^2(1-x)(1-x^2)}{(1-x)^2m_e^2+x\mu^2}+\frac{m_e^2(1-x)(1+x^2-4x)}{(1-x)^2m_e^2+x\mu^2}\bigg]\\
&= \frac{\alpha}{2\pi}\int dx\bigg[x\ln\frac{x\Lambda^2}{(1-x)^2m_e^2+x\mu^2}-\frac{2m_e^2x(1-x)(2-x)}{(1-x)^2m_e^2+x\mu^2}\bigg]\\
&= \frac{\alpha}{2\pi}\int dt\bigg[(1-t)\ln\frac{(1-t)\Lambda^2}{t^2m_e^2+(1-t)\mu^2}-\frac{2m_e^2t(1-t)(1+t)}{t^2m_e^2+(1-t)\mu^2}\bigg].
\end{aligned}
\tag{7.128}
$$

We can immediately conclude that $\delta F_1(0)=-\delta Z_2$.

7.5.3 The Renormalization Constant Z_1

In our calculation of the vertex correction we have used the bare propagator $i/(\gamma.p-m_e)$ which has a pole at the bare mass $m=m_e$ which is as we have seen is actually a divergent quantity. This calculation should be repeated with the physical propagator $iZ_2/(\gamma.p-m_r)$. This propagator is obtained by taking the sum of the Feynman diagrams shown on SELF and ONEPARTICLE.

We reconsider the problem of scattering of an electron from an external electromagnetic field. The probability amplitude is given by the formula (7.13). We rewrite this formula as [1]

$$
\begin{aligned}
<\vec{p}^{'}s^{'}\ \text{out}|\vec{p}s\ \text{in}> &= -\Big[\bar{u}^{s^{'}}(p^{'})(\gamma.p^{'}-m_e)\Big]_{\alpha^{'}}\int d^4x\int d^4x^{'}\ e^{-ipx+ip^{'}x^{'}}<\Omega|T(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_{\alpha}(x))|\Omega>\\
&\times\ \Big[(\gamma.p-m_e)u^s(p)\Big]_{\alpha}.
\end{aligned}
\tag{7.129}
$$

We sum up the quantum corrections to the two external legs by simply making the replacements

$$
\gamma.p^{'}-m_e\longrightarrow(\gamma.p^{'}-m_r)/Z_2\ ,\ \gamma.p-m_e\longrightarrow(\gamma.p-m_r)/Z_2. \tag{7.130}
$$

The probability for the spinor field to create or annihilate a particle is precisely Z_2 since $<\Omega|\hat{\psi}(0)|\vec{p},s>=\sqrt{Z_2}u^s(p)$. Thus one must also replace $u^s(p)$ and $\bar{u}^{s^{'}}(p^{'})$ by $\sqrt{Z_2}u^s(p)$ and $\sqrt{Z_2}\bar{u}^{s^{'}}(p^{'})$.

Furthermore from our previous experience we know that the 2-point function $\int d^4x\int d^4x^{'}\ e^{-ipx+ip^{'}x^{'}}<\Omega|T(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_{\alpha}(x))|\Omega>$ will be equal to the product of the two external propagators $iZ_2/(\gamma.p-$

[1]In writing this formula in this form we use the fact that $|0\ \text{out}>=|0\ \text{in}>=|0>$ and $|\Omega>=|0>/\sqrt{<0|0>}$. Recall that dividing by $<0|0>$ is equivalent to taking into account only connected Feynman graphs.

m_r) and $iZ_2/(\gamma.p^{'}-m_r)$ times the amputated electron-photon vertex $\int d^4x\int d^4x^{'}\ e^{-ipx+ip^{'}x^{'}}<\Omega|T(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_{\alpha}(x))|\Omega>_{\rm amp}$. Thus we make the replacement

$$<\Omega|T(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_{\alpha}(x))|\Omega>\longrightarrow\frac{iZ_2}{\gamma.p^{'}-m_r}<\Omega|T(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_{\alpha}(x))|\Omega>\frac{iZ_2}{\gamma.p-m_r}. \quad (7.131)$$

The formula of the probability amplitude $<\vec{p}^{'}s^{'}\ {\rm out}|\vec{p}s\ {\rm in}>$ becomes

$$<\vec{p}^{'}s^{'}\ {\rm out}|\vec{p}s\ {\rm in}>\ =\ Z_2\bar{u}^{s^{'}}(p^{'})_{\alpha^{'}}\int d^4x\int d^4x^{'}\ e^{-ipx+ip^{'}x^{'}}<\Omega|T(\hat{\psi}_{\alpha^{'}}(x^{'})\bar{\hat{\psi}}_{\alpha}(x))|\Omega>_{\rm amp}u^s(p)_{\alpha}. \quad (7.132)$$

The final result is that the amputated electron-photon vertex $\Gamma_\lambda(p^{'},p)$ must be multiplied by Z_2, viz

$$<\vec{p}^{'}s^{'}\ {\rm out}|\vec{p}s\ {\rm in}>\ =\ -ie\bigg(\bar{u}^{s^{'}}(p^{'})Z_2\Gamma_\lambda(p^{'},p)u^s(p)\bigg)A^{\lambda,{\rm backgr}}(q). \quad (7.133)$$

What we have done here is to add together the two Feynman diagrams VERTEXCOR. In the one-loop diagram the internal electron propagators are replaced by renormalized propagators.

In general an amputated Green's function with n incoming lines and m outgoing lines must be multiplied by a factor $(\sqrt{Z_2})^{n+m}$ in order to yield correctly the corresponding $S-$matrix element.

The calculation of the above probability amplitude will proceed exactly as before. The result by analogy with equation (7.71) must be of the form

$$<\vec{p}^{'}s^{'}\ {\rm out}|\vec{p}s\ {\rm in}>\ =\ -ie\bar{u}^{s^{'}}(p^{'})\bigg[\gamma_\lambda F_1^{'}(q^2)+\frac{i\sigma_{\lambda\gamma}q^\gamma}{2m_r}F_2^{'}(q^2)\bigg]u^s(p).A^{\lambda,{\rm backgr}}(q). \quad (7.134)$$

In other words

$$\begin{aligned}Z_2\Gamma_\lambda(p^{'},p)\ &=\ \gamma_\lambda F_1^{'}(q^2)+\frac{i\sigma_{\lambda\gamma}q^\gamma}{2m_r}F_2^{'}(q^2)\\ &=\ \gamma_\lambda F_1(q^2)+\frac{i\sigma_{\lambda\gamma}q^\gamma}{2m_r}F_2(q^2)+\gamma_\lambda\Delta F_1(q^2)+\frac{i\sigma_{\lambda\gamma}q^\gamma}{2m_r}\Delta F_2(q^2). \quad (7.135)\end{aligned}$$

We are interested in order α. Since $Z_2=1+\delta Z_2$ where $\delta Z_2=O(\alpha)$ we have $Z_2\Gamma_\lambda=\Gamma_\lambda+\delta Z_2\Gamma_\lambda=\Gamma_\lambda+\delta Z_2\gamma_\lambda$ to order α. By using also the fact that $F_2^{'}=O(\alpha)$ we must have $\Delta F_2=0$. We conclude that we must have $\Delta F_1=\delta Z_2$. Since $\delta Z_2=-\delta F_1(0)$ we have the final result

$$\begin{aligned}F_1^{'}(q^2)\ &=\ F_1(q^2)+\Delta F_1(q^2)\\ &=\ F_1(q^2)+\delta Z_2\\ &=\ F_1(q^2)-\delta F_1(0)\\ &=\ 1+\delta F_1(q^2)-\delta F_1(0)\\ &=\ F_1^{\rm ren}(q^2). \quad (7.136)\end{aligned}$$

We introduce a new renormalization constant Z_1 by the relation

$$Z_1\Gamma_\lambda(q=0)\ =\ \gamma_\lambda. \quad (7.137)$$

The requirement that $F_1^{\rm ren}(0)=1$ is equivalent to the statement that $Z_1=Z_2$.

7.6 Ward-Takahashi Identities

Ward-Takahashi Identities: Let us start by considering the $3-$point function $\partial_\mu T(\hat{j}^\mu(x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})$. For $y_0 > y_0^{'}$ we have explicitly

$$T(\hat{j}^\mu(x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})) = \theta(x_0-y_0)\hat{j}^\mu(x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})+\theta(y_0^{'}-x_0)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})\hat{j}^\mu(x) + \theta(y_0-x_0)\theta(x_0-y_0^{'})\hat{\psi}(y)\hat{j}^\mu(x)\bar{\hat{\psi}}(y^{'}). \quad (7.138)$$

Recall that $\hat{j}^\mu = e\bar{\hat{\psi}}\gamma^\mu\hat{\psi}$. We compute immediately that (using current conservation $\partial_\mu\hat{j}^\mu=0$)

$$\begin{aligned}\partial_\mu T(\hat{j}^\mu(x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})) &= \delta(x_0-y_0)\hat{j}^0(x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})-\delta(y_0^{'}-x_0)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})\hat{j}^0(x)\\ &- \delta(y_0-x_0)\theta(x_0-y_0^{'})\hat{\psi}(y)\hat{j}^0(x)\bar{\hat{\psi}}(y^{'})+\theta(y_0-x_0)\delta(x_0-y_0^{'})\hat{\psi}(y)\hat{j}^0(x)\bar{\hat{\psi}}(y^{'})\\ &= \delta(x_0-y_0)[\hat{j}^0(x),\hat{\psi}(y)]\bar{\hat{\psi}}(y^{'})-\delta(y_0^{'}-x_0)\hat{\psi}(y)[\bar{\hat{\psi}}(y^{'}),\hat{j}^0(x)]. \quad (7.139)\end{aligned}$$

We compute $[\hat{j}^0(x),\hat{\psi}(y)]=-e\delta^3(\vec{x}-\vec{y})\hat{\psi}(y)$ and $[\bar{\hat{\psi}}(y^{'}),\hat{j}^0(x)]=-e\delta^3(\vec{x}-\vec{y}^{'})\bar{\hat{\psi}}(y^{'})$. Hence we get

$$\partial_\mu T(\hat{j}^\mu(x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})) = -e\delta^4(x-y)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})+e\delta(y^{'}-x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'}). \quad (7.140)$$

The full result is clearly

$$\partial_\mu T(\hat{j}^\mu(x)\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})) = \Big(-e\delta^4(x-y)+e\delta(y^{'}-x)\Big)T(\hat{\psi}(y)\bar{\hat{\psi}}(y^{'})). \quad (7.141)$$

In general we would have

$$\begin{aligned}\partial_\mu T(\hat{j}^\mu(x)\hat{\psi}(y_1)\bar{\hat{\psi}}(y_1^{'})...\hat{\psi}(y_n)\bar{\hat{\psi}}(y_n^{'})\hat{A}^{\alpha_1}(z_1)...) &= \sum_{i=1}^n\Big(-e\delta^4(x-y_i)+e\delta(y_i^{'}-x)\Big)T(\hat{\psi}(y_1)\bar{\hat{\psi}}(y_1^{'})...\\ &\times \hat{\psi}(y_n)\bar{\hat{\psi}}(y_n^{'})\hat{A}^{\alpha_1}(z_1)...). \quad (7.142)\end{aligned}$$

These are the Ward-Takahashi identities. Another important application of these identities is

$$\partial_\mu T(\hat{j}^\mu(x)\hat{A}^{\alpha_1}(z_1)...)=0. \quad (7.143)$$

Exact Photon Propagator: The exact photon propagator is defined by

$$\begin{aligned}iD^{\mu\nu}(x-y) &= <0\text{ out}|T(\hat{A}^\mu(x)\hat{A}^\nu(y))|0\text{ in}>\\ &= <0\text{ in}|T(\hat{A}^\mu_{\rm in}(x)\hat{A}^\nu_{\rm in}(y)S)|0\text{ in}>\\ &= iD_F^{\mu\nu}(x-y)+\frac{(-i)^2}{2}\int d^4z_1\int d^4z_2<0\text{ in}|T(\hat{A}^\mu_{\rm in}(x)\hat{A}^\nu_{\rm in}(y)\hat{A}^{\rho_1}_{\rm in}(z_1)\hat{A}^{\rho_2}_{\rm in}(z_2))|0\text{ in}>\\ &\times <0\text{ in}|T(\hat{j}_{{\rm in},\rho_1}(z_1)\hat{j}_{{\rm in},\rho_2}(z_2))|0\text{ in}>+...\\ &= iD_F^{\mu\nu}(x-y)+(-i)^2\int d^4z_1 iD_F^{\mu\rho_1}(x-z_1)\int d^4z_2 iD_F^{\nu\rho_2}(y-z_2)<0\text{ in}|T(\hat{j}_{{\rm in},\rho_1}(z_1)\\ &\times \hat{j}_{{\rm in},\rho_2}(z_2))|0\text{ in}>+... \quad (7.144)\end{aligned}$$

This can be rewritten as

$$\begin{aligned}iD^{\mu\nu}(x-y) &= iD_F^{\mu\nu}(x-y)-i\int d^4z_1 iD_F^{\mu\rho_1}(x-z_1)<0\text{ in}|T(\hat{j}_{{\rm in},\rho_1}(z_1)\hat{A}^\nu_{\rm in}(y)\Big(-i\int d^4z_2\hat{A}^{\rho_2}_{\rm in}(z_2)\\ &\times \hat{j}_{{\rm in},\rho_2}(z_2)\Big))|0\text{ in}>+... \quad (7.145)\end{aligned}$$

This is indeed correct since we can write the exact photon propagator in the form

$$\begin{aligned} iD^{\mu\nu}(x-y) &= iD_F^{\mu\nu}(x-y) - i\int d^4z_1 iD_F^{\mu\rho_1}(x-z_1) < 0\text{ out}|T(\hat{j}_{\rho_1}(z_1)\hat{A}^\nu(y)|0\text{ in} > . \\ &= iD_F^{\mu\nu}(x-y) - i\int d^4z_1 iD_F^{\mu\rho_1}(z_1) < 0\text{ out}|T(\hat{j}_{\rho_1}(z_1+x)\hat{A}^\nu(y)|0\text{ in} > . \end{aligned} \tag{7.146}$$

See the Feynman diagram EXACTPHOTON. By using the identity (7.143) we see immediately that

$$i\partial_{\mu,x}D^{\mu\nu}(x-y) = i\partial_{\mu,x}D_F^{\mu\nu}(x-y). \tag{7.147}$$

In momentum space this reads

$$q_\mu D^{\mu\nu}(q) = q_\mu D_F^{\mu\nu}(q). \tag{7.148}$$

This expresses transversality of the vacuum polarization (more on this below).

Exact Vertex Function: Let us now discuss the exact vertex function $V^\mu(p^{'},p)$ defined by

$$-ie(2\pi)^4\delta^4(p^{'}-p-q)V^\mu(p^{'},p) = \int d^4x\int d^4x_1\int d^4y_1\; e^{i(p^{'}x_1-py_1-qx)}\; < \Omega|T(\hat{A}^\mu(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega > . \tag{7.149}$$

See the Feynman graph VERTEXEXACT1. We compute (with $D_F^{\mu\nu}(q) = -i\eta^{\mu\nu}/(q^2+i\epsilon)$)

$$\begin{aligned} \int d^4x e^{-iqx} < 0\text{ out}|T(\hat{A}^\mu(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|0\text{ in} > &= \int d^4x e^{-iqx} < 0\text{ in}|T(\hat{A}^\mu_{\text{in}}(x)\hat{\psi}_{\text{in}}(x_1)\bar{\hat{\psi}}_{\text{in}}(y_1)S)|0\text{ in} > \\ &= -i\int d^4xe^{-iqx}\int d^4z < 0\text{ in}|T(\hat{A}^\mu_{\text{in}}(x)\hat{A}^\nu_{\text{in}}(z)\hat{j}_{\text{in},\nu}(z) \\ &\times\; \hat{\psi}_{\text{in}}(x_1)\bar{\hat{\psi}}_{\text{in}}(y_1))|0\text{ in} > +... \\ &= -i\int d^4xe^{-iqx}\int d^4z iD_F^{\mu\nu}(x-z) < 0\text{ in}|T(\hat{j}_{\text{in},\nu}(z) \\ &\times\; \hat{\psi}_{\text{in}}(x_1)\bar{\hat{\psi}}_{\text{in}}(y_1))|0\text{ in} > +... \\ &= -iD_F^{\mu\nu}(q)\int d^4xe^{-iqx} < 0\text{ in}|T(\hat{j}_{\text{in},\nu}(x)\hat{\psi}_{\text{in}}(x_1) \\ &\times\; \bar{\hat{\psi}}_{\text{in}}(y_1))|0\text{ in} > +... \end{aligned} \tag{7.150}$$

This result holds to all orders of perturbation. In other words we must have

$$\int d^4xe^{-iqx} < \Omega|T(\hat{A}^\mu(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega > = -iD^{\mu\nu}(q)\int d^4xe^{-iqx} < \Omega|T(\hat{j}_\nu(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega > . \tag{7.151}$$

It is understood that $D^{\mu\nu}(q)$ is the full photon propagator. We must then have

$$\begin{aligned} -ie(2\pi)^4\delta^4(p^{'}-p-q)V^\mu(p^{'},p) &= -iD^{\mu\nu}(q)\int d^4x\int d^4x_1\int d^4y_1\; e^{i(p^{'}x_1-py_1-qx)}\; < \Omega|T(\hat{j}_\nu(x)\hat{\psi}(x_1) \\ &\times\; \bar{\hat{\psi}}(y_1))|\Omega > . \end{aligned} \tag{7.152}$$

In terms of the vertex function $\Gamma^{\mu}(p^{'},p)$ defined previously and the exact fermion propagators $S(p)$, $S(p^{'})$ and the exact photon propagator $D^{\mu\nu}(q)$ we have

$$V^{\mu}(p^{'},p) = D^{\mu\nu}(q)S(p^{'})\Gamma_{\nu}(p^{'},p)S(p). \tag{7.153}$$

This expression means that the vertex function can be decomposed into the QED proper vertex dressed with the full electron and photon propagators. See the Feynman graph VERTEXEXACT.

We have then

$$\begin{aligned}-ie(2\pi)^4\delta^4(p^{'}-p-q)D^{\mu\nu}(q)S(p^{'})\Gamma_{\nu}(p^{'},p)S(p) &= -iD^{\mu\nu}(q)\int d^4x\int d^4x_1\int d^4y_1\ e^{i(p^{'}x_1-py_1-qx)}\\ &\times\ <\Omega|T(\hat{j}_{\nu}(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega>. \end{aligned}\tag{7.154}$$

We contract this equation with q_{μ} we obtain

$$\begin{aligned}-ie(2\pi)^4\delta^4(p^{'}-p-q)q_{\mu}D^{\mu\nu}(q)S(p^{'})\Gamma_{\nu}(p^{'},p)S(p) &= -iq_{\mu}D^{\mu\nu}(q)\int d^4x\int d^4x_1\int d^4y_1\ e^{i(p^{'}x_1-py_1-qx)}\\ &\times\ <\Omega|T(\hat{j}_{\nu}(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega>. \end{aligned}\tag{7.155}$$

By using the identity $q_{\mu}D^{\mu\nu}(q)=q_{\mu}D_F^{\mu\nu}(q)=-iq^{\nu}/(q^2+i\epsilon)$ we obtain

$$\begin{aligned}-ie(2\pi)^4\delta^4(p^{'}-p-q)S(p^{'})q^{\nu}\Gamma_{\nu}(p^{'},p)S(p) &= -iq^{\nu}\int d^4x\int d^4x_1\int d^4y_1\ e^{i(p^{'}x_1-py_1-qx)}\\ &\times\ <\Omega|T(\hat{j}_{\nu}(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega>\\ &= -\int d^4x\int d^4x_1\int d^4y_1\ e^{i(p^{'}x_1-py_1-qx)}\\ &\times\ \partial^{\nu,x}<\Omega|T(\hat{j}_{\nu}(x)\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega>. \end{aligned}\tag{7.156}$$

By using the identity (7.141) we get

$$\begin{aligned}-ie(2\pi)^4\delta^4(p^{'}-p-q)S(p^{'})q^{\nu}\Gamma_{\nu}(p^{'},p)S(p) &= -\int d^4x\int d^4x_1\int d^4y_1\ e^{i(p^{'}x_1-py_1-qx)}\\ &\times\ (-e\delta^4(x-x_1)+e\delta^4(x-y_1))<\Omega|T(\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega>\\ &= e\int d^4x_1\int d^4y_1\ e^{i(p^{'}-q)x_1}\ e^{-ipy_1}<\Omega|T(\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega>\\ &- e\int d^4x_1\int d^4y_1\ e^{ip^{'}x_1}\ e^{-i(p+q)y_1}<\Omega|T(\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega>\\ &= e(2\pi)^4\delta^4(p^{'}-p-q)(S(p)-S(p^{'})). \end{aligned}\tag{7.157}$$

In the above equation we have made use of the Fourier transform

$$<\Omega|T(\hat{\psi}(x_1)\bar{\hat{\psi}}(y_1))|\Omega>=\int\frac{d^4k}{(2\pi)^4}S(k)\ e^{-ik(x_1-y_1)}. \tag{7.158}$$

We derive then the fundamental result

$$-iS(p^{'})q^{\nu}\Gamma_{\nu}(p^{'},p)S(p) = S(p)-S(p^{'}). \tag{7.159}$$

Equivalently we have

$$-iq^{\nu}\Gamma_{\nu}(p^{'},p) = S^{-1}(p^{'})-S^{-1}(p). \tag{7.160}$$

For our purposes this is the most important of all Ward-Takahashi identities.

We know that for p near mass shell, i.e. $p^2 = m_r^2$, the propagator $S(p)$ behaves as $S(p) = iZ_2/(\gamma.p - m_r)$. Since $p^{'} = p+q$ the momentum $p^{'}$ is near mass shell only if p is near mass shell and q goes to 0. Thus near mass shell we have

$$-iq^\nu \Gamma_\nu(p,p) \quad = \quad -iZ_2^{-1} q^\nu \gamma_\nu. \tag{7.161}$$

In other words

$$\Gamma_\nu(p,p) \quad = \quad Z_2^{-1}\gamma^\nu. \tag{7.162}$$

The renormalization constant Z_1 is defined precisely by

$$\Gamma_\nu(p,p) \quad = \quad Z_2^{-1}\gamma^\nu. \tag{7.163}$$

In other words we have

$$Z_1 = Z_2. \tag{7.164}$$

The above Ward-Takahashi identity guarantees $F_1^{\rm ren}(0) = 1$ to all orders in perturbation theory.

7.7 One-Loop Calculation III: Vacuum Polarization

7.7.1 The Renormalization Constant Z_3 and Renormalization of the Electric Charge

The next natural question we can ask is what is the structure of the exact $2-$point photon function. At tree level we know that the answer is given by the bare photon propagator, viz

$$\int d^4x e^{iq(x-y)} \ < \Omega|T(\hat{A}^\mu(x)\hat{A}^\nu(y))|\Omega > \quad = \quad \frac{-i\eta^{\mu\nu}}{q^2+i\epsilon} + ... \tag{7.165}$$

Recall the case of the electron bare propagator which was corrected at one-loop by the electron self-energy $-i\Sigma_2(p)$. By analogy the above bare photon propagator will be corrected at one-loop by the photon self-energy $i\Pi_2^{\mu\nu}(q)$ shown on figure 2POINTPH. By using Feynman rules we have

$$i\Pi_2^{\mu\nu}(q) = (-1)\int \frac{d^4k}{(2\pi)^4}{\rm tr}(-ie\gamma^\mu)\frac{i(\gamma.k+m_e)}{k^2-m_e^2+i\epsilon}(-ie\gamma^\nu)\frac{i(\gamma.(k+q)+m_e)}{(k+q)^2-m_e^2+i\epsilon}. \tag{7.166}$$

This self-energy is the essential ingredient in vacuum polarization diagrams. See for example (7.7).

Similarly to the electron case, the photon self-energy $i\Pi_2^{\mu\nu}(q)$ is only the first diagram (which is of order e^2) among the one-particle irreducible (1PI) diagrams with 2 photon lines which we will denote by $i\Pi^{\mu\nu}(q)$. See figure 2POINTPH1. By Lorentz invariance $i\Pi^{\mu\nu}(q)$ must be a linear combination of $\eta^{\mu\nu}$ and $q^{\mu\nu}$. Now the full $2-$point photon function will be obtained by the sum of all diagrams with an increasing number of insertions of the 1PI diagram $i\Pi^{\mu\nu}(q)$. This is shown on figure 2POINTPHE. The corresponding expression is

$$\begin{aligned}\int d^4x e^{iq(x-y)} \ < \Omega|T(\hat{A}^\mu(x)\hat{A}^\nu(y))|\Omega > \quad &= \quad \frac{-i\eta^{\mu\nu}}{q^2+i\epsilon} + \frac{-i\eta^\mu_\rho}{q^2+i\epsilon}i\Pi^{\rho\sigma}(q)\frac{-i\eta^\nu_\sigma}{q^2+i\epsilon} \\ &+ \quad \frac{-i\eta^\mu_\rho}{q^2+i\epsilon}i\Pi^{\rho\sigma}(q)\frac{-i\eta_{\sigma\lambda}}{q^2+i\epsilon}i\Pi^{\lambda\eta}(q)\frac{-i\eta^\nu_\eta}{q^2+i\epsilon} + \end{aligned} \tag{7.167}$$

By comparing with (7.146) we get

$$-i\int d^4x e^{iq(x-y)}\int d^4z_1 iD_F^{\mu\rho_1}(z_1) < 0 \text{ out}|T(\hat{j}_{\rho_1}(z_1+x)\hat{A}^{\nu}(y)|0 \text{ in}> = \frac{-i\eta^{\mu}_{\rho}}{q^2+i\epsilon}i\Pi^{\rho\sigma}(q)\frac{-i\eta^{\nu}_{\sigma}}{q^2+i\epsilon} + ... \quad (7.168)$$

By contracting both sides with q_μ and using current conservation $\partial_\mu\hat{j}^\mu=0$ we obtain the Ward identity

$$q^\mu\Pi_{\mu\nu}(q)=0. \quad (7.169)$$

Hence we must have

$$\Pi^{\mu\nu}(q)=(q^2\eta^{\mu\nu}-q^\mu q^\nu)\Pi(q^2). \quad (7.170)$$

It is straightforward to show that the exact 2−point photon function becomes

$$\int d^4x e^{iq(x-y)} < \Omega|T(\hat{A}^\mu(x)\hat{A}^\nu(y))|\Omega> = \frac{-i\eta^{\mu\nu}}{q^2+i\epsilon}+\frac{-i\eta^{\mu}_{\rho}}{q^2+i\epsilon}(\eta^{\rho\nu}-\frac{q^\rho q^\nu}{q^2})(\Pi+\Pi^2+...)$$
$$= \frac{-iq^\mu q^\nu}{(q^2)^2}+\frac{-i}{q^2+i\epsilon}\frac{1}{1-\Pi(q^2)}(\eta^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}). \quad (7.171)$$

This propagator has a single pole at $q^2=0$ if the function $\Pi(q^2)$ is regular at $q^2=0$. This is indeed true to all orders in perturbation theory. Physically this means that the photon remains massless. We define the renormalization constant Z_3 as the residue at the $q^2=0$ pole, viz

$$Z_3=\frac{1}{1-\Pi(0)}. \quad (7.172)$$

The terms proportional to $q^\mu q^\nu$ in the above exact propagator will lead to vanishing contributions inside a probability amplitude, i.e. when we connect the exact 2−point photon function to at least one electron line. This is another manifestation of the Ward-Takahashi identities. We give an example of this cancellation next.

The contribution of the tree level plus vacuum polarization diagrams to the probability amplitude of the process $e^-+e^+\longrightarrow\mu^-+\mu^+$ was given by

$$-e^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\bigg(\frac{-i\eta^{\mu\nu}}{q^2}+\frac{-i\eta^{\mu}_{\rho}}{q^2}i\Pi_2^{\rho\sigma}(q)\frac{-i\eta^{\nu}_{\sigma}}{q^2}\bigg)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)). \quad (7.173)$$

By using the exact 2−point photon function this becomes

$$-e^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\bigg(\frac{-iq^\mu q^\nu}{(q^2)^2}+\frac{-i}{q^2+i\epsilon}\frac{1}{1-\Pi(q^2)}(\eta^{\mu\nu}-\frac{q^\mu q^\nu}{q^2})\bigg)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)). \quad (7.174)$$

We can check that $\bar{u}^{s^{'}}(p^{'})\gamma_\mu q^\mu u^s(p)=\bar{u}^{s^{'}}(p^{'})(\gamma_\mu p^\mu-\gamma_\mu p^{'\mu})u^s(p)=0$. We get then the probability amplitude

$$-e^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\bigg(\frac{-i}{q^2+i\epsilon}\frac{1}{1-\Pi(q^2)}\eta^{\mu\nu}\bigg)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)). \quad (7.175)$$

For scattering with very low q^2 this becomes

$$-e^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\bigg(\frac{-i}{q^2+i\epsilon}\frac{1}{1-\Pi(0)}\eta^{\mu\nu}\bigg)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)) =$$
$$-e_R^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\bigg(\frac{-i}{q^2+i\epsilon}\eta^{\mu\nu}\bigg)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)). \quad (7.176)$$

This looks exactly like the tree level contribution with an electric charge e_R given by

$$e_R = e\sqrt{Z_3}. \quad (7.177)$$

The electric charge e_R is called the renormalized electric charge. This shift of the electric charge relative to tree level is a general feautre since the amplitude for any process with very low momentum transfer q^2 when we replace the bare photon propagator with the exact photon propagator will appear as a tree level process with the renoramlized electric charge e_R.

Using the definition of the renormalized electric charge e_R the above probability amplitude can now be put in the form

$$-e^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\bigg(\frac{-i}{q^2+i\epsilon}\frac{1}{1-\Pi(q^2)}\eta^{\mu\nu}\bigg)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)) =$$
$$-e_R^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\bigg(\frac{-i}{q^2+i\epsilon}\frac{1-\Pi(0)}{1-\Pi(q^2)}\eta^{\mu\nu}\bigg)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)) =$$
$$-e_{\rm eff}^2(2\pi)^4\delta^4(k+p-k^{'}-p^{'})(\bar{u}^{s^{'}}(p^{'})\gamma_\mu u^s(p))\bigg(\frac{-i}{q^2+i\epsilon}\eta^{\mu\nu}\bigg)(\bar{u}^{r^{'}}(k^{'})\gamma_\nu u^r(k)) \quad (7.178)$$

The effective charge $e_{\rm eff}$ is momentum dependent given by

$$e_{\rm eff}^2 = e_R^2\frac{1-\Pi(0)}{1-\Pi(q^2)} = \frac{e^2}{1-\Pi(q^2)}. \quad (7.179)$$

At one-loop order we have $\Pi=\Pi_2$ and thus the effective charge becomes

$$e_{\rm eff}^2 = \frac{e_R^2}{1-\Pi_2(q^2)+\Pi_2(0)}. \quad (7.180)$$

7.7.2 Dimensional Regularization

We now evaluate the loop integral $\Pi_2(q^2)$ given by

$$\Pi_2^{\mu\nu}(q) = ie^2\int\frac{d^4k}{(2\pi)^4}{\rm tr}\gamma^\mu\frac{(\gamma.k+m_e)}{k^2-m_e^2+i\epsilon}\gamma^\nu\frac{(\gamma.(k+q)+m_e)}{(k+q)^2-m_e^2+i\epsilon}. \quad (7.181)$$

This integral is quadratically UV divergent as one can see from the rough estimate

$$\Pi_2^{\mu\nu}(q) \sim \int_0^\Lambda k^3dk\frac{1}{k}\frac{1}{k}$$
$$\sim \frac{1}{2}\Lambda^2. \quad (7.182)$$

This can be made more precise using this naive cutoff procedure and we will indeed find that it is quadratically UV divergent. This is a severe divergence which is stronger than the logarithmic divergences we encountered in previous calculations. In any case a naive cutoff will break the Ward-Takahashi identity $Z_1 = Z_2$. As in previous cases the Pauli-Villars regularization can be used here and it will preserve the Ward-Takahashi identity $Z_1 = Z_2$. However this method is very complicated to implement in this case.

We will employ in this section a more powerful and more elegant regularization method known as dimensional regularization. The idea is simply to compute the loop integral $\Pi_2(q^2)$ not in 4 dimensions but in d dimensions. The result will be an analytic function in d. We are clearly interested in the limit $d \longrightarrow 4$.

We start as before by introducing Feynman parameters, namely

$$\begin{aligned} \frac{1}{k^2 - m_e^2 + i\epsilon} \frac{1}{(k+q)^2 - m_e^2 + i\epsilon} &= \int_0^1 dx \int_0^y \delta(x+y-1) \frac{1}{\bigg[x(k^2 - m_e^2 + i\epsilon) + y((k+q)^2 - m_e^2 + i\epsilon)\bigg]^2} \\ &= \int_0^1 dx \, \frac{1}{\bigg[(k+(1-x)q)^2 + x(1-x)q^2 - m_e^2 + i\epsilon\bigg]^2} \\ &= \int_0^1 dx \, \frac{1}{\bigg[l^2 - \Delta + i\epsilon\bigg]^2}. \end{aligned} \tag{7.183}$$

We have defined $l = k + (1-x)q$ and $\Delta = m_e^2 - x(1-x)q^2$. Furthermore

$$\begin{aligned} \mathrm{tr}\gamma^\mu(\gamma.k + m_e)\gamma^\nu(\gamma.(k+q) + m_e) &= 4k^\mu(k+q)^\nu + 4k^\nu(k+q)^\mu - 4\eta^{\mu\nu}(k.(k+q) - m_e^2) \\ &= 4(l^\mu - (1-x)q^\mu)(l^\nu + xq^\nu) + 4(l^\nu - (1-x)q^\nu)(l^\mu + xq^\mu) \\ &- 4\eta^{\mu\nu}((l - (1-x)q).(l + xq) - m_e^2) \\ &= 4l^\mu l^\nu - 4(1-x)xq^\mu q^\nu + 4l^\nu l^\mu - 4(1-x)xq^\nu q^\mu \\ &- 4\eta^{\mu\nu}(l^2 - x(1-x)q^2 - m_e^2) + ... \end{aligned} \tag{7.184}$$

We have now the $d-$dimensional loop integral

$$\begin{aligned} \Pi_2^{\mu\nu}(q) &= 4ie^2 \int \frac{d^dl}{(2\pi)^d} \bigg(l^\mu l^\nu + l^\nu l^\mu - 2(1-x)xq^\nu q^\mu - \eta^{\mu\nu}(l^2 - x(1-x)q^2 - m_e^2)\bigg) \\ &\times \int_0^1 dx \, \frac{1}{\bigg[l^2 - \Delta + i\epsilon\bigg]^2}. \end{aligned} \tag{7.185}$$

By rotational invariance in d dimensions we can replace $l^\mu l^\nu$ by $l^2\eta^{\mu\nu}/d$. Thus we get

$$\begin{aligned} \Pi_2^{\mu\nu}(q) &= 4ie^2 \int_0^1 dx \bigg[(\frac{2}{d} - 1)\eta^{\mu\nu} \int \frac{d^dl}{(2\pi)^d} \frac{l^2}{(l^2 - \Delta + i\epsilon)^2} \\ &- \big(2(1-x)xq^\mu q^\nu - \eta^{\mu\nu}(x(1-x)q^2 + m_e^2)\big) \int \frac{d^dl}{(2\pi)^d} \frac{1}{(l^2 - \Delta + i\epsilon)^2}\bigg]. \end{aligned} \tag{7.186}$$

Next we Wick rotate ($d^dl = id^dl_E$ and $l^2 = -l_E^2$) to obtain

$$\begin{aligned}
\Pi_2^{\mu\nu}(q) &= -4e^2\int_0^1 dx\bigg[(-\frac{2}{d}+1)\eta^{\mu\nu}\int\frac{d^dl_E}{(2\pi)^d}\frac{l_E^2}{(l_E^2+\Delta)^2} \\
&- \big(2(1-x)xq^\mu q^\nu - \eta^{\mu\nu}(x(1-x)q^2+m_e^2)\big)\int\frac{d^dl_E}{(2\pi)^d}\frac{1}{(l_E^2+\Delta)^2}\bigg].
\end{aligned} \tag{7.187}$$

We need to compute two $d-$dimensional integrals. These are

$$\begin{aligned}
\int\frac{d^dl_E}{(2\pi)^d}\frac{l_E^2}{(l_E^2+\Delta)^2} &= \frac{1}{(2\pi)^d}\int d\Omega_d\int r^{d-1}dr\frac{r^2}{(r^2+\Delta)^2} \\
&= \frac{1}{(2\pi)^d}\frac{1}{2}\int d\Omega_d\int (r^2)^{\frac{d}{2}}dr^2\frac{1}{(r^2+\Delta)^2} \\
&= \frac{1}{(2\pi)^d}\frac{1}{2}\frac{1}{\Delta^{1-\frac{d}{2}}}\int d\Omega_d\int_0^1 dx\ x^{-\frac{d}{2}}(1-x)^{\frac{d}{2}}.
\end{aligned} \tag{7.188}$$

$$\begin{aligned}
\int\frac{d^dl_E}{(2\pi)^d}\frac{1}{(l_E^2+\Delta)^2} &= \frac{1}{(2\pi)^d}\int d\Omega_d\int r^{d-1}dr\frac{1}{(r^2+\Delta)^2} \\
&= \frac{1}{(2\pi)^d}\frac{1}{2}\int d\Omega_d\int (r^2)^{\frac{d-2}{2}}dr^2\frac{1}{(r^2+\Delta)^2} \\
&= \frac{1}{(2\pi)^d}\frac{1}{2}\frac{1}{\Delta^{2-\frac{d}{2}}}\int d\Omega_d\int_0^1 dx\ x^{1-\frac{d}{2}}(1-x)^{\frac{d}{2}-1}.
\end{aligned} \tag{7.189}$$

In the above two equations we have used the change of variable $x = \Delta/(r^2+\Delta)$ and $dx/\Delta = -dr^2/(r^2+\Delta)^2$. We can also use the definition of the so-called beta function

$$B(\alpha,\beta) = \int_0^1 dx\ x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{7.190}$$

Also we can use the area of a $d-$dimensional unit sphere given by

$$\int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}. \tag{7.191}$$

We get then

$$\int\frac{d^dl_E}{(2\pi)^d}\frac{l_E^2}{(l_E^2+\Delta)^2} = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Delta^{1-\frac{d}{2}}}\frac{\Gamma(2-\frac{d}{2})}{\frac{2}{d}-1}. \tag{7.192}$$

$$\int\frac{d^dl_E}{(2\pi)^d}\frac{1}{(l_E^2+\Delta)^2} = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Delta^{2-\frac{d}{2}}}\Gamma(2-\frac{d}{2}). \tag{7.193}$$

With these results the loop integral $\Pi_2^{\mu\nu}(q)$ becomes

$$\begin{aligned}
\Pi_2^{\mu\nu}(q) &= -4e^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx\frac{1}{\Delta^{2-\frac{d}{2}}}\bigg[-\Delta\eta^{\mu\nu} - \big(2(1-x)xq^\mu q^\nu - \eta^{\mu\nu}(x(1-x)q^2+m_e^2)\big)\bigg] \\
&= -4e^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx\frac{2x(1-x)}{\Delta^{2-\frac{d}{2}}}(q^2\eta^{\mu\nu} - q^\mu q^\nu).
\end{aligned} \tag{7.194}$$

Therefore we conclude that the Ward-Takahashi identity is indeed maintained in dimensional regularization. The function $\Pi_2(q^2)$ is then given by

$$\Pi_2(q^2) = -4e^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx\frac{2x(1-x)}{\Delta^{2-\frac{d}{2}}}. \quad (7.195)$$

We want now to take the limit $d \longrightarrow 4$. We define the small parameter $\epsilon = 4-d$. We use the expansion of the gamma function near its pole $z=0$ given by

$$\Gamma(2-\frac{d}{2}) = \Gamma(\frac{\epsilon}{2}) = \frac{2}{\epsilon} - \gamma + O(\epsilon). \quad (7.196)$$

The number γ is given by $\gamma = 0.5772$ and is called the Euler-Mascheroni constant. It is not difficult to convince ourselves that the $1/\epsilon$ divergence in dimensional regularization corresponds to the logarithmic divergence $\ln \Lambda^2$ in Pauli-Villars regularization.

Thus near $d=4$ (equivalently $\epsilon = 0$) we get

$$\begin{aligned}\Pi_2(q^2) &= -\frac{4e^2}{(4\pi)^2}(\frac{2}{\epsilon} - \gamma + O(\epsilon))\int_0^1 dx\ 2x(1-x)(1-\frac{\epsilon}{2}\ln\Delta + O(\epsilon^2))\\ &= -\frac{2\alpha}{\pi}\int_0^1 dx\ x(1-x)(\frac{2}{\epsilon} - \ln\Delta - \gamma + O(\epsilon))\\ &= -\frac{2\alpha}{\pi}\int_0^1 dx\ x(1-x)(\frac{2}{\epsilon} - \ln(m_e^2 - x(1-x)q^2) - \gamma + O(\epsilon)). \quad (7.197)\end{aligned}$$

We will also need

$$\Pi_2(0) = -\frac{2\alpha}{\pi}\int_0^1 dx\ x(1-x)(\frac{2}{\epsilon} - \ln(m_e^2) - \gamma + O(\epsilon)). \quad (7.198)$$

Thus

$$\Pi_2(q^2) - \Pi_2(0) = -\frac{2\alpha}{\pi}\int_0^1 dx\ x(1-x)(\ln\frac{m_e^2}{m_e^2 - x(1-x)q^2} + O(\epsilon)). \quad (7.199)$$

This is finite in the limit $\epsilon \longrightarrow 0$. At very high energies (small distances) corresponding to $-q^2 >> m_e^2$ we get

$$\begin{aligned}\Pi_2(q^2) - \Pi_2(0) &= -\frac{2\alpha}{\pi}\int_0^1 dx\ x(1-x)(-\ln(1 + x(1-x)\frac{-q^2}{m_e^2}) + O(\epsilon))\\ &= \frac{\alpha}{3\pi}\bigg[\ln\frac{-q^2}{m_e^2} - \frac{5}{3} + O(\frac{m_e^2}{-q^2})\bigg]\\ &= \frac{\alpha_R}{3\pi}\bigg[\ln\frac{-q^2}{m_e^2} - \frac{5}{3} + O(\frac{m_e^2}{-q^2})\bigg]. \quad (7.200)\end{aligned}$$

At one-loop order the effective electric charge is

$$e_{\rm eff}^2 = \frac{e_R^2}{1 - \frac{\alpha_R}{3\pi}\big[\ln\frac{-q^2}{m_e^2} - \frac{5}{3} + O(\frac{m_e^2}{-q^2})\big]}. \quad (7.201)$$

The electromagnetic coupling constant depends therefore on the energy as follows

$$\alpha_{\rm eff}(\frac{-q^2}{m_e^2}) \quad = \quad \frac{\alpha_R}{1-\frac{\alpha_R}{3\pi}\big[\ln\frac{-q^2}{m_e^2}-\frac{5}{3}+O(\frac{m_e^2}{-q^2})\big]} \tag{7.202}$$

The effective electromagnetic coupling constant becomes large at high energies. We say that the electromagnetic coupling constant runs with energy or equivalently with distance.

7.8 Renormalization of QED

In this last section we will summarize all our results. The starting Lagrangian was

$$\mathcal{L} \quad = \quad -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\bar{\psi}(i\gamma^\mu\partial_\mu-m)\psi-e\bar{\psi}\gamma_\mu\psi A^\mu. \tag{7.203}$$

We know that the electron and photon two-point functions behave as

$$\int d^4x e^{ip(x-y)} \ <\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega>=\frac{iZ_2}{\gamma.p-m_r+i\epsilon}+... \tag{7.204}$$

$$\int d^4x e^{iq(x-y)} \ <\Omega|T(\hat{A}^\mu(x)\hat{A}^\nu(y))|\Omega> \quad = \quad \frac{-i\eta^{\mu\nu}Z_3}{q^2+i\epsilon}+.... \tag{7.205}$$

Let us absorb the field strength renormalization constants Z_2 and Z_3 in the fields as follows

$$\hat{\psi}_r=\hat{\psi}/\sqrt{Z_2}\ ,\ \hat{A}^\mu_r=\hat{A}^\mu/\sqrt{Z_3}. \tag{7.206}$$

The QED Lagrangian becomes

$$\mathcal{L} \quad = \quad -\frac{Z_3}{4}F_{r\mu\nu}F_r^{\mu\nu}+Z_2\bar{\psi}_r(i\gamma^\mu\partial_\mu-m)\psi_r-eZ_2\sqrt{Z_3}\bar{\psi}_r\gamma_\mu\psi_rA^\mu_r. \tag{7.207}$$

The renormalized electric charge is defined by

$$eZ_2\sqrt{Z_3}=e_RZ_1. \tag{7.208}$$

This reduces to the previous definition $e_R=e\sqrt{Z_3}$ by using Ward identity in the form

$$Z_1=Z_2. \tag{7.209}$$

We introduce the counter-terms

$$Z_1=1+\delta_1\ ,\ Z_2=1+\delta_2\ ,\ Z_3=1+\delta_3. \tag{7.210}$$

We also introduce the renormalized mass m_r and the counter-term δ_m by

$$Z_2m=m_r+\delta_m. \tag{7.211}$$

We have

$$\begin{aligned}\mathcal{L} \quad &= \quad -\frac{1}{4}F_{r\mu\nu}F_r^{\mu\nu}+\bar{\psi}_r(i\gamma^\mu\partial_\mu-m_r)\psi_r-e_R\bar{\psi}_r\gamma_\mu\psi_rA^\mu_r\\ &- \quad \frac{\delta_3}{4}F_{r\mu\nu}F_r^{\mu\nu}+\bar{\psi}_r(i\delta_2\gamma^\mu\partial_\mu-\delta_m)\psi_r-e_R\delta_1\bar{\psi}_r\gamma_\mu\psi_rA^\mu_r.\end{aligned} \tag{7.212}$$

By dropping total derivative terms we find

$$\begin{aligned}\mathcal{L} &= -\frac{1}{4}F_{r\mu\nu}F_r^{\mu\nu}+\bar{\psi}_r(i\gamma^\mu\partial_\mu-m_r)\psi_r-e_R\bar{\psi}_r\gamma_\mu\psi_rA_r^\mu\\ &- \frac{\delta_3}{2}A_{r\mu}(-\partial.\partial\,\eta^{\mu\nu}+\partial^\mu\partial^\nu)A_{r\nu}+\bar{\psi}_r(i\delta_2\gamma^\mu\partial_\mu-\delta_m)\psi_r-e_R\delta_1\bar{\psi}_r\gamma_\mu\psi_rA_r^\mu. \quad (7.213)\end{aligned}$$

There are three extra Feynman diagrams associated with the counter-terms δ_1, δ_2, δ_3 and δ_m besides the usual three Feynman diagrams associated with the photon and electron propagators and the QED vertex. The Feynman diagrams of renormalized QED are shown on figure RENQED.

The counter-terms will be determined from renormalization conditions. There are four counter-terms and thus one must have 4 renormalization conditions. The first two renormalization conditions correspond to the fact that the electron and photon field-strength renormalization constants are equal 1. Indeed we have by construction

$$\int d^4x e^{ip(x-y)} \quad <\Omega|T(\hat{\psi}_r(x)\ \hat{\bar{\psi}}_r(y))|\Omega>=\frac{i}{\gamma.p-m_r+i\epsilon}+... \quad (7.214)$$

$$\int d^4x e^{iq(x-y)} \quad <\Omega|T(\hat{A}_r^\mu(x)\hat{A}_r^\nu(y))|\Omega> \quad = \quad \frac{-i\eta^{\mu\nu}}{q^2+i\epsilon}+.... \quad (7.215)$$

Let us recall that the one-particle irreducible (1PI) diagrams with 2 photon lines is $i\Pi^{\mu\nu}(q)=i(\eta^{\mu\nu}q^2-q^\mu q^\nu)\Pi(q^2)$. We know that the residue of the photon propagator at $q^2=0$ is $1/(1-\Pi(0))$. Thus the first renormalization constant is

$$\Pi(q^2=0)=1. \quad (7.216)$$

The one-particle irreducible (1PI) diagrams with 2 electron lines is $-i\Sigma(\gamma.p)$. The residue of the electron propagator at $\gamma.p=m_r$ is $1/(1-(d\Sigma(\gamma.p)/d\gamma.p)|_{\gamma.p=m_r})$. Thus the second renormalization constant is

$$\frac{d\Sigma(\gamma.p)}{\gamma.p}|_{\gamma.p=m_r}=0. \quad (7.217)$$

Clearly the renormalized mass m_r must be defined by setting the self-energy $-i\Sigma(\gamma.p)$ at $\gamma.p=m_r$ to zero so it is not shifted by quantum effects in renormalized QED. In other words we must have the renormalization constant

$$\Sigma(\gamma.p=m_r)=0. \quad (7.218)$$

Lastly the renormalized electric charge e_R must also not be shifted by quantum effects in renormalized QED. The quantum correction to the electric charge is contained in the exact vertex function (the QED proper vertex) $-ie\Gamma^\mu(p^{'},p)$. Thus we must impose

$$\Gamma^\mu(p^{'}-p=0)=\gamma^\mu. \quad (7.219)$$

7.9 Exercises and Problems

Mott Formula and Bhabha Scattering:

- Use Feynman rules to write down the tree level probability amplitude for electron-muon scattering.

- Derive the unpolarized cross section of the electron-muon scattering at tree level in the limit $m_\mu \longrightarrow \infty$. The result is known as Mott formula.

- Repeat the above two questions for electron-electron scattering. This is known as Bhabha scattering.

Scattering from an External Electromagnetic Field: Compute the Feynman diagrams corresponding to the three first terms of equation (7.21).

Spinor Technology:

- Prove Gordon's identity (with $q = p - p^{'}$)

$$\bar{u}^{s^{'}}(p^{'})\gamma^\mu u^s(p) = \frac{1}{2m_e}\bar{u}^{s^{'}}(p^{'})\bigg[(p+p^{'})^\mu - i\sigma^{\mu\nu}q_\nu\bigg]u^s(p). \tag{7.220}$$

- Show that we can make the replacement

$$\begin{aligned}\bar{u}^{s^{'}}(p^{'})\bigg[(x\gamma.p + y\gamma.q)\gamma^\mu(x\gamma.p + (y-1)\gamma.q)\bigg]u^s(p) \quad \longrightarrow \quad & \bar{u}^{s^{'}}(p^{'})\bigg[m_e(x+y)(x+y-1)(2p^\mu - m_e\gamma^\mu) \\ - \quad & (x+y)(y-1)\bigg(2m_e(p+p^{'})^\mu + q^2\gamma^\mu - 3m_e^2 \\ \times \quad & \gamma^\mu\bigg) - m_e^2 y(x+y-1)\gamma^\mu + m_e y(y-1) \\ \times \quad & (2p^{'\mu} - m_e\gamma^\mu)\bigg]u^s(p).\end{aligned} \tag{7.221}$$

Spheres in d Dimensions: Show that the area of a $d-$dimensional unit sphere is given by

$$\int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}. \tag{7.222}$$

Renormalization Constant Z_2: Show that the probability for the spinor field to create or annihilate a particle is precisely Z_2.

Ward Identity: Consider a QED process which involves a single external photon with momentum k and polarization ϵ_μ. The probability amplitude of this process is of the form $i\mathcal{M}^\mu(k)\epsilon_\mu(k)$. Show that current conservation leads to the Ward identity $k_\mu\mathcal{M}^\mu(k) = 0$.

Hint: See Peskin and Schroeder.

Pauli-Villars Regulator Fields: Show that Pauli-Villars regularization is equivalent to the introduction of regulator fields with large masses. The number of regulator fields can be anything.

Hint: See Zinn-Justin.

Pauli-Villars Regularization:

- Use Pauli-Villars Regularization to compute $\Pi_2^{\mu\nu}(q^2)$.
- Show that the $1/\epsilon$ divergence in dimensional regularization corresponds to the logarithmic divergence $\ln\Lambda^2$ in Pauli-Villars regularization. Compare for example the value of the integral (7.193) in both schemes.

Uehling Potential and Lamb Shift:

- Show that the electrostatic potential can be given by the integral

$$V(\vec{x}) = \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{-e^2 e^{i\vec{q}\vec{x}}}{\vec{q}^2}. \tag{7.223}$$

- Compute the one-loop correction to the above potential due to the vacuum polarization.
- By approximating the Uehling potential by a delta function determine the Lamb shift of the levels of the Hydrogen atom.

Hard Cutoff Regulator:

- Use a naive cutoff to evaluate $\Pi_2^{\mu\nu}(q^2)$. What do you conclude.
- Show that a naive cutoff will not preserve the Ward-Takahashi identity $Z_1 = Z_2$.

Dimensional Regularization and QED Counter-terms:

- Reevaluate the electron self-energy $-i\Sigma(\gamma.p)$ at one-loop in dimensional regularization.
- Compute the counter-terms δ_m and δ_2 at one-loop.
- Use the expression of the photon self-energy $i\Pi^{\mu\nu}$ at one-loop computed in the lecture in dimensional regularization to evaluate the counter term δ_3.
- Reevaluate the vertex function $-ie\Gamma^\mu(p^{'},p)$ at one-loop in dimensional regularization.
- Compute the counter-term δ_1 at one-loop.
- Show explicitly that dimensional regularization will preserve the Ward-Takahashi identity $Z_1 = Z_2$.

Part III

Path Integrals, Gauge Fields and Renormalization Group

8
Path Integral Quantization of Scalar Fields

8.1 Feynman Path Integral

We consider a dynamical system consisting of a single free particle moving in one dimension. The coordinate is x and the canonical momentum is $p = m\dot{x}$. The Hamiltonian is $H = p^2/(2m)$. Quantization means that we replace x and p with operators X and P satisfying the canonical commutation relation $[X, P] = i\hbar$. The Hamiltonian becomes $H = P^2/(2m)$. These operators act in a Hilbert space $\mathcal{H}$. The quantum states which describe the dynamical system are vectors on this Hilbert space whereas observables which describe physical quantities are hermitian operators acting in this Hilbert space. This is the canonical or operator quantization.

We recall that in the Schrödinger picture states depend on time while operators are independent of time. The states satisfy the Schrödinger equation, viz

$$H|\psi_s(t)>= i\hbar\frac{\partial}{\partial t}|\psi_s(t)> . \tag{8.1}$$

Equivalently

$$|\psi_s(t)>= e^{-\frac{i}{\hbar}H(t-t_0)}|\psi_s(t_0)> . \tag{8.2}$$

Let $|x>$ be the eigenstates of X, i.e $X|x>= x|x>$. The completness relation is $\int dx|x><x| = 1$. The components of $|\psi_s(t)>$ in this basis are $<x|\psi_s(t)>$. Thus

$$|\psi_s(t)> \;=\; \int dx <x|\psi_s(t)> |x> . \tag{8.3}$$

$$\begin{aligned} <x|\psi_s(t)> &= <x|e^{-\frac{i}{\hbar}H(t-t_0)}|\psi_s(t_0)> \\ &= \int dx_0 G(x,t;x_0,t_0) <x_0|\psi_s(t_0)> . \end{aligned} \tag{8.4}$$

In above we have used the completness relation in the form $\int dx_0|x_0><x_0| = 1$. The Green function $G(x, t; x_0, t_0)$ is defined by

$$G(x,t;x_0,t_0) \;=\; <x|e^{-\frac{i}{\hbar}H(t-t_0)}|x_0> . \tag{8.5}$$

In the Heisenberg picture states are independent of time while operators are dependent of time. The Heisenberg states are related to the Schrödinger states by the relation

$$|\psi_H>=e^{\frac{i}{\hbar}H(t-t_0)}|\psi_s(t)>. \tag{8.6}$$

We can clearly make the identification $|\psi_H>=|\psi_s(t_0)>$. Let $X(t)$ be the position operator in the Heisenberg picture. Let $|x,t>$ be the eigenstates of $X(t)$ at time t, i.e $X(t)|x,t>=x|x,t>$. We set

$$|x,t>=e^{\frac{i}{\hbar}Ht}|x>\ ,\ |x_0,t_0>=e^{\frac{i}{\hbar}Ht_0}|x_0>. \tag{8.7}$$

From the facts $X(t)|x,t>=x|x,t>$ and $X|x>=x|x>$ we conclude that the Heisenberg operators are related to the Schrödinger operators by the relation

$$X(t)=e^{\frac{i}{\hbar}Ht}Xe^{-\frac{i}{\hbar}Ht}. \tag{8.8}$$

We immediately obtain the Heisenberg equation of motion

$$\frac{dX(t)}{dt}=e^{\frac{i}{\hbar}Ht}\frac{\partial X}{\partial t}e^{-\frac{i}{\hbar}Ht}+\frac{i}{\hbar}[H,X(t)]. \tag{8.9}$$

The Green function (8.5) can be put into the form

$$G(x,t;x_0,t_0) \;=\; <x,t|x_0,t_0>. \tag{8.10}$$

This is the transition amplitude from the point x_0 at time t_0 to the point x at time t which is the most basic object in the quantum theory.

We discretize the time interval $[t_0,t]$ such that $t_j=t_0+j\epsilon$, $\epsilon=(t-t_0)/N$, $j=0,1,...,N$, $t_N=t_0+N\epsilon=t$. The corresponding coordinates are $x_0,x_1,...,x_N$ with $x_N=x$. The corresponding momenta are $p_0,p_1,...,p_{N-1}$. The momentum p_j corresponds to the interval $[x_j,x_{j+1}]$. We can show

$$\begin{aligned}
G(x,t;x_0,t_0) &= <x,t|x_0,t_0>\\
&= \int dx_1 <x,t|x_1,t_1><x_1,t_1|x_0,t_0>\\
&= \int dx_1dx_2...dx_{N-1}\prod_{j=0}^{N-1}<x_{j+1},t_{j+1}|x_j,t_j>.
\end{aligned} \tag{8.11}$$

We compute (with $<p|x>=\exp(-ipx/\hbar)/\sqrt{2\pi\hbar}$)

$$\begin{aligned}
<x_{j+1},t_{j+1}|x_j,t_j> &= <x_{j+1}|(1-\frac{i}{\hbar}H\epsilon)|x_j>\\
&= \int dp_j <x_{j+1}|p_j><p_j|(1-\frac{i}{\hbar}H\epsilon)|x_j>\\
&= \int dp_j(1-\frac{i}{\hbar}H(p_j,x_j)\epsilon)<x_{j+1}|p_j><p_j|x_j>\\
&= \int \frac{dp_j}{2\pi\hbar}(1-\frac{i}{\hbar}H(p_j,x_j)\epsilon)\ e^{\frac{i}{\hbar}p_jx_{j+1}}e^{-\frac{i}{\hbar}p_jx_j}\\
&= \int \frac{dp_j}{2\pi\hbar}e^{\frac{i}{\hbar}(p_j\dot{x}_j-H(x_j,p_j))\epsilon}.
\end{aligned} \tag{8.12}$$

In above $\dot{x}_j = (x_{j+1}-x_j)/\epsilon$. Therefore by taking the limit $N \longrightarrow \infty$, $\epsilon \longrightarrow 0$ keeping $t-t_0 =$ fixed we obtain

$$G(x,t;x_0,t_0) = \int \frac{dp_0}{2\pi\hbar}\frac{dp_1dx_1}{2\pi\hbar}...\frac{dp_{N-1}dx_{N-1}}{2\pi\hbar}\; e^{\frac{i}{\hbar}\sum_{j=0}^{N-1}(p_j\dot{x}_j - H(p_j,x_j))\epsilon}$$
$$= \int \mathcal{D}p\mathcal{D}x\; e^{\frac{i}{\hbar}\int_{t_0}^t ds(p\dot{x}-H(p,x))}. \quad (8.13)$$

Now $\dot{x} = dx/ds$. In our case the Hamiltonian is given by $H = p^2/(2m)$. Thus by performing the Gaussian integral over p we obtain [1]

$$G(x,t;x_0,t_0) = \mathcal{N}\int \mathcal{D}x\; e^{\frac{i}{\hbar}\int_{t_0}^t dsL(\dot{x},x)}$$
$$= \mathcal{N}\int \mathcal{D}x\; e^{\frac{i}{\hbar}S[x]}. \quad (8.14)$$

In the above equation $S[x] = \int dt\; L(x,\dot{x}) = m\int dt\; \dot{x}^2/2$ is the action of the particle. As it turns out this fundamental result holds for all Hamiltonians of the form $H = p^2/(2m) + V(x)$ in which case $S[x] = \int dt\; L(x,\dot{x}) = \int dt\; (m\dot{x}^2/2 - V(x))$ [2].

This result is essentially the principle of linear superposition of quantum theory. The total probability amplitude for traveling from the point x_0 to the point x is equal to the sum of probability amplitudes for traveling from x_0 to x through all possible paths connecting these two points. Clearly a given path between x_0 and x is defined by a configuration $x(s)$ with $x(t_0) = x_0$ and $x(t) = x$. The corresponding probability amplitude (wave function) is $e^{\frac{i}{\hbar}S[x(s)]}$. In the classical limit $\hbar \longrightarrow 0$ only one path (the classical path) exists by the method of the stationary phase. The classical path is clearly the path of least action as it should be.

We note also that the generalization of the result (8.14) to matrix elements of operators is given by [3]

$$< x,t|T(X(t_1)...X(t_n))|x_0,t_0> = \mathcal{N}\int \mathcal{D}x\; x(t_1)...x(t_n)\; e^{\frac{i}{\hbar}S[x]}. \quad (8.15)$$

The T is the time-ordering operator defined by

$$T(X(t_1)X(t_2)) = X(t_1)X(t_2) \text{ if } t_1 > t_2. \quad (8.16)$$

$$T(X(t_1)X(t_2)) = X(t_2)X(t_1) \text{ if } t_1 < t_2. \quad (8.17)$$

[1]Exercise:

- Show that

$$\int dpe^{-ap^2+bp} = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}}.$$

- Use the above result to show that

$$\int \mathcal{D}p\; e^{\frac{i}{\hbar}\int_{t_0}^t ds(p\dot{x}-\frac{p^2}{2m})} = \mathcal{N}e^{\frac{i}{\hbar}\int_{t_0}^t ds\frac{m}{2}\dot{x}^2(s)}.$$

Determine the constant of normalization $\mathcal{N}$.

[2]Exercise: Repeat the analysis for a non-zero potential. See for example Peskin and Schroeder

[3]Exercise:Verify this explicitly. See for example Randjbar-Daemi lecture notes.

Let us now introduce the basis $|n>$. This is the eigenbasis of the Hamiltonian, viz $H|n>=E_n|n>$. We have the completness relation $\sum_n |n><n| = 1$. The matrix elements (8.15) can be rewritten as

$$\sum_{n,m} e^{-itE_n+it_0E_m} \quad <x|n><m|x_0><n|T(X(t_1)...X(t_n))|m> \quad = \quad \mathcal{N}\int \mathcal{D}x\ x(t_1)...x(t_n)\ e^{\frac{i}{\hbar}S[x]}. \tag{8.18}$$

In the limit $t_0 \longrightarrow -\infty$ and $t \longrightarrow \infty$ we observe that only the ground state with energy E_0 contributes, i.e. the rapid oscillation of the first exponential in this limit forces $n = m = 0$ [4]. Thus we obtain in this limit

$$e^{iE_0(t_0-t)} <x|0><0|x_0><0|T(X(t_1)...X(t_n))|0> \quad = \quad \mathcal{N}\int \mathcal{D}x\ x(t_1)...x(t_n)\ e^{\frac{i}{\hbar}S[x]}. \tag{8.20}$$

We write this as

$$<0|T(X(t_1)...X(t_n))|0> \quad = \quad \mathcal{N}^{'}\int \mathcal{D}x\ x(t_1)...x(t_n)\ e^{\frac{i}{\hbar}S[x]}. \tag{8.21}$$

In particular

$$<0|0> \quad = \quad \mathcal{N}^{'}\int \mathcal{D}x\ e^{\frac{i}{\hbar}S[x]}. \tag{8.22}$$

Hence

$$<0|T(X(t_1)...X(t_n))|0> \quad = \quad \frac{\int \mathcal{D}x\ x(t_1)...x(t_n)\ e^{\frac{i}{\hbar}S[x]}}{\int \mathcal{D}x\ e^{\frac{i}{\hbar}S[x]}}. \tag{8.23}$$

We introduce the path integral $Z[J]$ in the presence of a source $J(t)$ by

$$Z[J] \quad = \quad \int \mathcal{D}x\ e^{\frac{i}{\hbar}S[x]+\frac{i}{\hbar}\int dt J(t)x(t)}. \tag{8.24}$$

This path integral is the generating functional of all the matrix elements $<0|T(X(t_1)...X(t_n))|0>$. Indeed

$$<0|T(X(t_1)...X(t_n))|0> \quad = \quad \frac{1}{Z[0]}\bigg(\frac{\hbar}{i}\bigg)^n \frac{\delta^n Z[J]}{\delta J(t_1)...\delta J(t_n)}|_{J=0}. \tag{8.25}$$

From the above discussion $Z[0]$ is the vacuum-to-vacuum amplitude. Therefore $Z[J]$ is the vacuum-to-vacuum amplitude in the presence of the source $J(t)$.

---

[4] We consider the integral

$$I = \int_{-\infty}^{\infty} dx\ F(x)e^{i\phi(x)}. \tag{8.19}$$

The function $\phi(x)$ is a rapidly-varying function over the range of integration while $F(x)$ is slowly-varying by comparison. Evaluate this integral using the method of stationary phase.

8.2 Scalar Field Theory

8.2.1 Path Integral

A field theory is a dynamical system with N degrees of freedom where $N \longrightarrow \infty$. The classical description is given in terms of a Lagrangian and an action principle while the quantum description is given in terms of a path integral and correlation functions. In a scalar field theory the basic field has spin $j = 0$ with respect to Lorentz transformations.

It is well established that scalar field theories are relevant to critical phenomena and to the Higgs sector in the standard model of particle physics.

We start with the relativistic energy-momentum relation $p^\mu p_\mu = M^2c^2$ where $p^\mu = (p^0, \vec{p}) = (E/c, \vec{p})$. We adopt the metric $(1, -1, -1, -1)$, i.e. $p_\mu = (p_0, -\vec{p}) = (E/c, -\vec{p})$. Next we employ the correspondence principle $p_\mu \longrightarrow i\hbar\partial_\mu$ where $\partial_\mu = (\partial_0, \partial_i)$ and apply the resulting operator on a function ϕ. We obtain the Klein-Gordon equation

$$\partial_\mu \partial^\mu \phi = -m^2\phi \ , \ m^2 = \frac{M^2c^2}{\hbar^2}. \tag{8.26}$$

As a wave equation the Klein-Gordon equation is incompatible with the statistical interpretation of quantum mechanics. However the Klein-Gordon equation makes sense as an equation of motion of a classical scalar field theory with action and Lagrangian $S = \int dtL$, $L = \int d^3x\mathcal{L}$ where the lagrangian density $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2. \tag{8.27}$$

So in summary ϕ is not really a wave function but it is a dynamical variable which plays the same role as the coordinate x of the free particle discussed in the previous section.

The principle of least action applied to an action $S = \int dtL$ yields (with the assumption $\delta\phi|_{x_\mu=\pm\infty} = 0$) the result [5]

$$\frac{\delta S}{\delta\phi} = \frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} = 0. \tag{8.28}$$

It is not difficult to verify that this is the same equation as (8.26) if $L = \int d^3x\mathcal{L}$ and $\mathcal{L}$ is given by (8.27) [6].

The free scalar field theory is a collection of infinite number of decoupled harmonic oscillators. To see this fact we introduce the fourier transform $\tilde{\phi} = \tilde{\phi}(t, \vec{k})$ of $\phi = \phi(t, \vec{x})$ as follows

$$\phi = \phi(t, \vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^3}\ \tilde{\phi}(t, \vec{k})\ e^{i\vec{k}\vec{x}} \ , \ \tilde{\phi} = \tilde{\phi}(t, \vec{k}) = \int d^3\vec{x}\ \phi(t, \vec{x})\ e^{-i\vec{k}\vec{x}}. \tag{8.29}$$

Then the Lagrangian and the equation of motion can be rewritten as

$$L = \int \frac{d^3\vec{k}}{(2\pi)^3}\left(\frac{1}{2}\partial_0\tilde{\phi}\partial_0\tilde{\phi}^* - \frac{1}{2}\omega_k^2\tilde{\phi}\tilde{\phi}^*\right). \tag{8.30}$$

$$\partial_0^2\tilde{\phi} + \omega_k^2\tilde{\phi} = 0 \ , \ \omega_k^2 = \vec{k}^2 + m^2. \tag{8.31}$$

[5] Exercise: Verify this statement.
[6] Exercise: Verify this statement.

This is the equation of motion of a harmonic oscillator with frequency ω_k. Using box normalization the momenta become discrete and the measure $\int d^3\vec{k}/(2\pi)^3$ becomes $\sum_{\vec{k}}/V$. Reality of the scalar field ϕ implies that $\tilde{\phi}(t,\vec{k})=\tilde{\phi}^*(t,-\vec{k})$ and by writing $\tilde{\phi}=\sqrt{V}(X_k+iY_k)$ we end up with the Lagrangian

$$\begin{aligned} L &= \frac{1}{V}\sum_{k_1>0}\sum_{k_2>0}\sum_{k_3>0}\bigg(\partial_0\tilde{\phi}\partial_0\tilde{\phi}^*-\omega_k^2\tilde{\phi}\tilde{\phi}^*\bigg) \\ &= \sum_{k_1>0}\sum_{k_2>0}\sum_{k_3>0}\bigg((\partial_0X_k)^2-\omega_k^2X_k^2+(\partial_0Y_k)^2-\omega_k^2Y_k^2\bigg). \end{aligned} \tag{8.32}$$

The path integral of the two harmonic oscillators X_k and Y_k is immediately given by

$$Z[J_k,K_k] = \int \mathcal{D}X_k\mathcal{D}Y_k\ e^{\frac{i}{\hbar}S[X_k,Y_k]+\frac{i}{\hbar}\int dt(J_k(t)X_k(t)+K_k(t)Y_k(t))}. \tag{8.33}$$

The action $S[X_k,Y_k]$ is obviously given by

$$S[X_k,Y_k]=\int_{t_0\longrightarrow-\infty}^{t\longrightarrow+\infty}ds\bigg((\partial_0X_k)^2-\omega_k^2X_k^2+(\partial_0Y_k)^2-\omega_k^2Y_k^2\bigg) \tag{8.34}$$

The definition of the measures $\mathcal{D}X_k$ and $\mathcal{D}Y_k$ must now be clear from our previous considerations. We introduce the notation $X_k(t_i)=x_i^{(k)}$, $Y_k(t_i)=y_i^{(k)}$, $i=0,1,...,N-1,N$ with the time step $\epsilon=t_i-t_{i-1}=(t-t_0)/N$. Then as before we have (with $N\longrightarrow\infty$, $\epsilon\longrightarrow 0$ keeping $t-t_0$ fixed) the measures

$$\mathcal{D}X_k=\prod_{i=1}^{N-1}dx_i^{(k)}\ ,\ \mathcal{D}Y_k=\prod_{i=1}^{N-1}dy_i^{(k)}. \tag{8.35}$$

The path integral of the scalar field ϕ is the product of the path integrals of the harmonic oscillators X_k and Y_k with different $k=(k_1,k_2,k_3)$, viz

$$\begin{aligned} Z[J,K] &= \prod_{k_1>0}\prod_{k_2>0}\prod_{k_3>0}Z[J_k,K_k] \\ &= \int\prod_{k_1>0}\prod_{k_2>0}\prod_{k_3>0}\mathcal{D}X_k\mathcal{D}Y_k\ \exp\bigg(\frac{i}{\hbar}\sum_{k_1>0}\sum_{k_2>0}\sum_{k_3>0}S[X_k,Y_k] \\ &+ \frac{i}{\hbar}\int dt\sum_{k_1>0}\sum_{k_2>0}\sum_{k_3>0}(J_k(t)X_k(t)+K_k(t)Y_k(t))\bigg). \end{aligned} \tag{8.36}$$

The action of the scalar field is precisely the first term in the exponential, namely

$$\begin{aligned} S[\phi] &= \sum_{k_1>0}\sum_{k_2>0}\sum_{k_3>0}S[X_k,Y_k] \\ &= \int_{t_0\longrightarrow-\infty}^{t\longrightarrow+\infty}ds\sum_{k_1>0}\sum_{k_2>0}\sum_{k_3>0}\bigg((\partial_0X_k)^2-\omega_k^2X_k^2+(\partial_0Y_k)^2-\omega_k^2Y_k^2\bigg) \\ &= \int d^4x\bigg(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2\bigg). \end{aligned} \tag{8.37}$$

We remark also that (with $J(t,\vec{x})=\int d^3\vec{k}/(2\pi)^3\ \tilde{J}(t,\vec{k})\ e^{i\vec{k}\vec{x}}$, $\tilde{J}=\sqrt{V}(J_k+iK_k)$) we have

$$\sum_{k_1>0}\sum_{k_2>0}\sum_{k_3>0}(J_k(t)X_k(t)+K_k(t)Y_k(t))=\int d^3xJ(t,\vec{x})\phi(t,\vec{x}). \tag{8.38}$$

We write therefore the above path integral formally as

$$Z[J] = \int \mathcal{D}\phi \; e^{\frac{i}{\hbar}S[\phi]+\frac{i}{\hbar}\int d^4x J(x)\phi(x)}. \tag{8.39}$$

This path integral is the generating functional of all the matrix elements $<0|T(\Phi(x_1)...\Phi(x_n))|0>$ (also called $n-$point functions). Indeed

$$\begin{aligned} <0|T(\Phi(x_1)...\Phi(x_n))|0> &= \frac{1}{Z[0]}\bigg(\frac{\hbar}{i}\bigg)^n \frac{\delta^n Z[J]}{\delta J(x_1)...\delta J(x_n)}|_{J=0} \\ &= \frac{\int \mathcal{D}\phi \; \phi(x_1)...\phi(x_n) \; e^{\frac{i}{\hbar}S[\phi]}}{\int \mathcal{D}\phi \; e^{\frac{i}{\hbar}S[\phi]}}. \end{aligned} \tag{8.40}$$

The intercations are added by modifying the action appropriately. The only renormalizable interacting scalar field theory in $d=4$ dimensions is the quartic ϕ^4 theory. Thus we will only consider this model given by the action

$$S[\phi] = \int d^4x \bigg[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\bigg]. \tag{8.41}$$

8.2.2 The Free $2-$Point Function

It is more rigorous to perform the different computations of interest on an Euclidean spacetime. Euclidean spacetime is obtained from Minkowski spacetime via the so-called Wick rotation. This is also called the imaginary time formulation which is obtained by the substitutions $t \longrightarrow -i\tau$, $x^0 = ct \longrightarrow -ix^4 = -ic\tau$, $\partial_0 \longrightarrow i\partial_4$. Hence $\partial_\mu\phi\partial^\mu\phi \longrightarrow -(\partial_\mu\phi)^2$ and $iS \longrightarrow -S_E$ where

$$S_E[\phi] = \int d^4x \bigg[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4\bigg]. \tag{8.42}$$

The path integral becomes

$$Z_E[J] = \int \mathcal{D}\phi \; e^{-\frac{1}{\hbar}S_E[\phi]+\frac{1}{\hbar}\int d^4x J(x)\phi(x)}. \tag{8.43}$$

The Euclidean $n-$point functions are given by

$$\begin{aligned} <0|T(\Phi(x_1)...\Phi(x_n))|0>_E &= \frac{1}{Z[0]}\bigg(\hbar\bigg)^n \frac{\delta^n Z_E[J]}{\delta J(x_1)...\delta J(x_n)}|_{J=0} \\ &= \frac{\int \mathcal{D}\phi \; \phi(x_1)...\phi(x_n) \; e^{-\frac{1}{\hbar}S_E[\phi]}}{\int \mathcal{D}\phi \; e^{-\frac{1}{\hbar}S_E[\phi]}}. \end{aligned} \tag{8.44}$$

The action of a free scalar field is given by

$$S_E[\phi] = \int d^4x \; \bigg[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2\bigg] = \frac{1}{2}\int d^4x \; \phi\big[-\partial^2 + m^2\big]\phi. \tag{8.45}$$

The corresponding path integral is (after completing the square)

$$\begin{aligned} Z_E[J] &= \int \mathcal{D}\phi \; e^{-\frac{1}{\hbar}S_E[\phi]+\frac{1}{\hbar}\int d^4x J(x)\phi(x)} \\ &= e^{\frac{1}{2\hbar}\int d^4x (JKJ)(x)} \int \mathcal{D}\phi \; e^{-\frac{1}{2\hbar}\int d^4x \big(\phi - JK\big)\big(-\partial^2+m^2\big)\big(\phi - KJ\big)}. \end{aligned} \tag{8.46}$$

In above K is the operator defined by

$$K\big(-\partial^2+m^2\big)=\big(-\partial^2+m^2\big)K=1. \tag{8.47}$$

After a formal change of variable given by $\phi\longrightarrow\phi-KJ$ the path integral $Z[J]$ is reduced to (see next section for a rigorous treatment)

$$Z_E[J] \;=\; \mathcal{N}\ e^{\frac{1}{2\hbar}\int d^4x(J^TKJ)(x)}=\mathcal{N}\ e^{\frac{1}{2\hbar}\int d^4xd^4yJ(x)K(x,y)J(y)}. \tag{8.48}$$

The $\mathcal{N}$ is an unimportant normalization factor. The free 2−point function (the free propagator) is defined by

$$\begin{aligned}<0|T(\Phi(x_1)\Phi(x_2))|0>_E \;&=\; \frac{\int\mathcal{D}\phi\ \phi(x_1)\phi(x_2)\ e^{-\frac{1}{\hbar}S_E[\phi]}}{\int\mathcal{D}\phi\ e^{-\frac{1}{\hbar}S_E[\phi]}}\\ &=\; \frac{1}{Z[0]}\hbar^2\frac{\delta^2Z_E[J]}{\delta J(x_1)\delta J(x_2)}|_{J=0}.\end{aligned} \tag{8.49}$$

A direct calculation leads to

$$<0|T(\Phi(x_1)\Phi(x_2))|0>_E=\hbar K(x_1,x_2) \tag{8.50}$$

Clearly

$$\big(-\partial^2+m^2\big)K(x,y)=\delta^4(x-y). \tag{8.51}$$

Using translational invariance we can write

$$K(x,y)=K(x-y)=\int\frac{d^4k}{(2\pi)^4}\ \tilde{K}(k)\ e^{ik(x-y)}. \tag{8.52}$$

By construction $\tilde{K}(k)$ is the fourier transform of $K(x,y)$. It is trivial to compute that

$$\tilde{K}(k)=\frac{1}{k^2+m^2}. \tag{8.53}$$

The free euclidean 2−point function is therefore given by

$$<0|T(\Phi(x_1)\Phi(x_2))|0>_E=\int\frac{d^4k}{(2\pi)^4}\ \frac{\hbar}{k^2+m^2}\ e^{ik(x-y)} \tag{8.54}$$

8.2.3 Lattice Regularization

The above calculation of the 2−point function of a scalar field can be made more explicit and in fact more rigorous by working on an Euclidean lattice spacetime. The lattice provides a concrete non-perturbative definition of the theory.

We replace the Euclidean spacetime with a lattice of points $x_\mu=an_\mu$ where a is the lattice spacing. In the natural units $\hbar=c=1$ the action is dimensionless and hence the field is of dimension mass. We define a dimensionless field $\hat{\phi}_n$ by the relation $\hat{\phi}_n=a\phi_n$ where $\phi_n=\phi(x)$. The dimensionless mass parameter is $\hat{m}^2=m^2a^2$. The integral over spacetime will be replaced with a sum over the points of the lattice, i.e

$$\int d^4x=a^4\sum_n\ ,\ \sum_n=\sum_{n_1}...\sum_{n_4}. \tag{8.55}$$

The measure is therefore given by

$$\int \mathcal{D}\phi = \prod_n d\phi_n \ , \ \prod_n = \prod_{n_1} ... \prod_{n_4}. \tag{8.56}$$

The derivative can be replaced either with the forward difference or with the backward difference defined respectively by the equations

$$\partial_\mu \phi = \frac{\phi_{n+\hat{\mu}} - \phi_n}{a}. \tag{8.57}$$

$$\partial_\mu \phi = \frac{\phi_n - \phi_{n-\hat{\mu}}}{a}. \tag{8.58}$$

The $\hat{\mu}$ is the unit vector in the direction x_μ. The Laplacian on the lattice is defined such that

$$\partial^2 \phi = \frac{1}{a^2} \sum_\mu \big(\phi_{n+\hat{\mu}} + \phi_{n-\hat{\mu}} - 2\phi_n\big). \tag{8.59}$$

The free Euclidean action on the lattice is therefore

$$\begin{aligned} S_E[\hat{\phi}] &= \frac{1}{2} \int d^4x \ \phi\big[-\partial^2 + m^2\big]\phi \\ &= \frac{1}{2} \sum_{n,m} \hat{\phi}_n K_{nm} \hat{\phi}_m. \end{aligned} \tag{8.60}$$

$$K_{nm} = -\sum_\mu \bigg[\delta_{n+\hat{\mu},m} + \delta_{n-\hat{\mu},m} - 2\delta_{n,m}\bigg] + \hat{m}^2 \delta_{n,m}. \tag{8.61}$$

The path integral on the lattice is

$$Z_E[J] = \int \prod_n d\hat{\phi}_n \ e^{-S_E[\hat{\phi}] + \sum_n \hat{J}_n \hat{\phi}_n}. \tag{8.62}$$

The $n-$point functions on the lattice are given by

$$\begin{aligned} <0|T(\hat{\Phi}_s ... \hat{\Phi}_t)|0>_E &= \frac{1}{Z[0]} \frac{\delta^n Z_E[J]}{\delta \hat{J}_s ... \delta \hat{J}_t}|_{J=0} \\ &= \frac{\int \prod_n d\hat{\phi}_n \ \hat{\phi}_s ... \hat{\phi}_t \ e^{-S_E[\hat{\phi}]}}{\int \prod_n d\hat{\phi}_n \ e^{-S_E[\hat{\phi}]}}. \end{aligned} \tag{8.63}$$

The path integral of the free scalar field on the lattice can be computed in a closed form. We find [7]

[7]Exercise:

- Perform explicitly the Gaussian integral

$$I = \int \prod_{i=1}^N dx_i \ e^{-x_i D_{ij} x_j}. \tag{8.64}$$

Try to diagonalize the symmetric and invertible matrix D. It is also well advised to adopt an $i\epsilon$ prescription (i.e. make the replacement $D \longrightarrow D + i\epsilon$) in order to regularize the integral.

- Use the above result to determine the constant of normalization $\mathcal{N}$ in equation (8.63).

$$\begin{aligned} Z_E[J] &= e^{\frac{1}{2}\sum_{n,m} J_n K^{-1}_{nm} J_m} \int \prod_n d\hat{\phi}_n \; e^{-\frac{1}{2}\sum_{n,m}(\hat{\phi}-JK^{-1})_n K_{nm}(\hat{\phi}-K^{-1}J)_m} \\ &= \mathcal{N} \; e^{\frac{1}{2}\sum_{n,m} J_n K^{-1}_{nm} J_m}. \end{aligned} \tag{8.65}$$

The 2−point function is therefore given by

$$\begin{aligned} <0|T(\hat{\Phi}_s\hat{\Phi}_t)|0>_E &= \frac{1}{Z[0]}\frac{\delta^2 Z_E[J]}{\delta\hat{J}_s\delta\hat{J}_t}|_{J=0} \\ &= K^{-1}_{st}. \end{aligned} \tag{8.66}$$

We fourier transform on the lattice as follows

$$K_{st} = \int_{-\pi}^{\pi}\frac{d^4\hat{k}}{(2\pi)^4}\hat{K}(k) \; e^{i\hat{k}(s-t)}. \tag{8.67}$$

$$K^{-1}_{st} = \int_{-\pi}^{\pi}\frac{d^4\hat{k}}{(2\pi)^4}G(k) \; e^{i\hat{k}(s-t)}. \tag{8.68}$$

For $\hat{K}(k) = G(k) = 1$ we obtain the identity, viz

$$\delta_{st} = \int_{-\pi}^{\pi}\frac{d^4\hat{k}}{(2\pi)^4} \; e^{i\hat{k}(s-t)}. \tag{8.69}$$

Furthermore we can show that $K_{st}K^{-1}_{tr} = \delta_{sr}$ using the equations

$$(2\pi)^4\delta^4(\hat{k}-\hat{p}) = \sum_n \; e^{i(\hat{k}-\hat{p})n}. \tag{8.70}$$

$$G(k) = \hat{K}^{-1}(k) \tag{8.71}$$

Next we compute

$$\begin{aligned} K_{nm} &= -\sum_\mu \left[\delta_{n+\hat{\mu},m} + \delta_{n-\hat{\mu},m} - 2\delta_{n,m}\right] + \hat{m}^2\delta_{n,m} \\ &= -\int_{-\pi}^{\pi}\frac{d^4\hat{k}}{(2\pi)^4} \; e^{i\hat{k}(n-m)}\sum_\mu \left[e^{i\hat{k}\hat{\mu}} + e^{-i\hat{k}\hat{\mu}} - 2\right] + \hat{m}^2\int\frac{d^4\hat{k}}{(2\pi)^4} \; e^{i\hat{k}(n-m)}. \end{aligned} \tag{8.72}$$

Thus

$$\hat{K}(k) \;\; = \;\; 4\sum_\mu \sin^2(\frac{\hat{k}_\mu}{2}) + \hat{m}^2. \tag{8.73}$$

Hence

$$G(k) = \frac{1}{4\sum_\mu \sin^2(\frac{\hat{k}_\mu}{2}) + \hat{m}^2}. \tag{8.74}$$

The 2−point function is then given by

$$< 0|T(\hat{\Phi}_s\hat{\Phi}_t)|0 >_E= \int_{-\pi}^{\pi}\frac{d^4\hat{k}}{(2\pi)^4}\frac{1}{4\sum_{\mu}\sin^2(\frac{\hat{k}_{\mu}}{2})+\hat{m}^2}\ e^{i\hat{k}(s-t)} \tag{8.75}$$

In the continuum limit $a \longrightarrow 0$ we scale the fields as follows $\hat{\Phi}_s = a\phi(x)$, $\hat{\Phi}_t = a\phi(y)$ where $x = as$ and $y = at$. The momentum is scaled as $\hat{k} = ak$ and the mass is scaled as $\hat{m}^2 = a^2m^2$. In this limit the lattice mass $\hat{m}^2$ goes to zero and hence the correlation lenght $\hat{\xi} = 1/\hat{m}$ diverges. In other words the continuum limit is realized at a critical point of a second order phase transition. The physical 2−point function is given by

$$\begin{aligned} < 0|T(\hat{\Phi}(x)\hat{\Phi}(y))|0 >_E &= \lim_{a\longrightarrow 0}\frac{< 0|T(\hat{\Phi}_s\hat{\Phi}_t)|0 >_E}{a^2} \\ &= \int_{-\infty}^{\infty}\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\ e^{ik(x-y)}. \end{aligned} \tag{8.76}$$

This is the same result obtain from continuum considerations in the previous section.

8.3 The Effective Action

8.3.1 Formalism

We are interested in the ϕ^4 theory on a Minkowski spacetime given by the classical action

$$S[\phi] = \int d^4x\bigg[\frac{1}{2}\partial_{\mu}\phi\partial^{\mu}\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4\bigg]. \tag{8.77}$$

The quantum theory is given by the path integral

$$Z[J] = \int \mathcal{D}\phi\ e^{\frac{i}{\hbar}S[\phi]+\frac{i}{\hbar}\int d^4x J(x)\phi(x)}. \tag{8.78}$$

The functional $Z[J]$ generates all Green functions, viz

$$\begin{aligned} < 0|T(\Phi(x_1)...\Phi(x_n))|0 > &= \frac{1}{Z[0]}\bigg(\frac{\hbar}{i}\bigg)^n\frac{\delta^n Z[J]}{\delta J(x_1)...\delta J(x_n)}|_{J=0} \\ &= \frac{\int\mathcal{D}\phi\ \phi(x_1)...\phi(x_n)\ e^{\frac{i}{\hbar}S[\phi]}}{\int\mathcal{D}\phi\ e^{\frac{i}{\hbar}S[\phi]}}. \end{aligned} \tag{8.79}$$

The path integral $Z[J]$ generates disconnected as well as connected graphs and it generates reducible as well as irreducible graphs. Clearly the disconnected graphs can be obtained by putting togther connected graphs whereas reducible graphs can be decomposed into irreducible components. All connected Green functions can be generated from the functional $W[J]$ (vacuum energy) whereas all connected and irreducible Green functions (known also as the 1−particle irreducible) can be generated from the functional $\Gamma[\phi_c]$ (effective action). The vacuum energy $W[J]$ is defined through the equation

$$Z[J] = e^{\frac{i}{\hbar}W[J]}. \tag{8.80}$$

In order to define the effective action we introduce the notion of the classical field. This is defined by the equation

$$\phi_c(x) = \frac{\delta W[J]}{\delta J(x)}. \tag{8.81}$$

This is a functional of J. It becomes the vacuum expectation value of the field operator Φ at $J = 0$. Indeed we compute

$$\phi_c(x)|_{J=0} = \frac{\hbar}{i}\frac{1}{Z[0]}\frac{\delta Z[J]}{\delta J(x)}|_{J=0} =< 0|\Phi(x)|0 > . \tag{8.82}$$

The effective action $\Gamma[\phi_c]$ is the Legendre transform of $W[J]$ defined by

$$\Gamma[\phi_c] = W[J] - \int d^4x J(x)\phi_c(x). \tag{8.83}$$

This is the quantum analogue of the classical action $S[\phi]$. The effective action generates all the $1-$particle irreducible graphs from which the external legs have been removed. These are the connected, irreducible and amputated graphs.

The classical equations of motion are obtained from the principal of least action applied to the classical action $S[\phi] + \int d^4xJ(x)\phi(x)$. We obtain

$$\frac{\delta S[\phi]}{\delta\phi(x)} = -J(x). \tag{8.84}$$

Similarly the quantum equations of motion are obtained from the principal of least action applied to the quantum action $\Gamma[\phi_c]$. We obtain

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = 0. \tag{8.85}$$

In the presence of source this generalizes to

$$\frac{\delta\Gamma[\phi_c]}{\delta\phi_c(x)} = -J(x). \tag{8.86}$$

The proof goes as follows:

$$\begin{aligned}
\frac{\delta\Gamma}{\delta\phi_c(x)} &= \frac{\delta W}{\delta\phi_c(x)} - \int d^4y \frac{\delta J(y)}{\delta\phi_c(x)}\phi_c(y) - J \\
&= \frac{\delta W}{\delta\phi_c(x)} - \int d^4y \frac{\delta J(y)}{\delta\phi_c(x)}\frac{\delta W}{\delta J(y)} - J \\
&= -J(x)
\end{aligned} \tag{8.87}$$

A more explicit form of the quantum equation of motion can be obtained as follows. We start from the identity

$$\begin{aligned}
0 &= \int \mathcal{D}\phi\ \frac{\hbar}{i}\frac{\delta}{\delta\phi(x)}\ e^{\frac{i}{\hbar}S[\phi]+\frac{i}{\hbar}\int d^4xJ(x)\phi(x)} \\
&= \int \mathcal{D}\phi \left(\frac{\delta S}{\delta\phi(x)} + J\right)\ e^{\frac{i}{\hbar}S[\phi]+\frac{i}{\hbar}\int d^4xJ(x)\phi(x)} \\
&= \left(\frac{\delta S}{\delta\phi(x)}_{\phi=\frac{\hbar}{i}\frac{\delta}{\delta J}} + J\right)\ e^{\frac{i}{\hbar}W[J]} \\
&= e^{\frac{i}{\hbar}W[J]}\left(\frac{\delta S}{\delta\phi(x)}_{\phi=\frac{\hbar}{i}\frac{\delta}{\delta J}+\frac{\delta W}{\delta J}} + J\right).
\end{aligned} \tag{8.88}$$

In the last line above we have used the identity

$$F(\partial_x)\ e^{g(x)} = e^{g(x)}\ F(\partial_x g + \partial_x). \tag{8.89}$$

We obtain the equation of motion

$$\frac{\delta S}{\delta \phi(x)}_{\phi=\frac{\hbar}{i}\frac{\delta}{\delta J}+\frac{\delta W}{\delta J}} = -J = \frac{\delta \Gamma[\phi_c]}{\delta \phi_c}. \tag{8.90}$$

By the chaine rule we have

$$\begin{aligned}\frac{\delta}{\delta J(x)} &= \int d^4y \frac{\delta \phi_c(y)}{\delta J(x)}\frac{\delta}{\delta \phi_c(y)} \\ &= \int d^4y\ G^{(2)}(x,y)\frac{\delta}{\delta \phi_c(y)}.\end{aligned} \tag{8.91}$$

The $G^{(2)}(x,y)$ is the connected 2−point function in the presence of the source $J(x)$, viz

$$G^{(2)}(x,y) = \frac{\delta \phi_c(y)}{\delta J(x)} = \frac{\delta^2 W[J]}{\delta J(x)\delta J(y)}. \tag{8.92}$$

The quantum equation of motion becomes

$$\frac{\delta S}{\delta \phi(x)}_{\phi=\frac{\hbar}{i}\int d^4y\ G^{(2)}(x,y)\frac{\delta}{\delta \phi_c(y)}+\phi_c(x)} = -J = \frac{\delta \Gamma[\phi_c]}{\delta \phi_c}. \tag{8.93}$$

The connected $n-$point functions and the proper $n-$point vertices are defined as follows. The connected $n-$point functions are defined by

$$G^{(n)}(x_1,...,x_n) = G^{i_1...i_n} = \frac{\delta^n W[J]}{\delta J(x_1)...\delta J(x_n)}. \tag{8.94}$$

The proper $n-$point vertices are defined by

$$\Gamma^{(n)}(x_1,...,x_n) = \Gamma_{,i_1...i_n} = \frac{\delta^n \Gamma[\phi_c]}{\delta \phi_c(x_1)...\delta \phi_c(x_n)}. \tag{8.95}$$

These are connected 1−particle irreducible $n-$point functions from which the external legs are removed (amputated).

The proper 2−point vertex $\Gamma^{(2)}(x,y)$ is the inverse of the connected 2−point function $G^{(2)}(x,y)$. Indeed we compute

$$\begin{aligned}\int d^4z\ G^{(2)}(x,z)\Gamma^{(2)}(z,y) &= \int d^4z\ \frac{\delta \phi_c(z)}{\delta J(x)}\frac{\delta^2 \Gamma[\phi_c]}{\delta \phi_c(z)\delta \phi_c(y)} \\ &= -\int d^4z\ \frac{\delta \phi_c(z)}{\delta J(x)}\frac{\delta J(y)}{\delta \phi_c(z)} \\ &= -\delta^4(x-y).\end{aligned} \tag{8.96}$$

We write this as

$$G^{ik}\Gamma_{,kj} = -\delta^i_j. \tag{8.97}$$

We remark the identities

$$\frac{\delta G^{i_1...i_n}}{\delta J_{i_{n+1}}}=G^{i_1...i_n i_{n+1}}. \quad (8.98)$$

$$\frac{\delta\Gamma_{,i_1...i_n}}{\delta J_{i_{n+1}}}=\frac{\delta\Gamma_{,i_1...i_n}}{\delta\phi_c^k}\frac{\delta\phi_c^k}{\delta J_{i_{n+1}}}=\Gamma_{,i_1...i_nk}G^{ki_{n+1}}. \quad (8.99)$$

By differentiating (8.97) with respect to J_l we obatin

$$G^{ikl}\Gamma_{,kj}+G^{ik}\Gamma_{,kjr}G^{rl}=0. \quad (8.100)$$

Next by multiplying with G^{js} we get the 3−point connected function as

$$G^{isl}=G^{ik}G^{rl}G^{js}\Gamma_{,kjr}. \quad (8.101)$$

Now by differentiating (8.101) with respect to J_m we obatin the 4−point connected function as

$$G^{islm} = \bigg(G^{ikm}G^{rl}G^{js}+G^{ik}G^{rlm}G^{js}+G^{ik}G^{rl}G^{jsm}\bigg)\Gamma_{,kjr}+G^{ik}G^{rl}G^{js}\Gamma_{,kjrn}G^{nm}. \quad (8.102)$$

By using again (8.101) we get

$$\begin{aligned} G^{islm} &= \Gamma_{,k'j'r'}G^{ik'}G^{kj'}G^{mr'}G^{rl}G^{js}\Gamma_{,kjr}+\text{two permutations}\\ &+ G^{ik}G^{rl}G^{js}\Gamma_{,kjrn}G^{nm}. \end{aligned} \quad (8.103)$$

The diagrammatic representation of (8.94),(8.95),(8.97),(8.101) and (8.103) is shown on figure 1.

8.3.2 Perturbation Theory

In this section we will consider a general scalar field theory given by the action

$$S[\phi]=S_i\phi^i+\frac{1}{2!}S_{ij}\phi^i\phi^j+\frac{1}{3!}S_{ijk}\phi^i\phi^j\phi^k+\frac{1}{4!}S_{ijkl}\phi^i\phi^j\phi^k\phi^l+... \quad (8.104)$$

We need the first derivative of $S[\phi]$ with respect to ϕ^i, viz

$$S[\phi]_{,i}=S_i+S_{ij}\phi^j+\frac{1}{2!}S_{ijk}\phi^j\phi^k+\frac{1}{3!}S_{ijkl}\phi^j\phi^k\phi^l+\frac{1}{4!}S_{ijklm}\phi^j\phi^k\phi^l\phi^m+\frac{1}{5!}S_{ijklmn}\phi^j\phi^k\phi^l\phi^m\phi^n+... \quad (8.105)$$

Thus

$$\begin{aligned} \Gamma[\phi_c]_{,i}=S[\phi]_{,i}|_{\phi_i=\phi_{ci}+\frac{\hbar}{i}G^{ii_0}\frac{\delta}{\delta\phi_{ci_0}}} &= S_i+S_{ij}\phi_c^j+\frac{1}{2!}S_{ijk}\bigg(\phi_c^j+\frac{\hbar}{i}G^{jj_0}\frac{\delta}{\delta\phi_{cj_0}}\bigg)\phi_c^k\\ &+ \frac{1}{3!}S_{ijkl}\bigg(\phi_c^j+\frac{\hbar}{i}G^{jj_0}\frac{\delta}{\delta\phi_{cj_0}}\bigg)\bigg(\phi_c^k+\frac{\hbar}{i}G^{kk_0}\frac{\delta}{\delta\phi_{ck_0}}\bigg)\phi_c^l\\ &+ \frac{1}{4!}S_{ijklm}\bigg(\phi_c^j+\frac{\hbar}{i}G^{jj_0}\frac{\delta}{\delta\phi_{cj_0}}\bigg)\bigg(\phi_c^k+\frac{\hbar}{i}G^{kk_0}\frac{\delta}{\delta\phi_{ck_0}}\bigg)\bigg(\phi_c^l+\frac{\hbar}{i}G^{ll_0}\frac{\delta}{\delta\phi_{cl_0}}\bigg)\phi_c^m\\ &+ ... \end{aligned} \quad (8.106)$$

We find upto the first order in $\hbar$ the result [8]

$$\Gamma[\phi_c]_{,i}=S[\phi]_{,i}|_{\phi_i=\phi_{ci}+\frac{\hbar}{i}G^{ii_0}\frac{\delta}{\delta\phi_{ci_0}}} = S[\phi_c]_{,i}+\frac{1}{2}\frac{\hbar}{i}G^{jk}\bigg(S_{ijk}+S_{ijkl}\phi_c^l+\frac{1}{2}S_{ijklm}\phi_c^j\phi_c^m+...\bigg)+O\bigg(\Big(\frac{\hbar}{i}\Big)^2\bigg). \quad (8.107)$$

In other words

$$\Gamma[\phi_c]_{,i}=S[\phi]_{,i}|_{\phi_i=\phi_{ci}+\frac{\hbar}{i}G^{ii_0}\frac{\delta}{\delta\phi_{ci_0}}} = S[\phi_c]_{,i}+\frac{1}{2}\frac{\hbar}{i}G^{jk}S[\phi_c]_{,ijk}+O\bigg(\Big(\frac{\hbar}{i}\Big)^2\bigg). \quad (8.108)$$

We expand

$$\Gamma=\Gamma_0+\frac{\hbar}{i}\Gamma_1+\Big(\frac{\hbar}{i}\Big)^2\Gamma_2+... \quad (8.109)$$

$$G^{ij}=G_0^{ij}+\frac{\hbar}{i}G_1^{ij}+\Big(\frac{\hbar}{i}\Big)^2G_2^{ij}+... \quad (8.110)$$

Immediately we find

$$\Gamma_0[\phi_c]_{,i} = S[\phi_c]_{,i}. \quad (8.111)$$

$$\Gamma_1[\phi_c]_{,i} = \frac{1}{2}G_0^{jk}S[\phi_c]_{,ijk}. \quad (8.112)$$

Equation (8.111) can be trivially integrated. We obatin

$$\Gamma_0[\phi_c] = S[\phi_c]. \quad (8.113)$$

Let us recall the constraint $G^{ik}\Gamma_{,kj}=-\delta^i_j$. This is equivalent to the constraints

$$\begin{aligned}
&G_0^{ik}\Gamma_{0,kj}=-\delta^i_j\\
&G_0^{ik}\Gamma_{1,kj}+G_1^{ik}\Gamma_{0,kj}=0\\
&G_0^{ik}\Gamma_{2,kj}+G_1^{ik}\Gamma_{1,kj}+G_2^{ik}\Gamma_{0,kj}=0\\
&.\\
&.
\end{aligned} \quad (8.114)$$

The first constarint gives G_0^{ik} in terms of $\Gamma_0=S$ as

$$G_0^{ik}=-S^{-1}_{,ik}. \quad (8.115)$$

The second constarint gives G_1^{ik} in terms of Γ_0 and Γ_1 as

$$G_1^{ij}=G_0^{ik}G_0^{jl}\Gamma_{1kl}. \quad (8.116)$$

[8]Exercise: Verify this equation.

The third constraint gives G_2^{ik} in terms of Γ_0, Γ_1 and Γ_2. Hence the calculation of the 2−point function G^{ik} to all orders in perturbation theory requires the calculation the effective action to all orders in perturbation theory, viz the calculation of the Γ_n. In fact the knowledge of the effective action will allow us to calculate all proper $n-$point vertices to any order in perturbation theory.

We are now in a position to integrate equation (8.112). We have

$$\begin{aligned}\Gamma_1[\phi_c]_{,i} &= \frac{1}{2}G_0^{jk}\frac{\delta S[\phi_c]_{,jk}}{\delta\phi_{ci}}\\ &= -\frac{1}{2}G_0^{jk}\frac{\delta(G_0^{-1})_{jk}}{\delta\phi_{ci}}\\ &= -\frac{1}{2}\frac{\delta}{\delta\phi_{ci}}\ln\det G_0^{-1}.\end{aligned} \tag{8.117}$$

Thus

$$\Gamma_1[\phi_c] = -\frac{1}{2}\ln\det G_0^{-1}. \tag{8.118}$$

The effective action upto the 1−loop order is

$$\Gamma = \Gamma_0 + \frac{1}{2}\frac{\hbar}{i}\ln\det G_0 + ... \tag{8.119}$$

This is represented graphically by the first two diagrams on figure 2.

8.3.3 Analogy with Statistical Mechanics

We start by making a Wick rotation. The Euclidean vacuum energy, classical field, classical equation of motion, effective action and quantum equation of motion are defined by

$$Z_E[J] = e^{-\frac{1}{\hbar}W_E[J]}. \tag{8.120}$$

$$\phi_c(x)|_{J=0} = -\frac{\delta W_E[J]}{\delta J(x)}|_{J=0} =< 0|\Phi(x)|0 >_E . \tag{8.121}$$

$$\frac{\delta S_E[\phi]}{\delta\phi(x)} = J(x). \tag{8.122}$$

$$\Gamma_E[\phi_c] = W_E[J] + \int d^4x J(x)\phi_c(x). \tag{8.123}$$

$$\frac{\delta\Gamma_E[\phi_c]}{\delta\phi_c(x)} = J(x). \tag{8.124}$$

Let us now consider the following statistical mechanics problem. We consider a magnetic system consisting of spins $s(x)$. The spin energy density is $\mathcal{H}(s)$. The system is placed in a magnetic field H. The partition function of the system is defined by

$$Z[H]=\int \mathcal{D}s\ e^{-\beta\int dx\mathcal{H}(s)+\beta\int dx H(x)s(x)}. \tag{8.125}$$

The spin $s(x)$, the spin energy density $\mathcal{H}(s)$ and the magnetic field $H(x)$ play in statistical mechanics the role played by the scalar field $\phi(x)$, the Lagrangian density $\mathcal{L}(\phi)$ and the source $J(x)$ respectively in field theory. The free energy of the magnetic system is defined through the equation

$$Z[H]=e^{-\beta F[H]}. \tag{8.126}$$

This means that F in statistical mechanics is the analogue of W in field theory. The magnetization of the system is defined by

$$\begin{aligned}-\frac{\delta F}{\delta H}|_{\beta=\text{fixed}} &= \frac{1}{Z}\int dx\int\mathcal{D}s\ s(x)\ e^{-\beta\int dx(\mathcal{H}(s)-Hs(x))}\\ &= \int dx < s(x) >\\ &= M.\end{aligned} \tag{8.127}$$

Thus the magnetization M in statistical mechanics plays the role of the effective field $-\phi_c$ in field theory. In other words ϕ_c is the order parameter in the field theory. Finally the Gibbs free energy in statistical mechanics plays the role of the effective action $\Gamma[\phi_c]$ in field theory. Indeed G is the Legendre transform of F given by

$$G=F+MH. \tag{8.128}$$

Furthermore we compute

$$\frac{\delta G}{\delta M}=H. \tag{8.129}$$

The thermodynamically most stable state (the ground state) is the minimum of G. Similarly the quantum mechanically most stable state (the vacuum) is the minimum of Γ. The thermal fluctuations from one side correspond to quantum fluctuations on the other side.

8.4 The $O(N)$ Model

In this section we will consider a generalization of the ϕ^4 model known as the linear sigma model. We are interested in the $(\phi^2)^2$ theory with $O(N)$ symmetry given by the classical action

$$S[\phi]=\int d^4x\bigg[\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i-\frac{1}{2}m^2\phi_i^2-\frac{\lambda}{4!}(\phi_i^2)^2\bigg]. \tag{8.130}$$

This classical action is of the general form studied in the previous section, viz

$$S[\phi]=\frac{1}{2!}S_{IJ}\phi^I\phi^J+\frac{1}{4!}S_{IJKL}\phi^I\phi^J\phi^K\phi^L. \tag{8.131}$$

The index I stands for i and the spacetime index x, i.e $I=(i,x)$, $J=(j,y)$, $K=(k,z)$ and $L=(l,w)$. We have

$$\begin{aligned}S_{IJ}&=-\delta_{ij}(\Delta+m^2)\delta^4(x-y)\\ S_{IJKL}&=-\frac{\lambda}{3}\delta_{ijkl}\delta^4(y-x)\delta^4(z-x)\delta^4(w-x)\ ,\ \delta_{ijkl}=\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}.\end{aligned} \tag{8.132}$$

The effective action upto the 1−loop order is

$$\Gamma[\phi] = S[\phi] + \frac{1}{2}\frac{\hbar}{i}\ln\det G_0. \tag{8.133}$$

The proper n−point vertex is defined now by setting $\phi = 0$ after taking the n derivatives, viz

$$\Gamma^{(n)}_{i_1...i_n}(x_1,...,x_n) = \Gamma_{,I_1...I_n} = \frac{\delta^n\Gamma[\phi]}{\delta\phi_{i_1}(x_1)...\delta\phi_{i_n}(x_n)}|_{\phi=0}. \tag{8.134}$$

8.4.1 The 2−Point and 4−Point Proper Vertices

The proper 2−point vertex is defined by

$$\begin{aligned}\Gamma^{(2)}_{ij}(x,y) &= \frac{\delta^2\Gamma[\phi]}{\delta\phi_i(x)\delta\phi_j(y)}|_{\phi=0}\\ &= \frac{\delta^2 S[\phi]}{\delta\phi_i(x)\delta\phi_j(y)}|_{\phi=0} + \frac{\hbar}{i}\frac{\delta^2\Gamma_1[\phi]}{\delta\phi_i(x)\delta\phi_j(y)}|_{\phi=0}\\ &= -\delta_{ij}(\Delta+m^2)\delta^4(x-y) + \frac{\hbar}{i}\frac{\delta^2\Gamma_1[\phi]}{\delta\phi_i(x)\delta\phi_j(y)}|_{\phi=0}.\end{aligned} \tag{8.135}$$

The one-loop correction can be computed using the result

$$\Gamma_1[\phi]_{,j_0k_0} = \frac{1}{2}G_0^{mn}S[\phi]_{,j_0k_0mn} + \frac{1}{2}G_0^{mm_0}G_0^{nn_0}S[\phi]_{,j_0mn}S[\phi]_{,k_0m_0n_0}. \tag{8.136}$$

We get by setting $\phi = 0$ the result

$$\begin{aligned}\Gamma_1[\phi]_{,IJ} &= \frac{1}{2}G_0^{mn}S[\phi]_{,ijmn}\\ &= -\frac{\lambda}{6}\int d^4z d^4w G_0^{mn}(z,w)\bigg(\delta_{ij}\delta_{mn} + \delta_{im}\delta_{jn} + \delta_{in}\delta_{jm}\bigg)\delta^4(y-x)\delta^4(z-x)\delta^4(w-x)\\ &= -\frac{\lambda}{6}\bigg(\delta_{ij}G_0^{mm}(x,y) + 2G_0^{ij}(x,y)\bigg)\delta^4(x-y).\end{aligned} \tag{8.137}$$

We have

$$G_0^{IJ} = -S^{-1}_{,IJ}. \tag{8.138}$$

Since $S_{,IJ} = S_{IJ}$ and $S_{IJ} = -\delta_{ij}S(x,y)$ where $S(x,y) = (\Delta+m^2)\delta^4(x-y)$ we can write

$$G_0^{IJ} = \delta_{ij}G_0(x,y). \tag{8.139}$$

Clearly $\int d^4y G_0(x,y)S(y,z) = \delta^4(x-y)$. We obtain

$$\Gamma_1[\phi]_{,IJ} = -\frac{\lambda}{6}(N+2)\delta_{ij}G_0(x,y)\delta^4(x-y). \tag{8.140}$$

Now we compute the 4−point proper vertex. Clearly the first contribution will be given precisely by the second equation of (8.132). Indeed we have

$$\begin{aligned}\Gamma^{(4)}_{i_1...i_4}(x_1,...,x_4) &= \frac{\delta^4\Gamma[\phi]}{\delta\phi_{i_1}(x_1)...\delta\phi_{i_4}(x_4)}|_{\phi=0}\\ &= \frac{\delta^4 S[\phi]}{\delta\phi_{i_1}(x_1)...\delta\phi_{i_4}(x_4)}|_{\phi=0} + \frac{\hbar}{i}\frac{\delta^4\Gamma_1[\phi]}{\delta\phi_{i_1}(x_1)...\delta\phi_{i_4}(x_4)}|_{\phi=0}\\ &= S_{I_1...I_4} + \frac{\hbar}{i}\frac{\delta^2\Gamma_1[\phi]}{\delta\phi_{i_1}(x_1)...\delta\phi_{i_4}(x_4)}|_{\phi=0}.\end{aligned} \tag{8.141}$$

In order to compute the first correction we use the identity

$$\frac{\delta G_0^{mn}}{\delta\phi_{cl}} = G_0^{mm_0}G_0^{nn_0}S[\phi_c]_{,lm_0n_0}. \tag{8.142}$$

We compute

$$\begin{aligned}\Gamma_1[\phi]_{,j_0k_0ll_0} &= \bigg[\frac{1}{2}G_0^{mm_0}G_0^{nn_0}S[\phi]_{,j_0k_0mn}S[\phi]_{,ll_0m_0n_0} + \frac{1}{2}G_0^{mm_0}G_0^{nn_0}S[\phi]_{,j_0lmn}S[\phi]_{,k_0l_0m_0n_0}\\ &+ \frac{1}{2}G_0^{mm_0}G_0^{nn_0}S[\phi]_{,j_0l_0mn}S[\phi]_{,k_0lm_0n_0}\bigg]|_{\phi=0}.\end{aligned} \tag{8.143}$$

Thus

$$\begin{aligned}\frac{\delta^4\Gamma_1[\phi]}{\delta\phi_{i_1}(x_1)...\delta\phi_{i_4}(x_4)}|_{\phi=0} &= \frac{1}{2}\bigg(\frac{\lambda}{3}\bigg)^2\delta^4(x_1-x_2)\delta^4(x_3-x_4)\bigg((N+2)\delta_{i_1i_2}\delta_{i_3i_4}+2\delta_{i_1i_2i_3i_4}\bigg)G_0(x_1,x_3)^2\\ &+ \frac{1}{2}\bigg(\frac{\lambda}{3}\bigg)^2\delta^4(x_1-x_3)\delta^4(x_2-x_4)\bigg((N+2)\delta_{i_1i_3}\delta_{i_2i_4}+2\delta_{i_1i_2i_3i_4}\bigg)G_0(x_1,x_2)^2\\ &+ \frac{1}{2}\bigg(\frac{\lambda}{3}\bigg)^2\delta^4(x_1-x_4)\delta^4(x_2-x_3)\bigg((N+2)\delta_{i_1i_4}\delta_{i_2i_3}+2\delta_{i_1i_2i_3i_4}\bigg)G_0(x_1,x_2)^2.\end{aligned} \tag{8.144}$$

8.4.2 Momentum Space Feynman Graphs

The proper 2−point vertex upto the 1−loop order is

$$\Gamma_{ij}^{(2)}(x,y) = -\delta_{ij}(\Delta+m^2)\delta^4(x-y) - \frac{\hbar}{i}\frac{\lambda}{6}(N+2)\delta_{ij}G_0(x,y)\delta^4(x-y). \tag{8.145}$$

The proper 2−point vertex in momentum space $\Gamma_{ij}^{(2)}(p)$ is defined through the equations

$$\begin{aligned}\int d^4xd^4y\;\Gamma_{ij}^{(2)}(x,y)\;e^{ipx+iky} &= (2\pi)^4\delta^4(p+k)\Gamma_{ij}^{(2)}(p,k)\\ &= (2\pi)^4\delta^4(p+k)\Gamma_{ij}^{(2)}(p,-p)\\ &= (2\pi)^4\delta^4(p+k)\Gamma_{ij}^{(2)}(p).\end{aligned} \tag{8.146}$$

The delta function is due to translational invariance.

From the definition $S(x,y)=(\Delta+m^2)\delta^4(x-y)$ we have

$$S(x,y) = \int\frac{d^4p}{(2\pi)^4}(-p^2+m^2)\;e^{ip(x-y)}. \tag{8.147}$$

Then by using the equation $\int d^4y\;G_0(x,y)S(y,z)=\delta^4(x-y)$ we obtain

$$G_0(x,y) = \int\frac{d^4p}{(2\pi)^4}\frac{1}{-p^2+m^2}\;e^{ip(x-y)}. \tag{8.148}$$

We get

$$\Gamma_{ij}^{(2)}(p) = -\delta_{ij}(-p^2+m^2) - \frac{\hbar}{i}\frac{\lambda}{6}(N+2)\delta_{ij}\int\frac{d^4p_1}{(2\pi)^4}\frac{1}{-p_1^2+m^2}. \tag{8.149}$$

The corresponding Feynman diagrams are shown on figure 4.

The proper 4−point vertex upto the 1−loop order is

$$\begin{aligned}
\Gamma^{(4)}_{i_1...i_4}(x_1,...,x_4) &= -\frac{\lambda}{3}\bigg(\delta_{ij}\delta_{kl}+\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\bigg)\delta^4(y-x)\delta^4(z-x)\delta^4(w-x)+\frac{1}{2}\bigg(\frac{\hbar}{i}\bigg)\bigg(\frac{\lambda}{3}\bigg)^2\\
&\times \bigg[\delta^4(x_1-x_2)\delta^4(x_3-x_4)\bigg((N+2)\delta_{i_1i_2}\delta_{i_3i_4}+2\delta_{i_1i_2i_3i_4}\bigg)G_0(x_1,x_3)^2+\\
&\delta^4(x_1-x_3)\delta^4(x_2-x_4)\bigg((N+2)\delta_{i_1i_3}\delta_{i_2i_4}+2\delta_{i_1i_2i_3i_4}\bigg)G_0(x_1,x_2)^2+\\
&\delta^4(x_1-x_4)\delta^4(x_2-x_3)\bigg((N+2)\delta_{i_1i_4}\delta_{i_2i_3}+2\delta_{i_1i_2i_3i_4}\bigg)G_0(x_1,x_2)^2\bigg].
\end{aligned} \tag{8.150}$$

The proper 4−point vertex in momentum space $\Gamma^{(4)}_{i_1...i_4}(p_1...p_4)$ is defined through the equation

$$\int d^4x_1...d^4x_4\ \Gamma^{(4)}_{i_1...i_4}(x_1,...,x_4)\ e^{ip_1x_1+...+ip_4x_4} = (2\pi)^4\delta^4(p_1+...+p_4)\Gamma^{(2)}_{i_1...i_4}(p_1,...,p_4). \tag{8.151}$$

We find (with $p_{12}=p_1+p_2$ and $p_{14}=p_1+p_4$, etc)

$$\begin{aligned}
\Gamma^{(4)}_{i_1...i_4}(p_1,...,p_4) &= -\frac{\lambda}{3}\delta_{i_1i_2i_3i_4}\\
&+ \frac{\hbar}{i}\bigg(\frac{\lambda}{3}\bigg)^2\frac{1}{2}\bigg[\bigg((N+2)\delta_{i_1i_2}\delta_{i_3i_4}+2\delta_{i_1i_2i_3i_4}\bigg)\int_k\frac{1}{(-k^2+m^2)(-(p_{12}-k)^2+m^2)}\\
&+ 2\ \text{permutations}\bigg].
\end{aligned} \tag{8.152}$$

The corresponding Feynman diagrams are shown on figure 5.

8.4.3 Cut-off Regularization

At the one-loop order we have then

$$\Gamma^{(2)}_{ij}(p) = -\delta_{ij}(-p^2+m^2)-\frac{\hbar}{i}\frac{\lambda}{6}(N+2)\delta_{ij}I(m^2). \tag{8.153}$$

$$\begin{aligned}
\Gamma^{(4)}_{i_1...i_4}(p_1,...,p_4) &= -\frac{\lambda}{3}\delta_{i_1i_2i_3i_4}+\frac{\hbar}{i}\bigg(\frac{\lambda}{3}\bigg)^2\frac{1}{2}\bigg[\bigg((N+2)\delta_{i_1i_2}\delta_{i_3i_4}+2\delta_{i_1i_2i_3i_4}\bigg)J(p_{12}^2,m^2)\\
&+ 2\ \text{permutations}\bigg],
\end{aligned} \tag{8.154}$$

where

$$\Delta(k)=\frac{1}{-k^2+m^2}\ ,\ I(m^2)=\int\frac{d^4k}{(2\pi)^4}\Delta(k)\ ,\ J(p_{12}^2,m^2)=\int\frac{d^4k}{(2\pi)^4}\Delta(k)\Delta(p_{12}-k). \tag{8.155}$$

It is not difficult to convince ourselves that the first integral $I(m^2)$ diverges quadratically whereas the second integral $J(p_{12}^2,m^2)$ diverges logarithmically. To see this more carefully it is better we

Wick rotate to Euclidean signature. Formally this is done by writing $k_0 = ik_4$ which is consistent with $x^0 = -ix^4$. As a consequence we replace $k^2 = k_0^2 - \vec{k}^2$ with $-k_4^2 - \vec{k}^2 = -k^2$. The Euclidean expressions are

$$\Gamma^{(2)}_{ij}(p) = \delta_{ij}(p^2+m^2) + \hbar\frac{\lambda}{6}(N+2)\delta_{ij}I(m^2). \quad (8.156)$$

$$\begin{aligned}\Gamma^{(4)}_{i_1\ldots i_4}(p_1,...,p_4) &= \frac{\lambda}{3}\delta_{i_1i_2i_3i_4} - \hbar\bigg(\frac{\lambda}{3}\bigg)^2\frac{1}{2}\bigg[\bigg((N+2)\delta_{i_1i_2}\delta_{i_3i_4} + 2\delta_{i_1i_2i_3i_4}\bigg)J(p_{12}^2,m^2)\\ &+ 2 \text{ permutations}\bigg], \end{aligned} \quad (8.157)$$

where now

$$\Delta(k) = \frac{1}{k^2+m^2}\ ,\ I(m^2) = \int\frac{d^4k}{(2\pi)^4}\Delta(k)\ ,\ J(p_{12}^2,m^2) = \int\frac{d^4k}{(2\pi)^4}\Delta(k)\Delta(p_{12}-k). \quad (8.158)$$

Explicitly we have

$$\begin{aligned}I(m^2) &= \int_0^\infty d\alpha e^{-\alpha m^2}\int\frac{d^4k}{(2\pi)^4}e^{-\alpha k^2}\\ &= \int_0^\infty d\alpha e^{-\alpha m^2}\frac{1}{8\pi^2}\int k^3dke^{-\alpha k^2}\\ &= \frac{1}{16\pi^2}\int_0^\infty d\alpha\frac{e^{-\alpha m^2}}{\alpha^2}. \end{aligned} \quad (8.159)$$

To calculate the divergences we need to introduce a cut-off Λ. In principle we should use the regularized propagator

$$\Delta(k,\Lambda) = \frac{e^{-\frac{k^2}{\Lambda^2}}}{k^2+m^2}. \quad (8.160)$$

Alternatively we can introduce the cut-off Λ as follows

$$\begin{aligned}I(m^2,\Lambda) &= \frac{1}{16\pi^2}\int_{\frac{1}{\Lambda^2}}^\infty d\alpha\frac{e^{-\alpha m^2}}{\alpha^2}\\ &= \frac{1}{16\pi^2}\bigg(\Lambda^2 - m^2\int_{\frac{1}{\Lambda^2}}^\infty d\alpha\frac{e^{-\alpha m^2}}{\alpha}\bigg)\\ &= \frac{1}{16\pi^2}\bigg(\Lambda^2 + m^2\mathrm{Ei}(-\frac{m^2}{\Lambda^2})\bigg). \end{aligned} \quad (8.161)$$

This diverges quadratically. The exponential-integral function is defined by

$$Ei(x) = \int_{-\infty}^x\frac{e^t}{t}dt. \quad (8.162)$$

Also by using the same method we compute

$$\begin{aligned}J(p_{12}^2,m^2) &= \int d\alpha_1 d\alpha_2\ e^{-m^2(\alpha_1+\alpha_2)-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}p_{12}^2}\int\frac{d^4k}{(2\pi)^4}\ e^{-(\alpha_1+\alpha_2)k^2}\\ &= \frac{1}{(4\pi)^2}\int d\alpha_1 d\alpha_2\ \frac{e^{-m^2(\alpha_1+\alpha_2)-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}p_{12}^2}}{(\alpha_1+\alpha_2)^2}. \end{aligned} \quad (8.163)$$

We introduce the cut-off Λ as follows

$$\begin{aligned} J(p_{12}^2,m^2,\Lambda) &= \frac{1}{(4\pi)^2}\int_{\frac{1}{\Lambda^2}} d\alpha_1 d\alpha_2\; \frac{e^{-m^2(\alpha_1+\alpha_2)-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}p_{12}^2}}{(\alpha_1+\alpha a_2)^2}\\ &= \frac{1}{(4\pi)^2}\int_1 dx dx_2\; \frac{e^{-\frac{m^2}{\Lambda^2}(x+x_2)-\frac{xx_2}{x+x_2}\frac{p_{12}^2}{\Lambda^2}}}{(x+x_2)^2}. \end{aligned} \tag{8.164}$$

The integral can be rewritten as 2 times the integral over the symmetric region $x_2 > x$. We can also perform the change of variables $x_2 = xy$ to obtain

$$\begin{aligned} J(p_{12}^2,m^2,\Lambda) &= \frac{2}{(4\pi)^2}\int_1 \frac{dx}{x}\int_1 \frac{dy}{(1+y)^2}\; e^{-\frac{m^2}{\Lambda^2}x(1+y)-\frac{xy}{1+y}\frac{p_{12}^2}{\Lambda^2}}\\ &= \frac{2}{(4\pi)^2}\int_1 \frac{dx}{x}\int_0^{\frac{1}{2}} d\rho\; e^{-x\left(\frac{a}{\rho}+b(1-\rho)\right)}. \end{aligned} \tag{8.165}$$

In above $a = \frac{m^2}{\Lambda^2}$ and $b = \frac{p_{12}^2}{\Lambda^2}$. We have

$$\begin{aligned} J(p_{12}^2,m^2,\Lambda) &= \frac{2}{(4\pi)^2}\int_0^{\frac{1}{2}} d\rho\int_1^{\infty}\frac{dx}{x}\; e^{-x\left(\frac{a}{\rho}+b(1-\rho)\right)}\\ &= -\frac{1}{8\pi^2}\int_0^{\frac{1}{2}} d\rho\;\mathrm{Ei}\Big(-\frac{a}{\rho}-(1-\rho)b\Big). \end{aligned} \tag{8.166}$$

The exponential-integral function is such that

$$\mathrm{Ei}\Big(-\frac{a}{\rho}-(1-\rho)b\Big) = \mathbf{C} + \ln\Big(\frac{a}{\rho}+(1-\rho)b\Big) + \int_0^{\frac{a}{\rho}+(1-\rho)b} dt\;\frac{e^{-t}-1}{t}. \tag{8.167}$$

The last term leads to zero in the limit $\Lambda \longrightarrow \infty$ since $a, b \longrightarrow 0$ in this limit. The exponential-integral function becomes

$$\mathrm{Ei}\Big(-(1-\rho)b-\frac{a}{\rho}\Big) = \mathbf{C} + \ln\Big(\sqrt{a+\frac{b}{4}}+\sqrt{\frac{b}{4}}-\sqrt{b}\rho\Big) + \ln\Big(\sqrt{a+\frac{b}{4}}-\sqrt{\frac{b}{4}}+\sqrt{b}\rho\Big) - \ln\rho. \tag{8.168}$$

By using the integral $\int_0^1\; d\rho\;\ln(A+B\rho) = \frac{1}{B}\Big((A+B)\ln(A+B)-A\ln A\Big) - 1$ we find

$$\int_0^1 d\rho\;\mathrm{Ei}\Big(-(1-\rho)b-\frac{a}{\rho}\Big) = \mathbf{C} + \ln a + \sqrt{1+\frac{4a}{b}}\ln\Big(1+\frac{b}{2a}+\frac{1}{2a}\sqrt{b(b+4a)}\Big) + 1. \tag{8.169}$$

Hence we have

$$-\int_0^1 d\rho\;\mathrm{Ei}\Big(-(1-\rho)b-\frac{a}{\rho}\Big) = -\ln a + ... = \ln\frac{\Lambda^2}{m^2} + ... \tag{8.170}$$

Equivalently

$$J(p_{12}^2,m^2,\Lambda) = \frac{1}{16\pi^2}\ln\frac{\Lambda^2}{2m^2} + ... \tag{8.171}$$

This is the logarithmic divergence.

In summary we have found two divergences at one-loop order. A quadratic divergence in the proper 2−point vertex and a logarithmic divergence in the proper 4−point vertex. All higher n−point vertices are finite in the limit $\Lambda \longrightarrow \infty$.

8.4.4 Renormalization at 1−Loop

To renormalize the theory, i.e. to remove the above two divergences we will assume that:

- 1) The theory comes with a cut-off Λ so that the propagator of the theory is actually given by (8.160).
- 2) The parameters of the model m^2 and λ which are called from now on bare parameters will be assumed to depend implicitly on the cut-off Λ.
- 3) The renormalized (physical) parameters of the theory m_R^2 and λ_R will be determined from specific conditions imposed on the 2− and 4−proper vertices.

In the limit $\Lambda \longrightarrow \infty$ the renormalized parameters remain finite while the bare parameters diverge in such a way that the divergences coming from loop integrals are canceled. In this way the 2− and 4−proper vertices become finite in the large cut-off limit $\Lambda \longrightarrow \infty$.

Since only two vertices are divergent we will only need two conditions to be imposed. We choose the physical mass m_R^2 to correspond to the zero momentum value of the proper 2−point vertex, viz

$$\Gamma_{ij}^{(2)}(0) = \delta_{ij}m_R^2 = \delta_{ij}m^2 + \hbar\frac{\lambda}{6}(N+2)\delta_{ij}I(m^2,\Lambda). \tag{8.172}$$

We also choose the physical coupling constant λ_R^2 to correspond to the zero momentum value of the proper 4−point vertex, viz

$$\Gamma_{i_1\ldots i_4}^{(4)}(0,...,0) = \frac{\lambda_R}{3}\delta_{i_1i_2i_3i_4} = \frac{\lambda}{3}\delta_{i_1i_2i_3i_4} - \hbar\left(\frac{\lambda}{3}\right)^2\frac{N+8}{2}\delta_{i_1i_2i_3i_4}J(0,m^2,\Lambda). \tag{8.173}$$

We solve for the bare parameters in terms of the renormalized parameters we find

$$m^2 = m_R^2 - \hbar\frac{\lambda_R}{6}(N+2)I(m_R^2,\Lambda). \tag{8.174}$$

$$\frac{\lambda}{3} = \frac{\lambda_R}{3} + \hbar\left(\frac{\lambda_R}{3}\right)^2\frac{N+8}{2}J(0,m_R^2,\Lambda). \tag{8.175}$$

The 2− and 4−point vertices in terms of the renormalized parameters are

$$\Gamma_{ij}^{(2)}(p) = \delta_{ij}(p^2+m_R^2). \tag{8.176}$$

$$\Gamma_{i_1\ldots i_4}^{(4)}(p_1,...,p_4) = \frac{\lambda_R}{3}\delta_{i_1i_2i_3i_4} - \hbar\left(\frac{\lambda_R}{3}\right)^2\frac{1}{2}\bigg[\bigg((N+2)\delta_{i_1i_2}\delta_{i_3i_4} + 2\delta_{i_1i_2i_3i_4}\bigg)\bigg(J(p_{12}^2,m_R^2,\Lambda) - J(0,m_R^2,\Lambda)\bigg) + 2\text{ permutations}\bigg]. \tag{8.177}$$

8.5 Two-Loop Calculations

8.5.1 The Effective Action at 2−Loop

By extending equation (8.107) to the second order in $\hbar$ we get [9]

$$\Gamma[\phi_c]_{,i} = O(1)+O\Big(\frac{\hbar}{i}\Big)+\frac{1}{6}\Big(\frac{\hbar}{i}\Big)^2\Big[G^{jj_0}\frac{\delta G^{kl}}{\delta\phi_{cj_0}}\Big(S_{ijkl}+S_{ijklm}\phi_c^m+\frac{1}{2}S_{ijklmn}\phi_c^m\phi_c^n+...\Big) + \frac{3}{4}\Big(S_{ijklm}+S_{ijklmn}\phi_c^n+...\Big)G^{jk}G^{lm}\Big]+O\Big(\Big(\frac{\hbar}{i}\Big)^3\Big). \tag{8.178}$$

Equivalently

$$\Gamma[\phi_c]_{,i} = O(1)+O\Big(\frac{\hbar}{i}\Big)+\frac{1}{6}\Big(\frac{\hbar}{i}\Big)^2\Big[G^{jj_0}\frac{\delta G^{kl}}{\delta\phi_{cj_0}}S[\phi_c]_{,ijkl}+\frac{3}{4}S[\phi_c]_{,ijklm}G^{jk}G^{lm}\Big]+O\Big(\Big(\frac{\hbar}{i}\Big)^3\Big). \tag{8.179}$$

We use the identity

$$\frac{\delta G^{kl}}{\delta\phi_{cj_0}}=\frac{\delta G^{kl}}{\delta J_m}\frac{\delta J_m}{\delta\phi_{cj_0}} = -G^{klm}\Gamma_{,mj_0} = -G^{kk_0}G^{ll_0}G^{mm_0}\Gamma_{,k_0l_0m_0}\Gamma_{,mj_0}. \tag{8.180}$$

Thus

$$\Gamma[\phi_c]_{,i} = O(1)+O\Big(\frac{\hbar}{i}\Big)+\frac{1}{6}\Big(\frac{\hbar}{i}\Big)^2\Big[-G^{jj_0}G^{kk_0}G^{ll_0}G^{mm_0}\Gamma_{,k_0l_0m_0}\Gamma_{,mj_0}S[\phi_c]_{,ijkl} + \frac{3}{4}S[\phi_c]_{,ijklm}G^{jk}G^{lm}\Big]+O\Big(\Big(\frac{\hbar}{i}\Big)^3\Big). \tag{8.181}$$

By substituting the expansions (8.109) and (8.110) we get at the second order in $\hbar$ the equation

$$\Gamma_2[\phi_c]_{,i} = \frac{1}{2}G_1^{jk}S[\phi_c]_{,ijk}+\frac{1}{6}\Big[-G_0^{jj_0}G_0^{kk_0}G_0^{ll_0}G_0^{mm_0}\Gamma_{0,k_0l_0m_0}\Gamma_{0,mj_0}S[\phi_c]_{,ijkl} + \frac{3}{4}S[\phi_c]_{,ijklm}G_0^{jk}G_0^{lm}\Big]. \tag{8.182}$$

Next we compute G_1^{ij}. Therefore we must determine Γ_{1kl}. By differentiating equation (8.112) with respect to ϕ_{cl} we get

$$\Gamma_1[\phi_c]_{,kl} = \frac{1}{2}G_0^{mn}S[\phi_c]_{,klmn}+\frac{1}{2}\frac{\delta G_0^{mn}}{\delta\phi_{cl}}S[\phi_c]_{,kmn}. \tag{8.183}$$

By using the identity

$$\frac{\delta G_0^{mn}}{\delta\phi_{cl}}=G_0^{mm_0}G_0^{nn_0}S[\phi_c]_{,lm_0n_0}. \tag{8.184}$$

[9] Exercise:Verify this equation.

We get

$$\Gamma_1[\phi_c]_{,j_0k_0} = \frac{1}{2}G_0^{mn}S[\phi_c]_{,j_0k_0mn} + \frac{1}{2}G_0^{mm_0}G_0^{nn_0}S[\phi_c]_{,j_0mn}S[\phi_c]_{,k_0m_0n_0}. \quad (8.185)$$

Hence

$$G_1^{jk} = G_0^{jj_0}G_0^{kk_0}\bigg(\frac{1}{2}G_0^{mn}S[\phi_c]_{,j_0k_0mn} + \frac{1}{2}G_0^{mm_0}G_0^{nn_0}S[\phi_c]_{,j_0mn}S[\phi_c]_{,k_0m_0n_0}\bigg). \quad (8.186)$$

Equation (8.182) becomes

$$\begin{aligned}\Gamma_2[\phi_c]_{,i} &= \frac{1}{2}G_0^{jj_0}G_0^{kk_0}\bigg[\frac{1}{2}G_0^{mn}S[\phi_c]_{,j_0k_0mn} + \frac{1}{2}G_0^{mm_0}G_0^{nn_0}S[\phi_c]_{,j_0mn}S[\phi_c]_{,k_0m_0n_0}\bigg]S[\phi_c]_{,ijk}\\ &+ \frac{1}{6}\bigg[-G_0^{jj_0}G_0^{kk_0}G_0^{ll_0}G_0^{mm_0}S[\phi_c]_{,k_0l_0m_0}S[\phi_c]_{,mj_0}S[\phi_c]_{,ijkl} + \frac{3}{4}S[\phi_c]_{,ijklm}G_0^{jk}G_0^{lm}\bigg].\end{aligned} \quad (8.187)$$

Integration of this equation yields [10]

$$\Gamma_2[\phi_c] = \frac{1}{8}S[\phi_c]_{,ijkl}G_0^{ij}G_0^{kl} + \frac{1}{12}S[\phi_c]_{,ikm}G_0^{ij}G_0^{kl}G_0^{mn}S[\phi_c]_{,jln}. \quad (8.188)$$

The effective action upto the $2-$loop order is

$$\Gamma = \Gamma_0 + \frac{1}{2}\frac{\hbar}{i}\ln\det G_0 + \bigg(\frac{\hbar}{i}\bigg)^2\bigg(\frac{1}{8}S[\phi_c]_{,ijkl}G_0^{ij}G_0^{kl} + \frac{1}{12}S[\phi_c]_{,ikm}G_0^{ij}G_0^{kl}G_0^{mn}S[\phi_c]_{,jln}\bigg) + ... \quad (8.189)$$

This is represented graphically on figure 2.

8.5.2 The Linear Sigma Model at $2-$Loop

The proper $2-$point vertex upto $2-$loop is given by [11]

$$\Gamma_{ij}^{(2)}(x,y) = O(1) + O\bigg(\frac{\hbar}{i}\bigg) + \bigg(\frac{\hbar}{i}\bigg)^2\frac{\delta^2\Gamma_2[\phi]}{\delta\phi_i(x)\delta\phi_j(y)}|_{\phi=0}. \quad (8.190)$$

The $2-$loop correction can be computed using the result

$$\begin{aligned}\Gamma_2[\phi]_{,i} &= \frac{1}{2}G_0^{jj_0}G_0^{kk_0}\bigg[\frac{1}{2}G_0^{mn}S[\phi]_{,j_0k_0mn} + \frac{1}{2}G_0^{mm_0}G_0^{nn_0}S[\phi]_{,j_0mn}S[\phi]_{,k_0m_0n_0}\bigg]S[\phi]_{,ijk}\\ &+ \frac{1}{6}\bigg[-G_0^{jj_0}G_0^{kk_0}G_0^{ll_0}G_0^{mm_0}S[\phi]_{,k_0l_0m_0}S[\phi]_{,mj_0}S[\phi]_{,ijkl} + \frac{3}{4}S[\phi]_{,ijklm}G_0^{jk}G_0^{lm}\bigg].\end{aligned} \quad (8.191)$$

By setting $\phi=0$ we obtain

$$\begin{aligned}\Gamma_2[\phi]_{,IJ} &= \frac{1}{4}G_0^{i_0j_0}G_0^{kk_0}G_0^{mn}S[\phi]_{,j_0k_0mn}S[\phi]_{,ii_0kj} - \frac{1}{6}G_0^{i_0j_0}G_0^{kk_0}G_0^{ll_0}G_0^{mm_0}S[\phi]_{,k_0l_0m_0j}S[\phi]_{,mj_0}S[\phi]_{,ii_0kl}\\ &= \frac{1}{4}\bigg(\frac{\lambda}{3}\bigg)^2(N+2)^2\delta_{ij}\delta^4(x-y)G_0(w,w)\int d^4zG_0(x,z)G_0(y,z) + \frac{N+2}{2}\bigg(\frac{\lambda}{3}\bigg)^2\delta_{ij}G_0(x,y)^3.\end{aligned} \quad (8.192)$$

[10] Exercise:verify this result.
[11] Exercise: Verify all equations of this section.

We have then

$$\Gamma^{(2)}_{ij}(x,y) = O(1)+O\Big(\frac{\hbar}{i}\Big)+\Big(\frac{\hbar}{i}\Big)^2\Big(\frac{\lambda}{3}\Big)^2\frac{N+2}{2}\delta_{ij}\bigg(\frac{N+2}{2}\delta^4(x-y)G_0(w,w)\int d^4z G_0(x,z)G_0(y,z) + G_0(x,y)^3\bigg). \quad (8.193)$$

Next we write this result in momentum space. The proper 2−point vertex in momentum space $\Gamma^{(2)}_{ij}(p)$ is defined through the equations

$$\int d^4xd^4y\ \Gamma^{(2)}_{ij}(x,y)\ e^{ipx+iky} = (2\pi)^4\delta^4(p+k)\Gamma^{(2)}_{ij}(p). \quad (8.194)$$

We compute immediately

$$\Gamma^{(2)}_{ij}(p) = O(1)+O\Big(\frac{\hbar}{i}\Big)+\Big(\frac{\hbar}{i}\Big)^2\Big(\frac{\lambda}{3}\Big)^2\frac{N+2}{2}\delta_{ij}\bigg[\frac{N+2}{2}\int\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{1}{(-p_1^2+m^2)(-p_2^2+m^2)^2} + \int\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}\frac{1}{(-p_1^2+m^2)(-p_2^2+m^2)(-(p-p_1-p_2)^2+m^2)}\bigg]. \quad (8.195)$$

The corresponding Feynman diagrams are shown on figure 4.

The 4−point proper vertex upto 2−loop is given by

$$\Gamma^{(4)}_{i_1...i_4}(x_1,...,x_4) = O(1)+O\Big(\frac{\hbar}{i}\Big)+\Big(\frac{\hbar}{i}\Big)^2\frac{\delta^2\Gamma_2[\phi]}{\delta\phi_{i_1}(x_1)...\delta\phi_{i_4}(x_4)}|_{\phi=0}. \quad (8.196)$$

We compute

$$\begin{aligned}\Gamma_2[\phi]_{,ijkl}|_{\phi=0} &= \frac{1}{2}G_0^{j_1n_1}G_0^{j_0n_0}G_0^{k_1k_0}G_0^{m_1m_0}S_{,j_0k_0m_1m_0}\Big[S_{,ilj_1k_1}S_{,jkn_1n_0}+S_{,ikj_1k_1}S_{,ljn_1n_0}+S_{,ijj_1k_1}S_{,kln_1n_0}\Big]\\ &+ \frac{1}{4}G_0^{j_1j_0}G_0^{k_1k_0}G_0^{m_1m_0}G_0^{n_1n_0}S_{,j_0k_0m_1n_1}\Big[S_{,ilj_1k_1}S_{,jkm_0n_0}+S_{,ikj_1k_1}S_{,jlm_0n_0}+S_{,ijj_1k_1}S_{,klm_0n_0}\Big]\\ &+ \frac{1}{2}G_0^{j_1j_0}G_0^{k_1k_0}G_0^{m_1m_0}G_0^{n_1n_0}\Big[S_{,ilj_1k_1}S_{,jj_0m_1n_1}S_{,kk_0m_0n_0}+S_{,ikj_1k_1}S_{,jj_0m_1n_1}S_{,lk_0m_0n_0}\\ &+ S_{,ijj_1k_1}S_{,kj_0m_1n_1}S_{,lk_0m_0n_0}+S_{,ij_1k_1n_1}S_{,jkj_0m_0}S_{,lm_1k_0n_0}+S_{,ij_1k_1n_1}S_{,klj_0m_0}S_{,jm_1k_0n_0}\\ &+ S_{,ij_1k_1n_1}S_{,jlj_0m_0}S_{,km_1k_0n_0}\Big].\end{aligned} \quad (8.197)$$

Thus

$$\frac{\delta^4\Gamma_2[\phi]}{\delta\phi_{i_1}(x_1)...\delta\phi_{i_4}(x_4)}|_{\phi=0} = -\frac{1}{2}\left(\frac{\lambda}{3}\right)^3\Bigg[(N+2)\Big((N+2)\delta_{i_1i_4}\delta_{i_2i_3}+2\delta_{i_1i_2i_3i_4}\Big)\delta^4(x_1-x_4)\delta^4(x_2-x_3)$$
$$\times\ G_0(x_1,x_2)\int d^4z G_0(z,z)G_0(x_1,z)G_0(x_2,z)+2\text{ permutations}\Bigg]$$
$$-\ \frac{1}{4}\left(\frac{\lambda}{3}\right)^3\Bigg[\Big((N+2)(N+4)\delta_{i_1i_4}\delta_{i_2i_3}+4\delta_{i_1i_2i_3i_4}\Big)\delta^4(x_1-x_4)\delta^4(x_2-x_3)$$
$$\times\int d^4z G_0(x_1,z)^2G_0(x_2,z)^2+2\text{ permutations}\Bigg]$$
$$-\ \frac{1}{2}\left(\frac{\lambda}{3}\right)^3\Bigg[\Big(2(N+2)\delta_{i_1i_4}\delta_{i_2i_3}+(N+6)\delta_{i_1i_2i_3i_4}\Big)\delta^4(x_1-x_4)$$
$$\times\ G_0(x_1,x_2)G_0(x_1,x_3)G_0(x_2,x_3)^2+5\text{ permutations}\Bigg]. \qquad (8.198)$$

The proper 4−point vertex in momentum space $\Gamma^{(4)}_{i_1...i_4}(p_1...p_4)$ is defined through the equation

$$\int d^4x_1...d^4x_4\ \Gamma^{(4)}_{i_1...i_4}(x_1,...,x_4)\ e^{ip_1x_1+...+ip_4x_4} = (2\pi)^4\delta^4(p_1+...+p_4)\Gamma^{(2)}_{i_1...i_4}(p_1,...,p_4). \qquad (8.199)$$

Thus we obtain in momentum space (with $p_{12}=p_1+p_2$ and $p_{14}=p_1+p_4$, etc)

$$\Gamma^{(4)}_{i_1...i_4}(p_1,...,p_4) = O(1)+O\Big(\frac{\hbar}{i}\Big)$$
$$-\ \Big(\frac{\hbar}{i}\Big)^2\frac{N+2}{2}\left(\frac{\lambda}{3}\right)^3\Bigg[\Big((N+2)\delta_{i_1i_4}\delta_{i_2i_3}+2\delta_{i_1i_2i_3i_4}\Big)\int_l\frac{1}{-l^2+m^2}$$
$$\times\int_k\frac{1}{(-k^2+m^2)^2(-(p_{14}-k)^2+m^2)}+2\text{ permutations}\Bigg]$$
$$-\ \Big(\frac{\hbar}{i}\Big)^2\frac{1}{4}\left(\frac{\lambda}{3}\right)^3\Bigg[\Big((N+2)(N+4)\delta_{i_1i_4}\delta_{i_2i_3}+4\delta_{i_1i_2i_3i_4}\Big)\int_l\frac{1}{(-l^2+m^2)(-(p_{14}-l)^2+m^2)}$$
$$\times\int_k\frac{1}{(-k^2+m^2)(-(p_{14}-k)^2+m^2)}+2\text{ permutations}\Bigg]$$
$$-\ \Big(\frac{\hbar}{i}\Big)^2\frac{1}{2}\left(\frac{\lambda}{3}\right)^3\Bigg[\Big(2(N+2)\delta_{i_1i_4}\delta_{i_2i_3}+(N+6)\delta_{i_1i_2i_3i_4}\Big)\int_l\frac{1}{(-l^2+m^2)(-(p_{14}-l)^2+m^2)}$$
$$\times\int_k\frac{1}{(-k^2+m^2)(-(l-k+p_2)^2+m^2)}+5\text{ permutations}\Bigg]. \qquad (8.200)$$

The corresponding Feynman diagrams are shown on figure 5.

8.5.3 The 2−Loop Renormalization of the 2−Point Proper Vertex

The Euclidean expression of the proper 2−point vertex at 2−loop is given by

$$\Gamma^{(2)}_{ij}(p) = \delta_{ij}(p^2+m^2)+\hbar\frac{\lambda}{6}(N+2)\delta_{ij}I(m^2)-\hbar^2\left(\frac{\lambda}{3}\right)^2\frac{N+2}{2}\delta_{ij}\left[\frac{N+2}{2}I(m^2)J(0,m^2)+K(p^2,m^2)\right]. \qquad (8.201)$$

$$K(p^2,m^2)=\int \frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\Delta(k)\Delta(l)\Delta(k+l-p). \tag{8.202}$$

We compute

$$\begin{aligned} K(p^2,m^2) &= \int d\alpha_1 d\alpha_2 d\alpha_3\ e^{-m^2(\alpha_1+\alpha_2+\alpha_3)-\frac{\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3}p^2}\int\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\ e^{-(\alpha_1+\alpha_3)k^2}\ e^{-\frac{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3}{\alpha_1+\alpha_3}l^2}\\ &= \frac{1}{(4\pi)^4}\int d\alpha_1 d\alpha_2 d\alpha_3\ \frac{e^{-m^2(\alpha_1+\alpha_2+\alpha_3)-\frac{\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3}p^2}}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^2}. \end{aligned} \tag{8.203}$$

We have used the result

$$\int\frac{d^4k}{(2\pi)^4}\ e^{-ak^2}=\frac{1}{16\pi^2a^2}. \tag{8.204}$$

We introduce the cut-off Λ as follows

$$\begin{aligned} K(p^2,m^2) &= \frac{1}{(4\pi)^4}\int_{\frac{1}{\Lambda^2}} d\alpha_1 d\alpha_2 d\alpha_3\ \frac{e^{-m^2(\alpha_1+\alpha_2+\alpha_3)-\frac{\alpha_1\alpha_2\alpha_3}{\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3}p^2}}{(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3)^2}\\ &= \frac{m^2}{(4\pi)^4}\int_{\frac{m^2}{\Lambda^2}} dx_1dx_2dx_3\ \frac{e^{-x_1-x_2-x_3-\frac{x_1x_2x_3}{x_1x_2+x_1x_3+x_2x_3}\frac{p^2}{m^2}}}{(x_1x_2+x_1x_3+x_2x_3)^2}\\ &= \frac{m^2}{(4\pi)^4}(A+B\frac{p^2}{m^2}+C(\frac{p^2}{m^2})^2+...). \end{aligned} \tag{8.205}$$

We have

$$\begin{aligned} A &= \int_{\frac{m^2}{\Lambda^2}} dx_1dx_2dx_3\ \frac{e^{-x_1-x_2-x_3}}{(x_1x_2+x_1x_3+x_2x_3)^2}\\ &= \frac{\Lambda^2}{m^2}\int_1 dxdx_2dx_3\ \frac{e^{-\frac{m^2}{\Lambda^2}(x+x_2+x_3)}}{(xx_2+xx_3+x_2x_3)^2}. \end{aligned} \tag{8.206}$$

The integrand is symmetric in the three variables x, x_2 and x_3. The integral can be rewritten as 6 times the integral over the symmetric region $x_3>x_2>x$. We can also perform the change of variables $x_2=xy$ and $x_3=xyz$, i.e $dx_2dx_3=x^2ydydz$ to obtain

$$\begin{aligned} A &= \frac{\Lambda^2}{m^2}\int_1\frac{dx}{x^2}\frac{dy}{y}dz\ \frac{e^{-\frac{m^2}{\Lambda^2}x(1+y+yz)}}{(1+z+yz)^2}\\ &= 6\int_{\frac{m^2}{\Lambda^2}}^{\infty}\frac{dt}{t^2}\ e^{-t}\psi(t). \end{aligned} \tag{8.207}$$

$$\begin{aligned} \psi(t) &= \int_1^\infty\frac{dy}{y}\int_1^\infty dz\ \frac{e^{-ty(1+z)}}{(1+z+yz)^2}\\ &= \int_1^\infty\frac{dy}{y(y+1)}\ e^{-t\frac{y^2}{y+1}}\int_{y+2}^\infty\frac{dz}{z^2}\ e^{-z\frac{yt}{y+1}}\\ &= \int_1^\infty\frac{dy}{y(y+1)}\ e^{-t\frac{y^2}{y+1}}\bigg(\frac{e^{-t\frac{y(y+2)}{y+1}}}{y+2}+\frac{yt}{y+1}\mathrm{Ei}(-\frac{y(y+2)t}{y+1}\bigg). \end{aligned} \tag{8.208}$$

The most important contribution in the limit $\Lambda \longrightarrow \infty$ comes from the region $t \sim 0$. Thus near $t = 0$ we have

$$\begin{aligned}\psi(t) &= \int_1^\infty \frac{dy}{y(y+1)}\left(\frac{e^{-2ty}}{y+2} + \frac{yt}{y+1}\ e^{-t\frac{y^2}{y+1}}\left(\mathbf{C} + \ln t + \ln\frac{y(y+2)}{y+1} + O(t)\right)\right)\\ &= \psi_0 + \psi_1 t\ln t + \psi_2 t + ... \end{aligned} \tag{8.209}$$

$$\psi_0 = \int_1^\infty \frac{dy}{y(y+1)(y+2)} = \frac{1}{2}\ln\frac{4}{3}. \tag{8.210}$$

$$\psi_1 = \int_1^\infty \frac{dy}{(y+1)^2} = \frac{1}{2}. \tag{8.211}$$

$$\begin{aligned}\psi_2 &= -2\int_1^\infty \frac{dy}{(y+1)(y+2)} + \mathbf{C}\int_1^\infty \frac{dy}{(y+1)^2} + \int_1^\infty \frac{dy}{(y+1)^2}\ln\frac{y(y+2)}{y+1}\\ &= 2(\ln 2 - \ln 3) + \frac{\mathbf{C}}{2} + \int_2^\infty \frac{dy}{y^2}\ln\frac{y^2-1}{y}\\ &= 2(\ln 2 - \ln 3) + \frac{\mathbf{C}}{2} - \frac{1}{2} + \frac{3}{2}\ln 3 - \frac{1}{2}\ln 2\\ &= \frac{1}{2}(\mathbf{C} - 1 - \ln 3 + 3\ln 2). \end{aligned} \tag{8.212}$$

We have then

$$\begin{aligned}A &= 6\psi_0\int_{\frac{m^2}{\Lambda^2}} \frac{dt}{t^2}\ e^{-t} + 6\psi_1\int_{\frac{m^2}{\Lambda^2}} \frac{dt}{t}\ e^{-t}\ln t + 6\psi_2\int_{\frac{m^2}{\Lambda^2}} \frac{dt}{t}\ e^{-t} + ...\\ &= 6\psi_0\frac{\Lambda^2}{m^2} + 6\psi_1\int_{\frac{m^2}{\Lambda^2}} \frac{dt}{t}\ e^{-t}\ln t + 6(\psi_2 - \psi_0)\int_{\frac{m^2}{\Lambda^2}} \frac{dt}{t}\ e^{-t} + ...\\ &= 6\psi_0\frac{\Lambda^2}{m^2} + 6\psi_1\int_{\frac{m^2}{\Lambda^2}} \frac{dt}{t}\ e^{-t}\ln t + 6(\psi_2 - \psi_0)\frac{m^2}{\Lambda^2}\int_1 dt\ \ln t\ e^{-\frac{m^2}{\Lambda^2}t} + ...\\ &= 6\psi_0\frac{\Lambda^2}{m^2} - 3\psi_1\left(\ln\frac{\Lambda^2}{m^2}\right)^2 + 3\psi_1\int_{\frac{m^2}{\Lambda^2}} dt\ e^{-t}(\ln t)^2 - 6(\psi_2 - \psi_0)\mathrm{Ei}(-\frac{m^2}{\Lambda^2}) + ...\\ &= 6\psi_0\frac{\Lambda^2}{m^2} - 3\psi_1\left(\ln\frac{\Lambda^2}{m^2}\right)^2 + 6(\psi_2 - \psi_0)\ln(\frac{\Lambda^2}{m^2}) + ... \end{aligned} \tag{8.213}$$

Now we compute

$$\begin{aligned}
B &= -\int_{\frac{m^2}{\Lambda^2}} x_1x_2x_3dx_1dx_2dx_3\ \frac{e^{-x_1-x_2-x_3}}{(x_1x_2+x_1x_3+x_2x_3)^3}\\
&= -\int_1 xx_2x_3dxdx_2dx_3\ \frac{e^{-\frac{m^2}{\Lambda^2}(x+x_2+x_3)}}{(xx_2+xx_3+x_2x_3)^3}\\
&= -6\int_1^\infty xdx\int_x^\infty x_2dx_2\int_{x_2}^\infty x_3dx_3\ \frac{e^{-\frac{m^2}{\Lambda^2}(x+x_2+x_3)}}{(xx_2+xx_3+x_2x_3)^3}\\
&= -6\int_1^\infty \frac{dx}{x}\int_1^\infty dy\int_1^\infty zdz\ \frac{e^{-\frac{m^2}{\Lambda^2}x(1+y+yz)}}{(1+z+yz)^3}\\
&= -6\int_{\frac{m^2}{\Lambda^2}}^\infty \frac{dt}{t}e^{-t}\tilde{\psi}(t).
\end{aligned}\tag{8.214}$$

$$\tilde{\psi}(t) = \int_1^\infty dy\int_1^\infty zdz\ \frac{e^{-ty(1+z)}}{(1+z+yz)^3}.\tag{8.215}$$

It is not difficult to convince ourselves that only the constant part of $\tilde{\psi}$ leads to a divergence, i.e $\tilde{\psi}(0)=\frac{1}{12}$. We get

$$B = -6\int_{\frac{m^2}{\Lambda^2}}^\infty \frac{dt}{t}e^{-t}\tilde{\psi}(0) = -6\tilde{\psi}(0)\ln\frac{\Lambda^2}{m^2}.\tag{8.216}$$

Now we compute

$$\begin{aligned}
C &= \frac{1}{2}\int_{\frac{m^2}{\Lambda^2}} (x_1x_2x_3)^2dx_1dx_2dx_3\ \frac{e^{-x_1-x_2-x_3}}{(x_1x_2+x_1x_3+x_2x_3)^4}\\
&= \frac{m^2}{\Lambda^2}\int_1 (xx_2x_3)^2dxdx_2dx_3\ \frac{e^{-\frac{m^2}{\Lambda^2}(x+x_2+x_3)}}{(xx_2+xx_3+x_2x_3)^4}\\
&= 6\frac{m^2}{\Lambda^2}\int_1^\infty x^2dx\int_x^\infty x_2^2dx_2\int_{x_2}^\infty x_3^2dx_3\ \frac{e^{-\frac{m^2}{\Lambda^2}(x+x_2+x_3)}}{(xx_2+xx_3+x_2x_3)^4}\\
&= 6\frac{m^2}{\Lambda^2}\int_1^\infty dx\int_1^\infty ydy\int_1^\infty z^2dz\ \frac{e^{-\frac{m^2}{\Lambda^2}x(1+y+yz)}}{(1+z+yz)^4}\\
&= 6\int_{\frac{m^2}{\Lambda^2}}^\infty dt\ e^{-t}\int_1^\infty ydy\int_1^\infty z^2dz\ \frac{e^{-ty(1+z)}}{(1+z+yz)^4}.
\end{aligned}\tag{8.217}$$

This integral is well defined in the limit $\Lambda\longrightarrow\infty$. Furthermore it is positive definite.

In summay we have found that both $K(0,m^2)$ and $K^{'}(0,m^2)$ are divergent in the limit $\Lambda\longrightarrow\infty$, i.e $K(p^2,m^2)-K(0,m^2)$ is divergent at the two-loop order. This means that $\Gamma^{(2)}_{ij}(p)$ and $d\Gamma^{(2)}_{ij}(p)/dp^2$ are divergent at $p^2=0$ and hence in order to renormlaize the $2-$point proper vertex $\Gamma^{(2)}_{ij}(p)$ at the two-loop order we must impose two conditions on it. The first condition is the same as before namely we require that the value of the $2-$point proper vertex at zero momentum is precisely the physical or renormalized mass. The second condition is essentially a renormalization of the coefficient of the kinetic term, i.e $d\Gamma^{(2)}_{ij}(p)/dp^2$. Before we can write these two conditions we introduce a renormalization of the scalar field ϕ known also as wave function renormalization given by

$$\phi=\sqrt{Z}\phi_R. \tag{8.218}$$

This induces a renormalization of the $n-$point proper vertices. Indeed the effective action becomes

$$\begin{aligned}\Gamma[\phi] &= \sum_{n=0}\frac{1}{n!}\Gamma^{(n)}_{i_1...i_n}(x_1,...,x_n)\phi_{i_1}(x_1)...\phi_{i_n}(x_n)\\ &= \sum_{n=0}\frac{1}{n!}\Gamma^{(n)}_{i_1...i_nR}(x_1,...,x_n)\phi_{i_1R}(x_1)...\phi_{i_nR}(x_n).\end{aligned} \tag{8.219}$$

The renormalized $n-$point proper vertex $\Gamma^{(n)}_{i_1...i_nR}$ is given in terms of the bare $n-$point proper vertex $\Gamma^{(n)}_{i_1...i_n}$ by

$$\Gamma^{(n)}_{i_1...i_nR}(x_1,...,x_n)=Z^{\frac{n}{2}}\Gamma^{(n)}_{i_1...i_n}(x_1,...,x_n). \tag{8.220}$$

Thus the renormalized $2-$point proper vertex $\Gamma^{(2)}_{ijR}(p)$ in momentum space is given by

$$\Gamma^{(2)}_{ijR}(p)=Z\Gamma^{(2)}_{ij}(p). \tag{8.221}$$

Now we impose on the renormalized $2-$point proper vertex $\Gamma^{(2)}_{ijR}(p)$ the two conditions given by

$$\Gamma^{(2)}_{ijR}(p)|_{p=0}=Z\Gamma^{(2)}_{ij}(p)|_{p=0}=\delta_{ij}m_R^2. \tag{8.222}$$

$$\frac{d}{dp^2}\Gamma^{(2)}_{ijR}(p)|_{p=0}=Z\frac{d}{dp^2}\Gamma^{(2)}_{ij}(p)|_{p=0}=\delta_{ij}. \tag{8.223}$$

The second condition yields immediately

$$Z=\frac{1}{1-\hbar^2\big(\frac{\lambda}{3}\big)^2\frac{N+2}{2}K'(0,m^2,\Lambda)}=1+\hbar^2\bigg(\frac{\lambda}{3}\bigg)^2\frac{N+2}{2}K^{'}(0,m^2,\Lambda). \tag{8.224}$$

The first condition gives then

$$\begin{aligned}m^2 &= m_R^2-\hbar\frac{\lambda}{6}(N+2)I(m^2,\Lambda)+\hbar^2\bigg(\frac{\lambda}{3}\bigg)^2\frac{N+2}{2}\bigg[\frac{N+2}{2}I(m^2,\Lambda)J(0,m^2,\Lambda)+K(0,m^2,\Lambda)\\ &- m^2K^{'}(0,m^2,\Lambda)\bigg]\\ &= m_R^2-\hbar\frac{\lambda_R}{6}(N+2)I(m^2,\Lambda)+\hbar^2\bigg(\frac{\lambda_R}{3}\bigg)^2\frac{N+2}{2}\bigg[-3I(m_R^2,\Lambda)J(0,m_R^2,\Lambda)+K(0,m_R^2,\Lambda)\\ &- m_R^2K^{'}(0,m_R^2,\Lambda)\bigg]\\ &= m_R^2-\hbar\frac{\lambda_R}{6}(N+2)I(m_R^2,\Lambda)+\hbar^2\bigg(\frac{\lambda_R}{3}\bigg)^2\frac{N+2}{2}\bigg[-\frac{N+8}{2}I(m_R^2,\Lambda)J(0,m_R^2,\Lambda)+K(0,m_R^2,\Lambda)\\ &- m_R^2K^{'}(0,m_R^2,\Lambda)\bigg].\end{aligned} \tag{8.225}$$

In above we have used the relation between the bare coupling constant λ and the renormalized coupling constant λ_R at one-loop given by equation (8.175). We have also used the relation $I(m^2,\Lambda) = I(m_R^2,\Lambda) + \hbar\frac{\lambda_R}{6}(N+2)I(m^2,\Lambda)J(0,m_R^2,\Lambda)$ where we have assumed that $m^2 = m_R^2 - \hbar\frac{\lambda_R}{6}(N+2)I(m^2,\Lambda)$. We get therefore the 2−point proper vertex

$$\Gamma^{(2)}_{ijR}(p) = \delta_{ij}(p^2+m_R^2) - \hbar^2\bigg(\frac{\lambda_R}{3}\bigg)^2\frac{N+2}{2}\delta_{ij}\bigg(K(p^2,m_R^2,\Lambda) - K(0,m_R^2,\Lambda) - p^2K^{'}(0,m_R^2,\Lambda)\bigg). \tag{8.226}$$

8.5.4 The 2−Loop Renormalization of the 4−Point Proper Vertex

The Euclidean expression of the proper 4−point vertex at 2−loop is given by

$$\begin{aligned}
\Gamma^{(4)}_{i_1...i_4}(p_1,...,p_4) &= \frac{\lambda}{3}\delta_{i_1i_2i_3i_4} - \hbar\bigg(\frac{\lambda}{3}\bigg)^2\frac{1}{2}\bigg[\bigg((N+2)\delta_{i_1i_2}\delta_{i_3i_4} + 2\delta_{i_1i_2i_3i_4}\bigg)J(p_{12}^2,m^2)\\
&+ 2\text{ permutations}\bigg]\\
&+ \hbar^2\bigg(\frac{\lambda}{3}\bigg)^3\frac{N+2}{2}\bigg[\bigg((N+2)\delta_{i_1i_4}\delta_{i_2i_3} + 2\delta_{i_1i_2i_3i_4}\bigg)I(m^2)L(p_{14}^2,m^2)\\
&+ 2\text{ permutations}\bigg]\\
&+ \hbar^2\bigg(\frac{\lambda}{3}\bigg)^3\frac{1}{4}\bigg[\bigg((N+2)(N+4)\delta_{i_1i_4}\delta_{i_2i_3} + 4\delta_{i_1i_2i_3i_4}\bigg)J(p_{14}^2,m^2)^2\\
&+ 2\text{ permutations}\bigg]\\
&+ \hbar^2\bigg(\frac{\lambda}{3}\bigg)^3\frac{1}{2}\bigg[\bigg(2(N+2)\delta_{i_1i_4}\delta_{i_2i_3} + (N+6)\delta_{i_1i_2i_3i_4}\bigg)M(p_{14}^2,p_2^2,m^2)\\
&+ 5\text{ permutations}\bigg].
\end{aligned} \tag{8.227}$$

$$L(p_{14}^2,m^2) = \int\frac{d^4k}{(2\pi)^4}\Delta(k)^2\Delta(k-p_{14}). \tag{8.228}$$

$$M(p_{14}^2,p_2^2,m^2) = \int\frac{d^4l}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\Delta(l)\Delta(k)\Delta(l-p_{14})\Delta(l-k+p_2). \tag{8.229}$$

For simplicity we will not write explicitly the dependence on the cut-off Λ in the following. The renormalized 4−point proper vertex $\Gamma^{(4)}_{i_1i_2i_3i_4R}(p_1,p_2,p_3,p_4)$ in momentum space is given by

$$\Gamma^{(4)}_{i_1i_2i_3i_4R}(p_1,p_2,p_3,p_4) = Z^2\Gamma^{(4)}_{i_1i_2i_3i_4}(p_1,p_2,p_3,p_4). \tag{8.230}$$

We will impose the renormalization condition

$$\Gamma^{(4)}_{i_1...i_4R}(0,...,0) = \frac{\lambda_R}{3}\delta_{i_1i_2i_3i_4}. \tag{8.231}$$

We introduce a new renormalization constant Z_g defined by

$$Z_g\Gamma^{(4)}_{i_1...i_4}(0,...,0) = \frac{\lambda}{3}\delta_{i_1i_2i_3i_4}. \quad (8.232)$$

Equivalently this means

$$\frac{Z_g}{Z^2}\lambda_R = \lambda. \quad (8.233)$$

The constant Z is already known at two-loop. The constant Z_g at two-loop is computed to be

$$\begin{aligned} Z_g &= 1+\hbar\frac{\lambda}{6}(N+8)J(0,m^2)-\hbar^2\bigg(\frac{\lambda}{3}\bigg)^2\bigg[\frac{(N+2)(N+8)}{2}I(m^2)L(0,m^2) \\ &+ \frac{(N+2)(N+4)+12}{4}J(0,m^2)^2+(5N+22)M(0,0,m^2)\bigg]. \end{aligned} \quad (8.234)$$

We compute

$$\begin{aligned} \Gamma^{(4)}_{i_1i_2i_3i_4R}(p_1,p_2,p_3,p_4) &= Z^2\Gamma^{(4)}_{i_1i_2i_3i_4}(p_1,p_2,p_3,p_4) \\ &= Z_g\frac{\lambda_R}{3}\delta_{i_1i_2i_3i_4}+\Gamma^{(4)}_{i_1i_2i_3i_4}(p_1,p_2,p_3,p_4)|_{1-\text{loop}}+\Gamma^{(4)}_{i_1i_2i_3i_4}(p_1,p_2,p_3,p_4)|_{2-\text{loop}}. \end{aligned} \quad (8.235)$$

By using the relation $J(p_{12}^2,m^2)=J(p_{12}^2,m_R^2)+\hbar\frac{\lambda_R}{3}(N+2)I(m_R^2)L(p_{12}^2,m_R^2)$ we compute

$$\begin{aligned} Z_g\frac{\lambda_R}{3}\delta_{i_1i_2i_3i_4} &= \frac{\lambda_R}{3}\delta_{i_1i_2i_3i_4}+\hbar\bigg(\frac{\lambda_R}{3}\bigg)^2\frac{N+8}{2}\delta_{i_1i_2i_3i_4}J(0,m_R^2)-\hbar^2\bigg(\frac{\lambda_R}{3}\bigg)^3\delta_{i_1i_2i_3i_4}\bigg[\\ &\times \bigg(\frac{(N+2)(N+4)+12}{4}-\frac{(N+8)^2}{2}\bigg)J(0,m_R^2)^2+(5N+22)M(0,0,m_R^2)\bigg]. \end{aligned} \quad (8.236)$$

$$\begin{aligned} \Gamma^{(4)}_{i_1i_2i_3i_4}(p_1,p_2,p_3,p_4)|_{1-\text{loop}} &= -\hbar\bigg(\frac{\lambda_R}{3}\bigg)^2\frac{1}{2}\bigg[\bigg((N+2)\delta_{i_1i_2}\delta_{i_3i_4}+2\delta_{i_1i_2i_3i_4}\bigg)J(p_{12}^2,m_R^2) \\ &+ 2 \text{ permutations}\bigg] \\ &- \hbar^2\bigg(\frac{\lambda_R}{3}\bigg)^3\frac{N+8}{2}J(0,m_R^2)\bigg[\bigg((N+2)\delta_{i_1i_2}\delta_{i_3i_4}+2\delta_{i_1i_2i_3i_4}\bigg)J(p_{12}^2,m_R^2) \\ &+ 2 \text{ permutations}\bigg] \\ &- \hbar^2\bigg(\frac{\lambda_R}{3}\bigg)^3\frac{N+2}{2}I(m_R^2)\bigg[\bigg((N+2)\delta_{i_1i_2}\delta_{i_3i_4}+2\delta_{i_1i_2i_3i_4}\bigg)L(p_{12}^2,m_R^2) \\ &+ 2 \text{ permutations}\bigg]. \end{aligned} \quad (8.237)$$

We then find

$$\begin{aligned}\Gamma^{(4)}_{i_1...i_4R}(p_1,...,p_4) &= \frac{\lambda_R}{3}\delta_{i_1i_2i_3i_4} - \hbar\bigg(\frac{\lambda_R}{3}\bigg)^2\frac{1}{2}\bigg[\bigg((N+2)\delta_{i_1i_2}\delta_{i_3i_4}+2\delta_{i_1i_2i_3i_4}\bigg)(J(p_{12}^2,m_R^2)-J(0,m_R^2))\\
&+ 2 \text{ permutations}\bigg]\\
&+ \hbar^2\bigg(\frac{\lambda_R}{3}\bigg)^3\frac{1}{4}\bigg[\bigg((N+2)(N+4)\delta_{i_1i_4}\delta_{i_2i_3}+4\delta_{i_1i_2i_3i_4}\bigg)(J(p_{14}^2,m_R^2)-J(0,m_R^2))^2\\
&+ 2 \text{ permutations}\bigg]\\
&- \hbar^2\bigg(\frac{\lambda_R}{3}\bigg)^3\bigg[\bigg(2(N+2)\delta_{i_1i_4}\delta_{i_2i_3}+(N+6)\delta_{i_1i_2i_3i_4}\bigg)J(0,m_R^2)(J(p_{14}^2,m_R^2)-J(0,m_R^2))\\
&+ 2 \text{ permutations}\bigg]\\
&+ \hbar^2\bigg(\frac{\lambda_R}{3}\bigg)^3\frac{1}{2}\bigg[\bigg(2(N+2)\delta_{i_1i_4}\delta_{i_2i_3}+(N+6)\delta_{i_1i_2i_3i_4}\bigg)(M(p_{14}^2,p_2^2,m_R^2)-M(0,0,m_R^2))\\
&+ 5 \text{ permutations}\bigg].\end{aligned}\tag{8.238}$$

In the above last equation the combination $M(p_{14}^2,p_2^2,m_R^2)-M(0,0,m_R^2)-J(0,m_R^2)(J(p_{14}^2,m_R^2)-J(0,m_R^2))$ must be finite in the limit $\Lambda\longrightarrow\infty$ [12].

8.6 Renormalized Perturbation Theory

The $(\phi^2)^2$ theory with $O(N)$ symmetry studied in this chapter is given by the action

$$S=\int d^4x\bigg[\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i-\frac{1}{2}m^2\phi_i^2-\frac{\lambda}{4!}(\phi_i^2)^2\bigg].\tag{8.239}$$

This is called a bare action, the fields ϕ_i are the bare fields and the parameters m^2 and λ are the bare coupling constants of the theory.

Let us recall that the free 2−point function $<0|T(\hat{\phi}_{i,\rm in}(x)\hat{\phi}_{j,\rm in}(y))|0>$ is the probability amplitude for a free scalar particle to propagate from a spacetime point y to a spacetime x. In the interacting theory the 2−point function is $<\Omega|T(\hat{\phi}_i(x)\hat{\phi}_j(y))|\Omega>$ where $|\Omega>=|0>/\sqrt{<0|0>}$ is the ground state of the full Hamiltonian $\hat{H}$. On general grounds we can verify that the 2−point function $<\Omega|T(\hat{\phi}_i(x)\hat{\phi}_j(y))|\Omega>$ is given by

$$\int d^4x e^{ip(x-y)}\ <\Omega|T(\hat{\phi}_i(x)\hat{\phi}_j(y))|\Omega>=\frac{iZ\delta_{ij}}{p^2-m_R^2+i\epsilon}+....\tag{8.240}$$

The dots stands for regular terms at $p^2=m_R^2$ where m_R is the physical or renormalized mass. The residue or renormalization constant Z is called the wave function renormalization. Indeed the renormalized 2−point function $<\Omega|T(\hat{\phi}_R(x)\hat{\phi}_R(y))|\Omega>$ is given by

$$\int d^4x e^{ip(x-y)}\ <\Omega|T(\hat{\phi}_{iR}(x)\hat{\phi}_{jR}(y))|\Omega>=\frac{i\delta_{ij}}{p^2-m_R^2+i\epsilon}+....\tag{8.241}$$

[12] Exercise:show this result.

The physical or renormalized field ϕ_R is given by

$$\phi = \sqrt{Z}\phi_R. \tag{8.242}$$

As we have already discussed this induces a renormalization of the $n-$point proper vertices. Indeed the effective action becomes

$$\begin{aligned}\Gamma[\phi] &= \sum_{n=0} \frac{1}{n!}\int d^4x_1 ... \int d^4x_n \Gamma^{(n)}_{i_1...i_n}(x_1,...,x_n)\phi_{i_1}(x_1)...\phi_{i_n}(x_n) \\ &= \sum_{n=0} \frac{1}{n!}\int d^4x_1 ... \int d^4x_n \Gamma^{(n)}_{i_1...i_n R}(x_1,...,x_n)\phi_{i_1 R}(x_1)...\phi_{i_n R}(x_n).\end{aligned} \tag{8.243}$$

The renormalized $n-$point proper vertex $\Gamma^{(n)}_{i_1...i_n R}$ is given in terms of the bare $n-$point proper vertex $\Gamma^{(n)}_{i_1...i_n}$ by

$$\Gamma^{(n)}_{i_1...i_n R}(x_1,...,x_n) = Z^{\frac{n}{2}}\Gamma^{(n)}_{i_1...i_n}(x_1,...,x_n). \tag{8.244}$$

We introduce a renormalized coupling constant λ_R and a renormalization constant Z_g by

$$Z_g\lambda_R = Z^2\lambda. \tag{8.245}$$

The action takes the form

$$\begin{aligned}S &= \int d^4x \bigg[\frac{Z}{2}\partial_\mu\phi_{iR}\partial^\mu\phi_{iR} - \frac{Z}{2}m^2\phi_{iR}^2 - \frac{\lambda Z^2}{4!}(\phi_{iR}^2)^2\bigg] \\ &= S_R + \delta S.\end{aligned} \tag{8.246}$$

The renormalized action S_R is given by

$$S_R = \int d^4x \bigg[\frac{1}{2}\partial_\mu\phi_{iR}\partial^\mu\phi_{iR} - \frac{1}{2}m_R^2\phi_{iR}^2 - \frac{\lambda_R}{4!}(\phi_{iR}^2)^2\bigg]. \tag{8.247}$$

The action δS is given by

$$\delta S = \int d^4x \bigg[\frac{\delta_Z}{2}\partial_\mu\phi_{iR}\partial^\mu\phi_{iR} - \frac{1}{2}\delta_m\phi_{iR}^2 - \frac{\delta_\lambda}{4!}(\phi_{iR}^2)^2\bigg]. \tag{8.248}$$

The counterterms δ_Z, δ_m and δ_λ are given by

$$\delta_Z = Z - 1\ ,\ \delta_m = Zm^2 - m_R^2\ ,\ \delta_\lambda = \lambda Z^2 - \lambda_R = (Z_g - 1)\lambda_R. \tag{8.249}$$

The new Feynman rules derived from S_R and δS are shown on figure 7.

The so-called renormalized perturbation theory consists in the following. The renormalized or physical parameters of the theory m_R and λ_R are always assumed to be finite whereas the counterterms δ_Z, δ_m and δ_λ will contain the unobservable infinite shifts between the bare parameters m and λ and the physical parameters m_R and λ_R. The renormalized parameters are determined from imposing renormalization conditions on appropriate proper vertices. In this case we will impose on the 2−point proper vertex $\Gamma^{(2)}_{ijR}(p)$ and the 4−point proper vertex $\Gamma^{(2)}_{ijR}(p)$ the three conditions given by

$$\Gamma^{(2)}_{ijR}(p)|_{p=0} = \delta_{ij}m_R^2. \tag{8.250}$$

$$\frac{d}{dp^2}\Gamma^{(2)}_{ijR}(p)|_{p=0} = \delta_{ij}. \tag{8.251}$$

$$\Gamma^{(4)}_{i_1...i_4R}(0,...,0) \;=\; -\frac{\lambda_R}{3}\delta_{i_1i_2i_3i_4}. \tag{8.252}$$

As an example let us consider the 2−point and 4−point functions upto the 1−loop order. We have immediately the results

$$\Gamma^{(2)}_{Rij}(p) = \bigg[(p^2-m_R^2) - \frac{\hbar}{i}\frac{\lambda_R}{6}(N+2)I(m_R^2) + (\delta_Z p^2 - \delta_m)\bigg]\delta_{ij}. \tag{8.253}$$

$$\begin{aligned}\Gamma^{(4)}_{Ri_1...i_4}(p_1,...,p_4) \;&=\; -\frac{\lambda_R}{3}\delta_{i_1i_2i_3i_4} + \frac{\hbar}{i}\bigg(\frac{\lambda_R}{3}\bigg)^2\frac{1}{2}\bigg[\bigg((N+2)\delta_{i_1i_2}\delta_{i_3i_4} + 2\delta_{i_1i_2i_3i_4}\bigg)J(p_{12}^2,m_R^2)\\ &+\; 2\text{ permutations}\bigg] - \frac{\delta_\lambda}{3}\delta_{i_1i_2i_3i_4}.\end{aligned} \tag{8.254}$$

The first two terms in both $\Gamma^{(2)}_R$ and $\Gamma^{(4)}_R$ come from the renormalized action S_R and they are identical with the results obtained with the bare action S with the substitutions $m \longrightarrow m_R$ and $\lambda \longrightarrow \lambda_R$. The last terms in $\Gamma^{(2)}_R$ and $\Gamma^{(4)}_R$ come from the action δS. By imposing renormalization conditions we get (including a cut-off Λ)

$$\delta_Z = 0\ ,\ \delta_m = -\frac{\hbar}{i}\frac{\lambda_R}{6}(N+2)I_\Lambda(m_R^2)\ ,\ \delta_\lambda = \frac{\hbar}{i}\frac{\lambda_R^2}{6}(N+8)J_\Lambda(0,m_R^2). \tag{8.255}$$

In other words

$$\Gamma^{(2)}_{Rij}(p) = (p^2-m_R^2)\delta_{ij}. \tag{8.256}$$

$$\begin{aligned}\Gamma^{(4)}_{Ri_1...i_4}(p_1,...,p_4) \;&=\; -\frac{\lambda_R}{3}\delta_{i_1i_2i_3i_4} + \frac{\hbar}{i}\bigg(\frac{\lambda_R}{3}\bigg)^2\frac{1}{2}\bigg[\bigg((N+2)\delta_{i_1i_2}\delta_{i_3i_4} + 2\delta_{i_1i_2i_3i_4}\bigg)(J(p_{12}^2,m_R^2) - J(0,m_R^2))\\ &+\; 2\text{ permutations}\bigg].\end{aligned} \tag{8.257}$$

It is clear that the end result of renormalized perturbation theory upto 1−loop is the same as the somewhat "direct" renormalization employed in the previous sections to renormalize the perturbative expansion of $\Gamma^{(2)}_R$ and $\Gamma^{(4)}_R$ upto 1−loop. This result extends also to the 2−loop order[13].

Let us note at the end of this section that renormalization of higher n−point vertices should proceed along the same lines discussed above for the 2−point and 4−point vertices. The detail of this exercise will be omitted at this stage.

8.7 Effective Potential and Dimensional Regularization

Let us go back to our original $O(N)$ action which is given by

$$S[\phi] = \int d^4x\bigg[\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i + \frac{\mu^2}{2}\phi_i^2 - \frac{g}{4}(\phi_i^2)^2 + J_i\phi_i\bigg]. \tag{8.258}$$

[13]Exercise:Try this explicitly.

Now we expand the field as $\phi_i = \phi_{ci} + \eta_i$ where ϕ_{ci} is the classical field. We can always choose ϕ_c to point in the N direction, viz $\phi_c = (0, ..., 0, \phi_c)$. By translational invariance we may assume that ϕ_{ci} is a constant. The action becomes (where V is the spacetime volume)

$$\begin{aligned} S[\phi_c, \eta] &= V\bigg[\frac{\mu^2}{2}\phi_{ci}^2 - \frac{g}{4}(\phi_{ci}^2)^2 + J_i\phi_{ci}\bigg] + \int d^4x \bigg[\frac{1}{2}\partial_\mu \eta_i \partial^\mu \eta_i + \frac{\mu^2}{2}\eta_i^2 + (\mu^2 - g\phi_{cj}^2)\phi_{ci}\eta_i + J_i\eta_i \\ &- \frac{g}{2}\left[\phi_{ci}^2\eta_j^2 + 2(\phi_{ci}\eta_i)^2\right] - g(\phi_{ci}\eta_i)\eta_j^2 - \frac{g}{4}(\eta_i^2)^2\bigg]. \end{aligned} \tag{8.259}$$

In the spirit of renormalized perturbation theory we will think of the parameters μ^2 and g as renormalized parameters and add the counterterms

$$\delta S[\phi] = \int d^4x \bigg[\frac{1}{2}\delta_Z \partial_\mu \phi_i \partial^\mu \phi_i + \frac{1}{2}\delta_\mu \phi_i^2 - \frac{1}{4}\delta_g (\phi_i^2)^2 + \delta J_i \phi_i\bigg]. \tag{8.260}$$

The counterterm δJ_i is chosen so that the $1-$point vertex $\Gamma^{(1)}_{i_1}(x_1)$ is identically zero to all orders in perturbation theory. This is equivalent to the removal of all tadpole diagrams that contribute to $< \eta_i >$.

Let us recall the form of the effective action upto $1-$loop and the classical $2-$point function. These are given by

$$\Gamma = S + \frac{1}{2}\frac{\hbar}{i}\ln\det G_0 + ... \tag{8.261}$$

$$G_0^{ij} = -S^{-1}_{,ij}|_{\phi=\phi_c}. \tag{8.262}$$

The effective action can always be rewritten as the spacetime integral of an effective Lagrangian $\mathcal{L}_{\rm eff}$. For slowly varying fields the most important piece in this effective Lagrangian is the so-called effective potential which is the term with no dependence on the derivatives of the field. The effective Lagrangian takes the generic form

$$\mathcal{L}_{\rm eff}(\phi_c, \partial\phi_c, \partial\partial\phi_c, ...) = -V(\phi_c) + Z(\phi_c)\partial_\mu\phi_c\partial^\mu\phi_c + ... \tag{8.263}$$

For constant classical field we have

$$\Gamma(\phi_c) = -\int d^4x V(\phi_c) = -\bigg(\int d^4x\bigg)V(\phi_c). \tag{8.264}$$

We compute immediately

$$\begin{aligned} \frac{\delta^2 S}{\delta\eta_i(x)\delta\eta_j(y)}|_{\eta=0} &= \bigg[-\partial^2\delta_{ij} + \mu^2\delta_{ij} - g[\phi_{ck}^2\delta_{ij} + 2\phi_{ci}\phi_{cj}]\bigg]\delta^4(x-y) \\ &= \bigg[-\partial^2 - m_i^2\bigg]\delta_{ij}\delta^4(x-y). \end{aligned} \tag{8.265}$$

The masses m_i are given by

$$m_i^2 = g\phi_c^2 - \mu^2 \ , \ i, j \neq N \text{ and } m_i^2 = 3g\phi_c^2 - \mu^2 \ , \ i = j = N. \tag{8.266}$$

The above result can be put in the form

$$\frac{\delta^2 S}{\delta\eta_i(x)\delta\eta_j(y)}|_{\eta=0} = \int \frac{d^dp}{(2\pi)^d}\bigg[p^2 - m_i^2\bigg]\delta_{ij}e^{ip(x-y)}. \tag{8.267}$$

We compute

$$\begin{aligned}
\frac{1}{2}\frac{\hbar}{i}\ln\det G_0 &= -\frac{1}{2}\frac{\hbar}{i}\ln\det G_0^{-1}\\
&= \frac{i\hbar}{2}\ln\det\bigg(-\frac{\delta^2 S}{\delta\eta_i(x)\delta\eta_j(y)}|_{\eta=0}\bigg)\\
&= \frac{i\hbar}{2}Tr\ln\bigg(-\frac{\delta^2 S}{\delta\eta_i(x)\delta\eta_j(y)}|_{\eta=0}\bigg)\\
&= \frac{i\hbar}{2}\int d^4x <x|\ln\bigg(-\frac{\delta^2 S}{\delta\eta_i(x)\delta\eta_j(y)}|_{\eta=0}\bigg)|x>\\
&= \frac{i\hbar}{2}V\int\frac{d^4p}{(2\pi)^4}\ln\bigg((-p^2+m_i^2)\delta_{ij}\bigg)\\
&= \frac{i\hbar}{2}V\bigg[(N-1)\int\frac{d^4p}{(2\pi)^4}\ln\bigg(-p^2-\mu^2+g\phi_c^2\bigg)+\int\frac{d^4p}{(2\pi)^4}\ln\bigg(-p^2-\mu^2+3g\phi_c^2\bigg)\bigg].
\end{aligned}\tag{8.268}$$

The basic integral we need to compute is

$$I(m^2)=\int\frac{d^4p}{(2\pi)^4}\ln\bigg(-p^2+m^2\bigg).\tag{8.269}$$

This is clearly divergent. We will use here the powerful method of dimensional regularization to calculate this integral. This consists in 1) performing a Wick rotation $k^0\longrightarrow k^4=-ik^0$ and 2) continuing the number of dimensions from 4 to $d\neq 4$. We have then

$$I(m^2)=i\int\frac{d^dp_E}{(2\pi)^d}\ln\bigg(p_E^2+m^2\bigg).\tag{8.270}$$

We use the identity

$$\frac{\partial}{\partial\alpha}x^{-\alpha}|_{\alpha=0}=-\ln x.\tag{8.271}$$

We get then

$$\begin{aligned}
I(m^2) &= -i\frac{\partial}{\partial\alpha}\bigg(\int\frac{d^dp_E}{(2\pi)^d}\frac{1}{(p_E^2+m^2)^\alpha}\bigg)|_{\alpha=0}\\
&= -i\frac{\partial}{\partial\alpha}\bigg(\frac{\Omega_{d-1}}{(2\pi)^d}\int dp_E\frac{p_E^{d-1}}{(p_E^2+m^2)^\alpha}\bigg)|_{\alpha=0}.
\end{aligned}\tag{8.272}$$

The Ω_{d-1} is the solid angle in d dimensions, i.e. the area of a sphere S^{d-1}. It is given by [14]

$$\Omega_{d-1}=\frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}.\tag{8.273}$$

We make the change of variables $x=p_E^2$ then the change of variables $t=m^2/(x+m^2)$. We get

$$\begin{aligned}
I(m^2) &= -i\frac{\partial}{\partial\alpha}\bigg(\frac{\Omega_{d-1}}{2(2\pi)^d}\int_0^\infty dx\frac{x^{\frac{d}{2}-1}}{(x+m^2)^\alpha}\bigg)|_{\alpha=0}\\
&= -i\frac{\partial}{\partial\alpha}\bigg(\frac{\Omega_{d-1}(m^2)^{\frac{d}{2}-\alpha}}{2(2\pi)^d}\int_0^1 dt\, t^{\alpha-1-\frac{d}{2}}(1-t)^{\frac{d}{2}-1}\bigg)|_{\alpha=0}.
\end{aligned}\tag{8.274}$$

[14] Derive this result.

We use the result

$$\int_0^1 dt\; t^{\alpha-1}(1-t)^{\beta-1} = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{8.275}$$

We get then

$$\begin{aligned} I(m^2) &= -i\frac{\partial}{\partial\alpha}\bigg(\frac{\Omega_{d-1}(m^2)^{\frac{d}{2}-\alpha}}{2(2\pi)^d}\frac{\Gamma(\alpha-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(\alpha)}\bigg)|_{\alpha=0} \\ &= -i\frac{\partial}{\partial\alpha}\bigg(\frac{1}{(4\pi)^{\frac{d}{2}}}(m^2)^{\frac{d}{2}-\alpha}\frac{\Gamma(\alpha-\frac{d}{2})}{\Gamma(\alpha)}\bigg)|_{\alpha=0}. \end{aligned} \tag{8.276}$$

Now we use the result that

$$\Gamma(\alpha)\longrightarrow\frac{1}{\alpha}\;,\;\alpha\longrightarrow 0. \tag{8.277}$$

Thus

$$I(m^2) \;=\; -i\frac{1}{(4\pi)^{\frac{d}{2}}}(m^2)^{\frac{d}{2}}\Gamma(-\frac{d}{2}). \tag{8.278}$$

By using this result we have

$$\begin{aligned} \frac{1}{2}\frac{\hbar}{i}\ln\det G_0 &= \frac{i\hbar}{2}V\bigg(-i\frac{1}{(4\pi)^{\frac{d}{2}}}\Gamma(-\frac{d}{2})\bigg)\bigg[(N-1)(-\mu^2+g\phi_c^2)^{\frac{d}{2}}+(-\mu^2+3g\phi_c^2)^{\frac{d}{2}}\bigg] \\ &= \frac{\hbar}{2}V\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\bigg[(N-1)(-\mu^2+g\phi_c^2)^{\frac{d}{2}}+(-\mu^2+3g\phi_c^2)^{\frac{d}{2}}\bigg]. \end{aligned} \tag{8.279}$$

The effective potential including counterterms is given by

$$\begin{aligned} V(\phi_c) &= -\frac{\mu^2}{2}\phi_c^2+\frac{g}{4}(\phi_c^2)^2-\frac{\hbar}{2}\frac{\Gamma(-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\bigg[(N-1)(-\mu^2+g\phi_c^2)^{\frac{d}{2}}+(-\mu^2+3g\phi_c^2)^{\frac{d}{2}}\bigg] \\ &- \frac{1}{2}\delta_\mu\phi_c^2+\frac{1}{4}\delta_g(\phi_c^2)^2. \end{aligned} \tag{8.280}$$

Near $d=4$ we use the approximation given by (with $\epsilon=4-d$ and $\gamma=0.5772$ is Euler-Mascheroni constant)

$$\begin{aligned} \Gamma(-\frac{d}{2}) &= \frac{1}{\frac{d}{2}(\frac{d}{2}-1)}\Gamma(\frac{\epsilon}{2}) \\ &= \frac{1}{2}[\frac{2}{\epsilon}-\gamma+\frac{3}{2}+O(\epsilon)]. \end{aligned} \tag{8.281}$$

This divergence can be absorbed by using appropriate renormalization conditions. We remark that the classical minimum is given by $\phi_c=v=\sqrt{\mu^2/g}$. We will demand that the value of the minimum of $V_{\rm eff}$ remains given by $\phi_c=v$ at the one-loop order by imposing the condition

$$\frac{\partial}{\partial\phi_c}V(\phi_c)|_{\phi_c=v}=0. \tag{8.282}$$

As we will see in the next section this is equivalent to saying that the sum of all tadpole diagrams is 0. This condition leads immediately to [15]

$$\delta_\mu - \delta_g v^2 = \hbar g \frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \frac{3}{(2\mu^2)^{1-\frac{d}{2}}}. \tag{8.283}$$

The second renormalization condition is naturally chosen to be given by

$$\frac{\partial^4}{\partial \phi_c^4} V(\phi_c)|_{\phi_c=v} = \frac{g}{4}4!. \tag{8.284}$$

This leads to the result [16]

$$\delta_g = \hbar g^2 (N+8) \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}. \tag{8.285}$$

As a consequence we obtain [17]

$$\delta_\mu = \hbar g \mu^2 (N+2) \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}. \tag{8.286}$$

After substituting back in the potential we get [18]

$$\begin{aligned} V(\phi_c) &= -\frac{\mu^2}{2}\phi_c^2 + \frac{g}{4}(\phi_c^2)^2 + \frac{\hbar}{4(4\pi)^2}\bigg[(N-1)(-\mu^2+g\phi_c^2)^2\bigg(\ln(-\mu^2+g\phi_c^2) - \frac{3}{2}\bigg) \\ &+ (-\mu^2+3g\phi_c^2)^2\bigg(\ln(-\mu^2+3g\phi_c^2) - \frac{3}{2}\bigg)\bigg]. \end{aligned} \tag{8.287}$$

In deriving this result we have used in particular the equation

$$\Gamma(-\frac{d}{2})\frac{(m^2)^{\frac{d}{2}}}{(4\pi)^{\frac{d}{2}}} = \frac{m^4}{2(4\pi)^2}\bigg[\frac{2}{\epsilon} + \ln 4\pi - \ln m^2 - \gamma + \frac{3}{2} + O(\epsilon)\bigg]. \tag{8.288}$$

8.8 Spontaneous Symmetry Breaking

8.8.1 Example: The $O(N)$ Model

We are still interested in the $(\phi^2)^2$ theory with $O(N)$ symmetry in d dimensions ($d=4$ is of primary importance but other dimensions are important as well) given by the classical action (with the replacements $m^2=-\mu^2$ and $\lambda/4! = g/4$)

$$S[\phi] = \int d^dx \bigg[\frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i + \frac{1}{2}\mu^2\phi_i^2 - \frac{g}{4}(\phi_i^2)^2\bigg]. \tag{8.289}$$

This scalar field can be in two different phases depending on the value of m^2. The "symmetric phase" characterized by the "order parameter" $\phi_{ic}(J=0) \equiv <\phi_i> = 0$ and the "broken phase" with $\phi_{ic} \neq 0$. This corresponds to the spontaneous symmetry breaking of $O(N)$ down to $O(N-1)$

[15] Exercise: Verify explicitly.
[16] Exercise: Verify explicitly.
[17] Exercise: Verify explicitly.
[18] Exercise: Verify explicitly.

and the appearance of massless particles called Goldstone bosons in $d \geq 3$. For $N = 1$, it is the Z_2 symmetry $\phi \longrightarrow -\phi$ which is broken spontaneously. This is a very concrete instance of Goldstone theorem. In "local" scalar field theory in $d \leq 2$ there can be no spontaneous symmetry breaking according to the Wagner-Mermin-Coleman theorem. To illustrate these points we start from the classical potential

$$V[\phi] = \int d^d x \bigg[-\frac{1}{2}\mu^2 \phi_i^2 + \frac{g}{4}(\phi_i^2)^2 \bigg]. \tag{8.290}$$

This has a Mexican-hat shape. The minimum of the system is a configuration which must minimize the potential and also is uniform so that it minimizes also the Hamiltonian. The equation of motion is

$$\phi_j(-\mu^2 + g\phi_i^2) = 0. \tag{8.291}$$

For $\mu^2 < 0$ the minimum is unique given by the vector $\phi_i = 0$ whereas for $\mu^2 > 0$ we can have as solution either the vector $\phi_i < 0$ (which in fact is not a minimum) or any vector ϕ_i such that

$$\phi_i^2 = \frac{\mu^2}{g}. \tag{8.292}$$

As one may check any of these vectors is a minimum. In other words we have an infinitely degenerate ground state given by the sphere S^{N-1}. The ground state is conventionally chosen to point in the N direction by adding to the action a symmetry breaking term of the form

$$\Delta S = \epsilon \int d^d x \phi_N \ , \ \epsilon > 0. \tag{8.293}$$

$$(-\mu^2 + g\phi_i^2)\phi_j = \epsilon \delta_{jN}. \tag{8.294}$$

The solution is clearly of the form

$$\phi_i = v\delta_{iN}. \tag{8.295}$$

The coefficient v is given by

$$(-\mu^2 + gv^2)v = \epsilon \Rightarrow v = \sqrt{\frac{\mu^2}{g}} \ , \ \epsilon \longrightarrow 0. \tag{8.296}$$

We expand around this solution by writing

$$\phi_k = \pi_k \ , \ k = 1, ..., N-1 \ , \ \phi_N = v + \sigma. \tag{8.297}$$

By expanding the potential around this solution we get

$$V[\phi] = \int d^d x \bigg[\frac{1}{2}(-\mu^2 + gv^2)\pi_k^2 + \frac{1}{2}(-\mu^2 + 3gv^2)\sigma^2 + v(-\mu^2 + gv^2)\sigma + gv\sigma^3 + gv\sigma\pi_k^2 + \frac{g}{2}\sigma^2\pi_k^2 + \frac{g}{4}\sigma^4 + \frac{g}{4}(\pi_k^2)^2 \bigg]. \tag{8.298}$$

We have therefore one massive field (the σ) and $N-1$ massless fields (the pions π_k) for $\mu^2 > 0$. Indeed

$$m_\pi^2 = -\mu^2 + gv^2 \equiv 0 \ , \ m_\sigma^2 = -\mu^2 + 3gv^2 \equiv 2\mu^2. \tag{8.299}$$

For $\mu^2 < 0$ we must have $v = 0$ and thus $m_\pi^2 = m_\sigma^2 = -\mu^2$.

It is well known that the $O(4)$ model provides a very good approximation to the dynamics of the real world pions with masses $m_+ = m_- = 139.6$ Mev, $m_0 = 135$Mev which are indeed much less than the mass of the 4th particle (the sigma particle) which has mass $m_\sigma = 900$ Mev. The $O(4)$ model can also be identified with the Higgs sector of the standard model.

The action around the "broken phase" solution is given by

$$S[\phi] = \int d^dx \bigg[\frac{1}{2}\partial_\mu \pi_k \partial^\mu \pi_k + \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma - \mu^2\sigma^2 - gv\sigma^3 - gv\sigma\pi_k^2 - \frac{g}{2}\sigma^2\pi_k^2 - \frac{g}{4}\sigma^4 - \frac{g}{4}(\pi_k^2)^2\bigg]. \tag{8.300}$$

We use the counterterms

$$\begin{aligned} \delta S[\phi] &= \int d^dx \bigg[\frac{1}{2}\delta_Z \partial_\mu \phi_i \partial^\mu \phi_i + \frac{1}{2}\delta_\mu \phi_i^2 - \frac{1}{4}\delta_g(\phi_i^2)^2\bigg] \\ &= \int d^dx \bigg[\frac{1}{2}\delta_Z \partial_\mu \pi_i \partial^\mu \pi_i - \frac{1}{2}(-\delta_\mu + \delta_g v^2)\pi_i^2 + \frac{1}{2}\delta_Z \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2}(-\delta_\mu + 3\delta_g v^2)\sigma^2 \\ &- (-\delta_\mu v + \delta_g v^3)\sigma - \delta_g v\sigma\pi_i^2 - \delta_g v\sigma^3 - \frac{1}{4}\delta_g(\pi_i^2)^2 - \frac{1}{2}\delta_g \sigma^2\pi_i^2 - \frac{1}{4}\delta_g\sigma^4\bigg]. \end{aligned} \tag{8.301}$$

We compute the 1−point proper vertex of the sigma field. We start from the result

$$\Gamma_{,i} = S_{,i} + \frac{1}{2}\frac{\hbar}{i}G_0^{jk}S_{,ijk} + ... \tag{8.302}$$

$$G_0^{ij} = -S_{,ij}^{-1}|_{\phi=\phi_c}. \tag{8.303}$$

We compute immediately

$$\begin{aligned} \frac{\delta\Gamma}{\delta\sigma}|_{\sigma=\pi_i=0} &= 0 + \frac{1}{2}\frac{\hbar}{i}[G_0^{\sigma\sigma}S_{,\sigma\sigma\sigma} + G_0^{\pi_i\pi_j}S_{,\sigma\pi_i\pi_j}] \\ &= \frac{1}{2}\frac{\hbar}{i}\bigg[\int \frac{d^dk}{(2\pi)^d}\frac{1}{-k^2+2\mu^2}(-3!gv) + \int \frac{d^dk}{(2\pi)^d}\frac{\delta_{ij}}{-k^2+\xi^2}(-2gv\delta_{ij})\bigg] \\ &= -3igv\hbar \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-2\mu^2} - igv(N-1)\hbar\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\xi^2}. \end{aligned} \tag{8.304}$$

In the above equation we have added a small mass ξ^2 for the pions to control the infrared behavior. We need to compute

$$\begin{aligned} \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2} &= -i\int \frac{d^dk_E}{(2\pi)^d}\frac{1}{k_E^2+m^2} \\ &= -\frac{i\Omega_{d-1}}{2(2\pi)^d}\int \frac{x^{\frac{d}{2}-1}}{x+m^2}dx \\ &= -\frac{i\Omega_{d-1}}{2(2\pi)^d}(m^2)^{\frac{d}{2}-1}\int_0^1 t^{-\frac{d}{2}}(1-t)^{\frac{d}{2}-1}dt \\ &= -\frac{i}{(4\pi)^{\frac{d}{2}}}(m^2)^{\frac{d}{2}-1}\Gamma(1-\frac{d}{2}). \end{aligned} \tag{8.305}$$

We get

$$\frac{\delta\Gamma}{\delta\sigma}|_{\sigma=\pi_i=0} = -gv\hbar\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}(\frac{3}{(2\mu^2)^{1-\frac{d}{2}}}+\frac{N-1}{(\xi)^{1-\frac{d}{2}}}). \tag{8.306}$$

By adding the contribution of the counterterms we get

$$\frac{\delta\Gamma}{\delta\sigma}|_{\sigma=\pi_i=0} = -(-\delta_\mu v+\delta_g v^3)-gv\hbar\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}(\frac{3}{(2\mu^2)^{1-\frac{d}{2}}}+\frac{N-1}{(\xi^2)^{1-\frac{d}{2}}}). \tag{8.307}$$

The corresponding Feynman diagrams are shown on figure 8. We will impose the renormalization condition

$$\frac{\delta\Gamma}{\delta\sigma} = 0. \tag{8.308}$$

This is equivalent to the statement that the sum of all tadpole diagrams giving the 1−point proper vertex for the σ field vanishes. In other words we do not allow any quantum shifts in the vacuum expectation value of ϕ_N which is given by $<\phi_N>=v$. We get then

$$(-\delta_\mu+\delta_g v^2) = -g\hbar\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}(\frac{3}{(2\mu^2)^{1-\frac{d}{2}}}+\frac{N-1}{(\xi^2)^{1-\frac{d}{2}}}). \tag{8.309}$$

Next we consider the $\pi\pi$ amplitude. We use the result

$$\Gamma_{,j_0k_0} = S_{,j_0k_0}+\frac{\hbar}{i}\bigg[\frac{1}{2}G_0^{mn}S[\phi]_{,j_0k_0mn}+\frac{1}{2}G_0^{mm_0}G_0^{nn_0}S[\phi]_{,j_0mn}S[\phi]_{,k_0m_0n_0}\bigg]. \tag{8.310}$$

We compute immediately (including again a small mass ξ^2 for the pions)

$$S_{,j_0k_0} = -\delta_{j_0k_0}(\Delta+\xi^2)\delta^d(x-y). \tag{8.311}$$

$$\begin{aligned}\frac{1}{2}G_0^{mn}S[\phi]_{,j_0k_0mn} &= \frac{1}{2}\int d^dz d^dw[G_0^{\sigma\sigma}(z,w)][-2g\delta_{j_0k_0}\delta^d(x-y)\delta^d(x-z)\delta^d(x-w)]\\ &+ \frac{1}{2}\int d^dz d^dw[\delta_{mn}G_0^{\pi\pi}(z,w)][-3!\frac{g}{3}\delta_{j_0k_0mn}\delta^d(x-y)\delta^d(x-z)\delta^d(x-w)]\\ &= -g\delta_{j_0k_0}G_0^{\sigma\sigma}(x,x)\delta^d(x-y)-(N+1)g\delta_{j_0k_0}G_0^{\pi\pi}(x,x)\delta^d(x-y).\end{aligned} \tag{8.312}$$

$$\begin{aligned}\frac{1}{2}G_0^{mm_0}G_0^{nn_0}S[\phi]_{,j_0mn}S[\phi]_{,k_0m_0n_0} &= \frac{(2)}{2}\int d^dz\int d^dz_0\int d^dw\int d^dw_0[\delta_{mm_0}G_0^{\pi\pi}(z,z_0)][G_0^{\sigma\sigma}(w,w_0)]\\ &\times [-2gv\delta_{j_0m}\delta^d(x-z)\delta^d(x-w)][-2gv\delta_{k_0m_0}\delta^d(y-z_0)\delta^d(y-w_0)]\\ &= 4g^2v^2G_0^{\pi\pi}(x,y)G_0^{\sigma\sigma}(x,y).\end{aligned} \tag{8.313}$$

Thus we get

$$\begin{aligned}\Gamma_{j_0k_0}^{\pi\pi}(x,y) &= -\delta_{j_0k_0}(\Delta+\xi^2)\delta^d(x-y)+\frac{\hbar}{i}\bigg[-g\delta_{j_0k_0}G_0^{\sigma\sigma}(x,x)\delta^d(x-y)-(N+1)g\delta_{j_0k_0}G_0^{\pi\pi}(x,x)\delta^d(x-y)\\ &+ 4g^2v^2\delta_{j_0k_0}G_0^{\pi\pi}(x,y)G_0^{\sigma\sigma}(x,y)\bigg].\end{aligned} \tag{8.314}$$

Recall also that

$$G_0^{\pi\pi}(x,y)=\int\frac{d^dp}{(2\pi)^d}\frac{1}{-p^2+\xi^2}e^{ip(x-y)}\ ,\ G_0^{\sigma\sigma}(x,y)=\int\frac{d^dp}{(2\pi)^d}\frac{1}{-p^2+2\mu^2}e^{ip(x-y)}. \quad (8.315)$$

The Fourier transform is defined by

$$\int d^dx\int d^dy\Gamma^{\pi\pi}_{j_0k_0}(x,y)e^{ipx}e^{iky}=(2\pi)^d\delta^d(p+k)\Gamma^{\pi\pi}_{j_0k_0}(p). \quad (8.316)$$

We compute then

$$\begin{aligned}\Gamma^{\pi\pi}_{j_0k_0}(p) &= \delta_{j_0k_0}(p^2-\xi^2)+\frac{\hbar}{i}\delta_{j_0k_0}\bigg[-g\int\frac{d^dk}{(2\pi)^d}\frac{1}{-k^2+2\mu^2}-(N+1)g\int\frac{d^dk}{(2\pi)^d}\frac{1}{-k^2+\xi^2}\\ &+ 4g^2v^2\int\frac{d^4k}{(2\pi)^d}\frac{1}{-k^2+\xi^2}\frac{1}{-(k+p)^2+2\mu^2}\bigg]. \quad (8.317)\end{aligned}$$

By adding the contribution of the counterterms we get

$$\begin{aligned}\Gamma^{\pi\pi}_{j_0k_0}(p) &= \delta_{j_0k_0}(p^2-\xi^2)+\frac{\hbar}{i}\delta_{j_0k_0}\bigg[-g\int\frac{d^dk}{(2\pi)^d}\frac{1}{-k^2+2\mu^2}-(N+1)g\int\frac{d^dk}{(2\pi)^d}\frac{1}{-k^2+\xi^2}\\ &+ 4g^2v^2\int\frac{d^4k}{(2\pi)^d}\frac{1}{-k^2+\xi^2}\frac{1}{-(k+p)^2+2\mu^2}\bigg]+(\delta_Zp^2+\delta_\mu-\delta_gv^2)\delta_{j_0k_0}. \quad (8.318)\end{aligned}$$

The corresponding Feynman diagrams are shown on figure 8. After some calculation we obtain

$$\begin{aligned}\Gamma^{\pi\pi}_{j_0k_0}(p) &= \delta_{j_0k_0}(p^2-\xi^2)-2\hbar g\delta_{j_0k_0}\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}[(\xi^2)^{\frac{d}{2}-1}-(2\mu^2)^{\frac{d}{2}-1}]\\ &+ \frac{\hbar}{i}\delta_{j_0k_0}\bigg[4g^2v^2\int\frac{d^4k}{(2\pi)^d}\frac{1}{-k^2+\xi^2}\frac{1}{-(k+p)^2+2\mu^2}\bigg]+\delta_Zp^2\delta_{j_0k_0}. \quad (8.319)\end{aligned}$$

The last integral can be computed using Feynman parameters x_1, x_2 intrdouced by the identity

$$\frac{1}{A_1A_2}=\int_0^1dx_1\int_0^1dx_2\frac{1}{(x_1A_1+x_2A_2)^2}\delta(x_1+x_2-1). \quad (8.320)$$

We have then (with $s=2$, $l=k+(1-x_1)p$ and $M^2=\xi^2x_1+2\mu^2(1-x_1)-p^2x_1(1-x_1)$ and after a Wick rotation)

$$\begin{aligned}
\int \frac{d^dk}{(2\pi)^d}\frac{1}{-k^2+\xi^2}\frac{1}{-(k+p)^2+2\mu^2} &= \int \frac{d^dk}{(2\pi)^d}\int_0^1 dx_1\int_0^1 dx_2\frac{1}{\big[x_1(k^2-\xi^2)+x_2((k+p)^2-2\mu^2)\big]^s}\delta(x_1+x_2-1)\\
&= \int_0^1 dx_1\int_0^1 dx_2\delta(x_1+x_2-1)\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2-M^2)^s}\\
&= i(-1)^s\int_0^1 dx_1\int_0^1 dx_2\delta(x_1+x_2-1)\int\frac{d^dl_E}{(2\pi)^d}\frac{1}{(l_E^2+M^2)^s}\\
&= \frac{i(-1)^s\Omega_{d-1}}{(2\pi)^d}\int_0^1 dx_1\int_0^1 dx_2\delta(x_1+x_2-1)\int\frac{l_E^{d-1}dl_E}{(l_E^2+M^2)^s}\\
&= \frac{i(-1)^s\Omega_{d-1}}{2(2\pi)^d}\int_0^1 dx_1\int_0^1 dx_2\delta(x_1+x_2-1)\int\frac{x^{\frac{d}{2}-1}dx}{(x+M^2)^s}\\
&= \frac{i(-1)^s\Omega_{d-1}}{2(2\pi)^d}(M^2)^{\frac{d}{2}-s}\int_0^1 dx_1\int_0^1 dx_2\delta(x_1+x_2-1)\int_0^1(1-t)^{\frac{d}{2}-1}t^{s-\frac{d}{2}-1}dt\\
&= \frac{i(-1)^s\Omega_{d-1}}{2(2\pi)^d}(M^2)^{\frac{d}{2}-s}\int_0^1 dx_1\int_0^1 dx_2\delta(x_1+x_2-1)\frac{\Gamma(s-\frac{d}{2})\Gamma(\frac{d}{2})}{\Gamma(s)}\\
&= \int_0^1 dx_1\int_0^1 dx_2\delta(x_1+x_2-1)\frac{i(-1)^s}{(4\pi)^{\frac{d}{2}}}(M^2)^{\frac{d}{2}-s}\frac{\Gamma(s-\frac{d}{2})}{\Gamma(s)}.
\end{aligned}\tag{8.321}$$

Using this result we have

$$\begin{aligned}
\Gamma^{\pi\pi}_{j_0k_0}(p) &= \delta_{j_0k_0}(p^2-\xi^2)-2\hbar g\delta_{j_0k_0}\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}[(\xi^2)^{\frac{d}{2}-1}-(2\mu^2)^{\frac{d}{2}-1}]\\
&+ 4\hbar g^2v^2\delta_{j_0k_0}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx_1\big[\xi^2x_1+2\mu^2(1-x_1)-p^2x_1(1-x_1)\big]^{\frac{d}{2}-2}+\delta_Zp^2\delta_{j_0k_0}.
\end{aligned}\tag{8.322}$$

By studying the amplitudes $\sigma\sigma$, $\sigma\sigma\pi\pi$ and $\pi\pi\pi\pi$ we can determine that the counterterm δ_Z is finite at one-loop whereas the counterterm δ_g is divergent [19]. This means in particular that the divergent part of the above remaining integral does not depend on p. We simply set $p^2=0$ and study

$$\begin{aligned}
\Gamma^{\pi\pi}_{j_0k_0}(0) &= \delta_{j_0k_0}(-\xi^2)-2\hbar g\delta_{j_0k_0}\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}[(\xi^2)^{\frac{d}{2}-1}-(2\mu^2)^{\frac{d}{2}-1}]+4\hbar g^2v^2\delta_{j_0k_0}\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx_1\\
&\times \big[\xi^2x_1+2\mu^2(1-x_1)\big]^{\frac{d}{2}-2}.
\end{aligned}\tag{8.323}$$

We get (using $gv^2=\mu^2$)

$$\Gamma^{\pi\pi}_{j_0k_0}(0) = \delta_{j_0k_0}(-\xi^2)-2\hbar g\delta_{j_0k_0}\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}[(\xi^2)^{\frac{d}{2}-1}-(2\mu^2)^{\frac{d}{2}-1}]+2\hbar g\delta_{j_0k_0}\frac{\Gamma(1-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\frac{2\mu^2}{2\mu^2-\xi^2}[(\xi^2)^{\frac{d}{2}-1}-(2\mu^2)^{\frac{d}{2}-1}].\tag{8.324}$$

This vanishes exactly in the limit $\xi\longrightarrow 0$ and therefore the pions remain massless at one-

[19] Exercise: Show this result explicitly. You need to figure out and then use two more renormalization conditions.

loop[20]. This is a manifestation of the Goldstone's theorem which states that there must exist $N-1$ massless particles associated with the $N-1$ broken symmetries of the breaking pattern $O(N) \longrightarrow O(N-1)$.

8.8.2 Glodstone's Theorem

Spontaneous symmetry breaking of a continuous symmetry leads always to massless particles called Goldstone bosons. The number of massless Goldstone bosons which appear is precisely equal to the number of symmetry generators broken spontaneously. This is a general result known as Goldstone's theorem. For example in the case of the $O(N)$ model studied in the previous sections the continuous symmetries are precisely $O(N)$ transformations, i.e. rotations in N dimensions which rotate the different components of the scalar field into each other. There are in this case $N(N-1)/2$ independent rotations and hence $N(N-1)/2$ generators of the group $O(N)$. Under the symmetry breaking pattern $O(N) \longrightarrow O(N-1)$ the number of broken symmetries is exactly $N(N-1)/2 - (N-1)(N-2)/2 = N-1$ and hence there must appear $N-1$ massless Goldstone bosons in the low energy spectrum of the theory which have been already verified explicitly upto the one-loop order. This holds also true at any arbitrary order in perturbation theory. Remark that for $N=1$ there is no continuous symmetry and there are no massless Goldstone particles associated to the symmetry breaking pattern $\phi \longrightarrow -\phi$. We sketch now a general proof of Goldstone's theorem.

A typical Lagrangian density of interest is of the form

$$\mathcal{L}(\phi) = \text{terms with derivatives}(\phi) - V(\phi). \tag{8.325}$$

The minimum of V is denoted ϕ_0 and satisfies

$$\frac{\partial}{\partial \phi_a} V(\phi)|_{\phi=\phi_0} = 0. \tag{8.326}$$

Now we expand V around the minimum ϕ_0 upto the second order in the fields. We get

$$\begin{aligned} V(\phi) &= V(\phi_0) + \frac{1}{2}(\phi-\phi_0)_a(\phi-\phi_0)_b \frac{\partial^2}{\partial\phi_a\partial\phi_b} V(\phi)|_{\phi=\phi_0} + ... \\ &= V(\phi_0) + \frac{1}{2}(\phi-\phi_0)_a(\phi-\phi_0)_b m^2_{ab}(\phi_0) + ... \end{aligned} \tag{8.327}$$

The matrix $m^2_{ab}(\phi_0)$ called the mass matrix is clearly a symmetric matrix which is also positive since ϕ_0 is assumed to be a minimum configuration.

A general continuous symmetry will transform the scalar field ϕ infinitesimally according to the generic law

$$\phi_a \longrightarrow \phi_a^{'} = \phi_a + \alpha\Delta_a(\phi). \tag{8.328}$$

The parameter α is infinitesimal and Δ_a are some functions of ϕ. The invariance of the Lagrangian density is given by the condition

$$\text{terms with derivatives}(\phi) - V(\phi) = \text{terms with derivatives}(\phi+\alpha\Delta(\phi)) - V(\phi+\alpha\Delta(\phi)). \tag{8.329}$$

[20]Exercise: Show this result directly. Start by showing that

$$\frac{\hbar}{i}\bigg[4g^2v^2\int\frac{d^4k}{(2\pi)^d}\frac{1}{-k^2+\xi^2}\frac{1}{-k^2+2\mu^2}\bigg] = 2i\hbar g[I(\xi^2)-I(2\mu^2)]\ ,\ I(m^2)=\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2}.$$

For constant fields this condition reduces to

$$V(\phi) = V(\phi + \alpha\Delta(\phi)). \tag{8.330}$$

Equivalently

$$\Delta_a(\phi)\frac{\partial}{\partial\phi_a}V(\phi) = 0. \tag{8.331}$$

By differentiating with respect to ϕ_b and setting $\phi = \phi_0$ we get

$$m^2_{ab}(\phi_0)\Delta_b(\phi_0) = 0. \tag{8.332}$$

The symmetry transformations, as we have seen, leave always the Lagrangian density invariant which was actually our starting point. In the case that the above symmetry transformation leaves also the ground state configuration ϕ_0 invariant we must have $\Delta(\phi_0) = 0$ and thus the above equation becomes trivial. However, in the case that the symmetry transformation does not leave the ground state configuration ϕ_0 invariant, which is precisely the case of a spontaneously broken symmetry, $\Delta_b(\phi_0)$ is an eigenstate of the mass matrix $m^2_{ab}(\phi_0)$ with 0 eigenvalue which is exactly the massless Goldstone particle.

9

Path Integral Quantization of Dirac and Vector Fields

9.1 Free Dirac Field

9.1.1 Canonical Quantization

The Dirac field ψ describes particles of spin $\hbar/2$. The Dirac field ψ is a $4-$component object which transforms as spinor under the action of the Lorentz group. The classical equation of motion of a free Dirac field is the Dirac equation. This is given by

$$(i\hbar\gamma^{\mu}\partial_{\mu}-mc)\psi=0. \tag{9.1}$$

Equivalently the complex conjugate field $\bar{\psi}=\psi^{+}\gamma^{0}$ obeys the equation

$$\bar{\psi}(i\hbar\gamma^{\mu}\overleftarrow{\partial_{\mu}}+mc)=0. \tag{9.2}$$

These two equations are the Euler-Lagrange equations derived from the action

$$S=\int d^4x\bar{\psi}(i\hbar c\gamma^{\mu}\partial_{\mu}-mc^2)\psi. \tag{9.3}$$

The Dirac matrices γ^{μ} satisfy the usual Dirac algebra $\{\gamma^{\mu},\gamma^{\nu}\}=2\eta^{\mu\nu}$. The Dirac equation admits positive-energy solutions (associated with particles) denoted by spinors $u^i(p)$ and negative-energy solutions (associated with antiparticles) denoted by spinors $v^i(p)$.

The spinor field can be put in the form (with $\omega(\vec{p})=E/\hbar=\sqrt{\vec{p}^2c^2+m^2c^4}/\hbar$)

$$\psi(x) \;=\; \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i\bigg(e^{-\frac{i}{\hbar}px}u^{(i)}(\vec{p})b(\vec{p},i)+e^{\frac{i}{\hbar}px}v^{(i)}(\vec{p})d(\vec{p},i)^{+}\bigg). \tag{9.4}$$

The conjugate field is $\Pi(x)=i\hbar\psi^{+}$.

In the quantum theory (canonical quantization) the coefficients $b(\vec{p},i)$ and $d(\vec{p},i)$ become operators $\hat{b}(\vec{p},i)$ and $\hat{d}(\vec{p},i)$ and as a consequence the spinor field $\psi(x)$ and the conjugate field $\Pi(x)$ become operators $\hat{\psi}(x)$ and $\hat{\Pi}(x)$ respectively. In order to have a stable ground state the operators $\hat{\psi}(x)$ and $\hat{\Pi}(x)$ must satisfy the anticommutation (rather than commutation) relations given by

$$\{\hat{\psi}_\alpha(x^0,\vec{x}),\hat{\Pi}_\beta(x^0,\vec{y})\}=i\hbar\delta_{\alpha\beta}\delta^3(\vec{x}-\vec{y}). \tag{9.5}$$

Equivalently

$$\begin{aligned}
&\{\hat{b}(\vec{p},i),\hat{b}(\vec{q},j)^+\}=\hbar\delta_{ij}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q})\\
&\{\hat{d}(\vec{p},i)^+,\hat{d}(\vec{q},j)\}=\hbar\delta_{ij}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q})\\
&\{\hat{b}(\vec{p},i),\hat{d}(\vec{q},j)\}=\{\hat{d}(\vec{q},j)^+,\hat{b}(\vec{p},i)\}=0.
\end{aligned} \tag{9.6}$$

We find that excited particle states are obtained by acting with $\hat{b}(\vec{p},i)^+$ on the vacuum $|0>$ whereas excited antiparticle states are obtained by acting with $\hat{d}(\vec{p},i)^+$. The vacuum state $|0>$ is the eigenstate with energy equal 0 of the Hamiltonian

$$\hat{H} = \int\frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\sum_i\bigg(\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},i)+\hat{d}(\vec{p},i)^+\hat{d}(\vec{p},i)\bigg). \tag{9.7}$$

The Feynman propagator for a Dirac spinor field is defined by

$$(S_F)_{ab}(x-y) = <0|T\hat{\psi}_a(x)\bar{\hat{\psi}}_b(y)|0>. \tag{9.8}$$

The time-ordering operator is defined by

$$\begin{aligned}
&T\hat{\psi}(x)\hat{\psi}(y)=+\hat{\psi}(x)\hat{\psi}(y)\ ,\ x^0>y^0\\
&T\hat{\psi}(x)\hat{\psi}(y)=-\hat{\psi}(y)\hat{\psi}(x)\ ,\ x^0<y^0.
\end{aligned} \tag{9.9}$$

Explicitly we have [1]

$$(S_F)_{ab}(x-y) = \frac{\hbar}{c}\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i(\gamma^\mu p_\mu+mc)_{ab}}{p^2-m^2c^2+i\epsilon}e^{-\frac{i}{\hbar}p(x-y)}. \tag{9.10}$$

9.1.2 Fermionic Path Integral and Grassmann Numbers

Let us now expand the spinor field as

$$\psi(x^0,\vec{x})=\frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\chi(x^0,\vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}}. \tag{9.11}$$

The Lagrangian in terms of χ and χ^+ is given by

$$\begin{aligned}
L &= \int d^3x\mathcal{L}\\
&= \int d^3x\bar{\psi}(i\hbar c\gamma^\mu\partial_\mu-mc^2)\psi\\
&= \frac{c}{\hbar^2}\int\frac{d^3p}{(2\pi\hbar)^3}\bar{\chi}(x^0,\vec{p})(i\hbar\gamma^0\partial_0-\gamma^ip^i-mc)\chi(x^0,\vec{p}).
\end{aligned} \tag{9.12}$$

[1] There was a serious (well not really serious) error in our computation of the scalar propagator in the first semester which propagated to an error in the Dirac propoagtor. This must be corrected there and also in the previous chapter of this current semester in which we did not follow the factors of $\hbar$ and c properly. In any case the coefficient $\hbar/c$ apperaing in front of this propagator is now correct.

We use the identity

$$\gamma^0(\gamma^i p^i + mc)\chi(x^0,\vec{p}) = \frac{\hbar\omega(\vec{p})}{c}\chi(x^0,\vec{p}). \quad (9.13)$$

We get then

$$L = \frac{1}{\hbar}\int \frac{d^3p}{(2\pi\hbar)^3}\chi^+(x^0,\vec{p})(i\partial_t - \omega(\vec{p}))\chi(x^0,\vec{p}). \quad (9.14)$$

Using the box normalization the momenta become discrete and the measure $\int d^3\vec{p}/(2\pi\hbar)^3$ becomes the sum $\sum_{\vec{p}}/V$. Thus the Lagrangian becomes with $\theta_p(t) = \chi(x^0,\vec{p})/\sqrt{\hbar V}$ given by

$$L = \sum_{\vec{p}} \theta_p^+(t)(i\partial_t - \omega(\vec{p}))\theta_p(t). \quad (9.15)$$

For a single momentum $\vec{p}$ the Lagrangian of the theory simplifies to the single term

$$L_p = \theta_p^+(t)(i\partial_t - \omega(\vec{p}))\theta_p(t). \quad (9.16)$$

We will simplify further by thinking of $\theta_p(t)$ as a single component field. The conjugate variable is $\pi_p(t) = i\theta_p^+(t)$. In the quantum theory we replace θ_p and π_p with operators $\hat{\theta}_p$ and $\hat{\pi}_p$. The canonical commutation relations are

$$\{\hat{\theta}_p, \hat{\pi}_p\} = i\hbar, \{\hat{\theta}_p, \hat{\theta}_p\} = \{\hat{\pi}_p, \hat{\pi}_p\} = 0. \quad (9.17)$$

There several remarks here:

- In the limit $\hbar \longrightarrow 0$, the operators reduce to fields which are anticommuting classical functions. In other words even classical fermion fields must be represented by anticommuting numbers which are known as Grassmann numbers.
- There is no eigenvalues of the operators $\hat{\theta}_p$ and $\hat{\pi}_p$ in the set of complex numbers except 0. The non-zero eigenvalues must be therefore anticommuting Grassmann numbers.
- Obviously given two anticommuting Grassmann numbers α and β we have immediately the following fundamental properties

$$\alpha\beta = -\beta\alpha \ , \ \alpha^2 = \beta^2 = 0. \quad (9.18)$$

The classical equation of motion following from the Lagrangian L_p is $i\partial_t\theta_p = \omega(\vec{p})\theta_p$. An immediate solution is given by

$$\hat{\theta}_p(t) = \hat{b}_p \exp(-i\omega(\vec{p})t). \quad (9.19)$$

Thus

$$\{\hat{b}_p, \hat{b}_p^+\} = \hbar, \{\hat{b}_p, \hat{b}_p\} = \{\hat{b}_p^+, \hat{b}_p^+\} = 0. \quad (9.20)$$

The Hilbert space contains two states $|0>$ (the vacuum) and $|1>= \hat{b}_p^+|0>$ (the only exited state). Indeed we clearly have $\hat{b}_p^+|1>= 0$ and $\hat{b}_p|1>= \hbar|0>$. We define the (coherent) states at time $t = 0$ by

$$|\theta_p(0)>= e^{\hat{b}_p^+\theta_p(0)}|0> \ , \ <\theta_p(0)| = e^{\theta^+(0)\hat{b}_p} <0|. \quad (9.21)$$

The number $\theta_p(0)$ must be anticommuting Grassmann number, i.e. it must satisfy $\theta_p(0)^2 = 0$ whereas the number $\theta^+(0)$ is the complex conjugate of $\theta_p(0)$ which should be taken as independent and hence $(\theta_p^+(0))^2 = 0$ and $\theta_p^+(0)\theta_p(0) = -\theta_p(0)\theta_p^+(0)$. We compute immediately that

$$\hat{\theta}_p(0)|\theta_p(0)>=\theta_p(0)|\theta_p(0)> \ , \ <\theta_p(0)|\hat{\theta}_p(0)^+=<\theta_p(0)|\theta_p^+(0). \tag{9.22}$$

The Feynman propagator for the field $\theta_p(t)$ is defined by

$$S(t-t^{'})=<0|T(\hat{\theta}_p(t)\hat{\theta}_p^+(t^{'}))|0> . \tag{9.23}$$

We compute immediately (with $\epsilon > 0$) [2]

$$S(t-t^{'})=\hbar e^{-i\omega(\vec{p})(t-t^{'})}\equiv \hbar^2\int\frac{dp^0}{2\pi\hbar}\frac{i}{p^0-\hbar\omega(\vec{p})+i\epsilon}e^{-\frac{i}{\hbar}p^0(t-t^{'})} \ , \ t>t^{'}. \tag{9.24}$$

$$S(t-t^{'})=0\equiv \hbar^2\int\frac{dp^0}{2\pi\hbar}\frac{i}{p^0-\hbar\omega(\vec{p})+i\epsilon}e^{-\frac{i}{\hbar}p^0(t-t^{'})} \ , \ t<t^{'}. \tag{9.25}$$

The anticommuting Grassmann numbers have the following properties:

- A general function $f(\theta)$ of a single anticommuting Grassmann number can be expanded as

$$f(\theta)=A+B\theta. \tag{9.26}$$

- The integral of $f(\theta)$ is therefore

$$\int d\theta f(\theta)=\int d\theta(A+B\theta). \tag{9.27}$$

We demand that this integral is invariant under the shift $\theta\longrightarrow\theta+\eta$. This leads immediately to the so-called Berezin integration rules

$$\int d\theta=0 \ , \ \int d\theta\theta=1. \tag{9.28}$$

- The differential $d\theta$ anticommutes with θ, viz

$$d\theta\theta=-\theta d\theta. \tag{9.29}$$

- We have immediately

$$\int d\theta d\eta\eta\theta=1. \tag{9.30}$$

- The most general function of two anticommuting Grassmann numbers θ and θ^+ is

$$f(\theta,\theta^+)=A+B\theta+C\theta^++D\theta^+\theta. \tag{9.31}$$

- Given two anticommuting Grassmann numbers θ and η we have

$$(\theta\eta)^+=\eta^+\theta^+.=-\theta^+\eta^+. \tag{9.32}$$

[2]Exercise: Verify this result using the residue theorem.

- We compute the integrals

$$\int d\theta^+ d\theta e^{-\theta^+ b\theta} = \int d\theta^+ d\theta (1-\theta^+ b\theta) = b. \tag{9.33}$$

$$\int d\theta^+ d\theta \theta\theta^+ e^{-\theta^+ b\theta} = \int d\theta^+ d\theta \theta\theta^+ (1-\theta^+ b\theta) = 1. \tag{9.34}$$

It is instructive to compare the first integral with the bosonic integral

$$\int dz^+ dz e^{-z^+ bz} = \frac{2\pi}{b}. \tag{9.35}$$

- We consider now a general integral of the form

$$\int \prod_i d\theta_i^+ d\theta_i f(\theta^+, \theta). \tag{9.36}$$

Consider the unitary transformation $\theta_i \longrightarrow \theta_i^{'} = U_{ij}\theta_j$ where $U^+U = 1$. It is rather obvious that [3]

$$\prod_i d\theta_i^{'} = \det U \prod_i d\theta_i. \tag{9.37}$$

Hence $\prod_i d\theta_i^{'+} d\theta_i^{'} = \prod_i d\theta_i^+ d\theta_i$ since $U^+U = 1$. On the other hand, by expanding the function $f(\theta^+, \theta)$ and integrating out we immediately see that the only non-zero term will be exactly of the form $\prod_i \theta_i^+ \theta_i$ which is also invariant under the unitary transformation U. Hence

$$\int \prod_i d\theta_i^{'+} d\theta_i^{'} f(\theta^{'+}, \theta^{'}) = \int \prod_i d\theta_i^+ d\theta_i f(\theta^+, \theta). \tag{9.38}$$

- Consider the above integral for

$$f(\theta^+, \theta) = e^{-\theta^+ M\theta}. \tag{9.39}$$

M is a Hermitian matrix. By using the invariance under $U(N)$ we can diagonalize the matrix M without changing the value of the integral. The eigenvalues of M are denoted m_i. The integral becomes

$$\int \prod_i d\theta_i^+ d\theta_i e^{-\theta^+ M\theta} = \int \prod_i d\theta_i^+ d\theta_i e^{-\theta_i^+ m_i \theta_i} = \prod_i m_i = \det M. \tag{9.40}$$

Again it is instructive to compare with the bosonic integral

$$\int \prod_i dz_i^+ dz_i e^{-z^+ Mz} = \frac{(2\pi)^n}{\det M}. \tag{9.41}$$

[3]Exercise: Verify this fact.

- We consider now the integral

$$\int\prod_i d\theta_i^+ d\theta_i e^{-\theta^+M\theta-\theta^+\eta-\eta^+\theta} = \int\prod_i d\theta_i^+ d\theta_i e^{-(\theta^++\eta^+M^{-1})M(\theta+M^{-1}\eta)+\eta^+M^{-1}\eta}$$
$$= \det M e^{\eta^+M^{-1}\eta}. \quad (9.42)$$

- Let us consider now the integral

$$\int\prod_i d\theta_i^+ d\theta_i \theta_k\theta_l^+ e^{-\theta^+M\theta} = \frac{\delta}{\delta\eta_l}\frac{\delta}{\delta\eta_k^+}\bigg(\int\prod_i d\theta_i^+ d\theta_i e^{-\theta^+M\theta-\theta^+\eta-\eta^+\theta}\bigg)_{\eta=\eta^+=0}$$
$$= \det M(M^{-1})_{kl}. \quad (9.43)$$

  In the above equation we have always to remember that the order of differentials and variables is very important since they are anticommuting objects.

- In the above equation we observe that if the matrix M has eigenvalue 0 then the result is 0 since the determinant vanishes in this case.

We go back now to our original problem. We want to express the propagator $S(t-t^{'}) =< 0|T(\hat{\theta}_p(t)\hat{\theta}_p^+(t^{'}))|0>$ as a path integral over the classical fields $\theta_p(t)$ and $\theta_p^+(t)$ which must be complex anticommuting Grassmann numbers. By analogy with what happens in scalar field theory we expect the path integral to be a functional integral of the probability amplitude $\exp(iS_p/\hbar)$ where S_p is the action $S_p=\int dtL_p$ over the classical fields $\theta_p(t)$ and $\theta_p^+(t)$ (which are taken to be complex anticommuting Grassmann numbers instead of ordinary complex numbers). In the presence of sources $\eta_p(t)$ and $\eta_p^+(t)$ this path integral reads

$$Z[\eta_p,\eta_p^+] = \int\mathcal{D}\theta_p^+\mathcal{D}\theta_p\ \exp\bigg(\frac{i}{\hbar}\int dt\theta_p^+(i\partial_t-\omega(\vec{p}))\theta_p+\frac{i}{\hbar}\int dt\eta_p^+\theta_p+\frac{i}{\hbar}\int dt\theta_p^+\eta_p\bigg). \quad (9.44)$$

By using the result (9.42) we know immediately that

$$Z[\eta_p,\eta_p^+] = \det M e^{\eta^+M^{-1}\eta}\ ,\ M=-\frac{i}{\hbar}(i\partial_t-\omega(\vec{p}))\ ,\ \eta=-\frac{i}{\hbar}\eta_p\ ,\ \eta^+=-\frac{i}{\hbar}\eta_p^+. \quad (9.45)$$

In other words

$$Z[\eta_p,\eta_p^+] = \det M e^{-\frac{1}{\hbar^2}\int dt\int dt^{'}\eta_p^+(t)M^{-1}(t,t^{'})\eta_p(t^{'})}. \quad (9.46)$$

From one hand we have

$$\Big(\frac{\hbar}{i}\Big)^2\bigg(\frac{1}{Z}\frac{\delta^2}{\delta\eta_p(t^{'})\delta\eta_p^+(t)}Z\bigg)_{\eta_p=\eta_p^+=0} = \frac{\int\mathcal{D}\theta_p^+\mathcal{D}\theta_p\ \theta_p(t)\theta_p^+(t^{'})\exp\Big(\frac{i}{\hbar}\int dt\theta_p^+(i\partial_t-\omega(\vec{p}))\theta_p\Big)}{\int\mathcal{D}\theta_p^+\mathcal{D}\theta_p\ \exp\Big(\frac{i}{\hbar}\int dt\theta_p^+(i\partial_t-\omega(\vec{p}))\theta_p\Big)}$$
$$\equiv <\theta_p(t)\theta_p^+(t^{'})>. \quad (9.47)$$

From the other hand

$$\Big(\frac{\hbar}{i}\Big)^2\bigg(\frac{1}{Z}\frac{\delta^2}{\delta\eta_p(t^{'})\delta\eta_p^+(t)}Z\bigg)_{\eta_p=\eta_p^+=0} = M^{-1}(t,t^{'}). \quad (9.48)$$

Therefore

$$< \theta_p(t)\theta_p^+(t^{'}) >= M^{-1}(t,t^{'}). \tag{9.49}$$

We have

$$\begin{aligned} M(t,t^{'}) &= -\frac{i}{\hbar}(i\partial_t-\omega(\vec{p}))\delta(t-t^{'}) \\ &= \frac{1}{\hbar^2}\int\frac{dp^0}{2\pi\hbar}\frac{p^0-\omega(\vec{p})}{i}e^{-\frac{i}{\hbar}p^0(t-t^{'})}. \end{aligned} \tag{9.50}$$

The inverse is therefore given by

$$< \theta_p(t)\theta_p^+(t^{'}) >= M^{-1}(t,t^{'}) = \hbar^2\int\frac{dp^0}{2\pi\hbar}\frac{i}{p^0-\omega(\vec{p})+i\epsilon}e^{-\frac{i}{\hbar}p^0(t-t^{'})}. \tag{9.51}$$

We conclude therefore that

$$< 0|T(\hat{\theta}_p(t)\hat{\theta}_p^+(t^{'}))|0 >= \frac{\int\mathcal{D}\theta_p^+\mathcal{D}\theta_p\ \theta_p(t)\theta_p^+(t^{'})\exp\bigg(\frac{i}{\hbar}\int dt\theta_p^+(i\partial_t-\omega(\vec{p}))\theta_p\bigg)}{\int\mathcal{D}\theta_p^+\mathcal{D}\theta_p\ \exp\bigg(\frac{i}{\hbar}\int dt\theta_p^+(i\partial_t-\omega(\vec{p}))\theta_p\bigg)}. \tag{9.52}$$

9.1.3 The Electron Propagator

We are now ready to state our main punch line. The path integral of a free Dirac field in the presence of non-zero sources must be given by the functional integral

$$Z[\eta,\bar{\eta}] = \int\mathcal{D}\bar{\psi}\mathcal{D}\psi\ \exp\bigg(\frac{i}{\hbar}S_0[\psi,\bar{\psi}]+\frac{i}{\hbar}\int d^4x\bar{\eta}\psi+\frac{i}{\hbar}\int d^4x\bar{\psi}\eta\bigg). \tag{9.53}$$

$$S_0[\psi,\bar{\psi}] = \int d^4x\bar{\psi}(i\hbar c\gamma^\mu\partial_\mu-mc^2)\psi. \tag{9.54}$$

The Dirac spinor ψ and its Dirac conjugate spinor $\bar{\psi}=\psi^+\gamma^0$ must be treated as independent complex spinors with components which are Grassmann-valued functions of x. Indeed by taking $\chi_i(x)$ to be an orthonormal basis of $4-$component Dirac spinors (for example it can be constructed out of the $u^i(p)$ and $v^i(p)$ in an obvious way) we can expand ψ and $\bar{\psi}$ as $\psi=\sum_i\theta_i\chi_i(x)$ and $\bar{\psi}_i=\sum_i\theta_i^+\bar{\chi}_i$ respectively. The coefficients θ_i and θ_i^+ must then be complex Grassmann numbers. The measure appearing in the above integral is therefore

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \prod_i\mathcal{D}\theta_i^+\mathcal{D}\theta_i \tag{9.55}$$

The path integral $Z[\eta,\bar{\eta}]$ is the generating functional of all correlation functions of the fields ψ and $\bar{\psi}$. Indeed we have

$$\begin{aligned} < \psi_{\alpha_1}(x_1)...\psi_{\alpha_n}(x_n)\bar{\psi}_{\beta_1}(y_1)...\bar{\psi}_{\beta_n}(y_n) > &\equiv \frac{\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\ \psi_{\alpha_1}(x_1)...\psi_{\alpha_n}(x_n)\bar{\psi}_{\beta_1}(y_1)...\bar{\psi}_{\beta_n}(y_n)\exp\frac{i}{\hbar}S_0[\psi,\bar{\psi}]}{\int\mathcal{D}\bar{\psi}\mathcal{D}\psi\ \exp\frac{i}{\hbar}S_0[\psi,\bar{\psi}]}. \\ &= \bigg(\frac{\hbar^{2n}}{Z[\eta,\bar{\eta}]}\frac{\delta^{2n}Z[\eta,\bar{\eta}]}{\delta\bar{\eta}_{\alpha_1}(x_1)...\delta\bar{\eta}_{\alpha_1}(x_1)\delta\eta_{\beta_1}(y_1)...\delta\eta_{\beta_1}(y_1)}\bigg)_{\eta=\bar{\eta}=0}. \end{aligned} \tag{9.56}$$

For example the 2−point function is given by

$$< \psi_\alpha(x)\bar{\psi}_\beta(y) > \equiv \left(\frac{\hbar^2}{Z[\eta,\bar{\eta}]}\frac{\delta^2 Z[\eta,\bar{\eta}]}{\delta\bar{\eta}_\alpha(x)\delta\eta_\beta(y)}\right)_{\eta=\bar{\eta}=0}. \quad (9.57)$$

However by comparing the path integral $Z[\eta,\bar{\eta}]$ with the path integral (9.42) we can make the identification

$$M_{ij} \longrightarrow -\frac{i}{\hbar}(i\hbar c\gamma^\mu\partial_\mu - mc^2)_{\alpha\beta}\delta^4(x-y)\ ,\ \eta_i \longrightarrow -\frac{i}{\hbar}\eta_\alpha\ ,\ \eta_i^+ \longrightarrow -\frac{i}{\hbar}\bar{\eta}_\alpha. \quad (9.58)$$

We define

$$\begin{aligned} M_{\alpha\beta}(x,y) &= -\frac{i}{\hbar}(i\hbar c\gamma^\mu\partial_\mu - mc^2)_{\alpha\beta}\delta^4(x-y) \\ &= -\frac{ic}{\hbar}\int\frac{d^4p}{(2\pi\hbar)^4}(\gamma^\mu p_\mu - mc)_{\alpha\beta}e^{-\frac{i}{\hbar}p(x-y)} \\ &= \frac{c}{i\hbar}\int\frac{d^4p}{(2\pi\hbar)^4}\left(\frac{p^2-m^2c^2}{\gamma^\mu p_\mu + mc}\right)_{\alpha\beta}e^{-\frac{i}{\hbar}p(x-y)}. \end{aligned} \quad (9.59)$$

By using equation (9.42) we can deduce immediately the value of the path integral $Z[\eta,\bar{\eta}]$. We find

$$Z[\eta,\bar{\eta}] = \det M\exp\bigg(-\frac{1}{\hbar^2}\int d^4x\int d^4y\bar{\eta}_\alpha(x)M^{-1}_{\alpha\beta}(x,y)\eta_\beta(y)\bigg). \quad (9.60)$$

Hence the electron propagator is

$$< \psi_\alpha(x)\bar{\psi}_\beta(y) >= M^{-1}_{\alpha\beta}(x,y). \quad (9.61)$$

From the form of the Laplacian (9.59) we get immediately the propagator (including also an appropriate Feynman prescription)

$$< \psi_\alpha(x)\bar{\psi}_\beta(y) > = i\frac{\hbar}{c}\int\frac{d^4p}{(2\pi\hbar)^4}\frac{(\gamma^\mu p_\mu + mc)_{\alpha\beta}}{p^2-m^2c^2+i\epsilon}e^{-\frac{i}{\hbar}p(x-y)}. \quad (9.62)$$

9.2 Free Abelian Vector Field

9.2.1 Maxwell's Action

The electric and magnetic fields $\vec{E}$ and $\vec{B}$ generated by a charge density ρ and a current density $\vec{J}$ are given by the Maxwell's equations written in the Heaviside-Lorentz system as

$$\vec{\nabla}\vec{E} = \rho\ ,\ \text{Gauss}'\text{ s Law}. \quad (9.63)$$

$$\vec{\nabla}\vec{B} = 0\ ,\ \text{No} - \text{Magnetic Monopole Law}. \quad (9.64)$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}\ ,\ \text{Faraday}'\text{ s Law}. \quad (9.65)$$

$$\vec{\nabla}\times\vec{B} = \frac{1}{c}(\vec{J} + \frac{\partial\vec{E}}{\partial t})\ ,\ \text{Ampere} - \text{Maxwell}'\text{ s Law}. \quad (9.66)$$

The Lorentz force law expresses the force exerted on a charge q moving with a velocity $\vec{u}$ in the presence of an electric and magnetic fields $\vec{E}$ and $\vec{B}$. This is given by

$$\vec{F} = q(\vec{E} + \frac{1}{c}\vec{u} \times \vec{B}). \quad (9.67)$$

The continuity equation expresses local conservation of the electric charge. It reads

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\vec{J} = 0. \quad (9.68)$$

The so-called field strength tensor is a second-rank antisymmetric tensor $F_{\mu\nu}$ defined by

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \quad (9.69)$$

The dual field strength tensor is also a second-rank antisymmetric tensor $\tilde{F}_{\mu\nu}$ defined by

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}. \quad (9.70)$$

In terms of $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ Maxwell's equations will take the form

$$\partial_\mu F^{\mu\nu} = \frac{1}{c}J^\nu \ , \ \partial_\mu \tilde{F}^{\mu\nu} = 0. \quad (9.71)$$

The 4−vector current density J^μ is given by $J^\mu = (c\rho, J_x, J_y, J_z)$. The first equation yields Gauss's and Ampere-Maxwell's laws whereas the second equation yields Maxwell's third equation $\vec{\nabla}\vec{B} = 0$ and Faraday's law. The continuity equation and the Lorentz force law respectively can be rewritten in the covariant forms

$$\partial_\mu J^\mu = 0. \quad (9.72)$$

$$K^\mu = \frac{q}{c}\frac{dx_\nu}{d\tau}F^{\mu\nu}. \quad (9.73)$$

The electric and magnetic fields $\vec{E}$ and $\vec{B}$ can be expressed in terms of a scalar potential V and a vector potential $\vec{A}$ as

$$\vec{B} = \vec{\nabla} \times \vec{A}. \quad (9.74)$$

$$\vec{E} = -\frac{1}{c}(\vec{\nabla}V + \frac{\partial \vec{A}}{\partial t}). \quad (9.75)$$

We construct the 4−vector potential A^μ as

$$A^\mu = (V/c, \vec{A}). \quad (9.76)$$

The field tensor $F_{\mu\nu}$ can be rewritten in terms of A_μ as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{9.77}$$

This equation is actually equivalent to the two equations (9.74) and (9.75). The homogeneous Maxwell's equation $\partial_\mu \tilde{F}^{\mu\nu} = 0$ is automatically solved by this ansatz. The inhomogeneous Maxwell's equation $\partial_\mu F^{\mu\nu} = J^\nu/c$ becomes

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{1}{c} J^\nu. \tag{9.78}$$

These equations of motion should be derived from a local Lagrangian density $\mathcal{L}$, i.e. a Lagrangian which depends only on the fields and their first derivatives at the point $\vec{x}$. Indeed it can be easily proven that the above equations of motion are the Euler-Lagrange equations of motion corresponding to the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu. \tag{9.79}$$

The free Maxwell's action is

$$S_0[A] = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu}. \tag{9.80}$$

The total Maxwell's action will include a non-zero source and is given by

$$S[A] = -\frac{1}{4} \int d^4x F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} \int d^4x J_\mu A^\mu. \tag{9.81}$$

9.2.2 Gauge Invariance and Canonical Quantization

We have a gauge freedom in choosing A^μ given by local gauge transformations of the form (with λ any scalar function)

$$A^\mu \longrightarrow A^{'\mu} = A^\mu + \partial^\mu \Lambda. \tag{9.82}$$

Indeed under this transformation we have

$$F^{\mu\nu} \longrightarrow F^{'\mu\nu} = F^{\mu\nu}. \tag{9.83}$$

These local gauge transformations form a (gauge) group. In this case the group is just the abelian $U(1)$ unitary group. The invariance of the theory under these transformations is termed a gauge invariance. The $4-$vector potential A^μ is called a gauge potential or a gauge field. We make use of the invariance under gauge transformations by working with a gauge potential A^μ which satisfies some extra conditions. This procedure is known as gauge fixing. Some of the gauge conditions so often used are

$$\partial_\mu A^\mu = 0 \text{ , Lorentz Gauge.} \tag{9.84}$$

$$\partial_i A^i = 0 \text{ , Coulomb Gauge.} \tag{9.85}$$

$$A^0 = 0 \text{ , Temporal Gauge.} \tag{9.86}$$

$$A^3 = 0 \ , \ \text{Axial Gauge.} \tag{9.87}$$

$$A^0 + A^1 = 0 \ , \ \text{Light Cone Gauge.} \tag{9.88}$$

The form of the equations of motion (9.78) strongly suggest we impose the Lorentz condition. In the Lorentz gauge the equations of motion (9.78) become

$$\partial_\mu \partial^\mu A^\nu = \frac{1}{c} J^\nu. \tag{9.89}$$

Clearly we still have a gauge freedom $A^\mu \longrightarrow A^{'\mu} = A^\mu + \partial^\mu \phi$ where $\partial_\mu \partial^\mu \phi = 0$. In other words if A^μ satisfies the Lorentz gauge $\partial_\mu A^\mu = 0$ then $A^{'\mu}$ will also satisfy the Lorentz gauge, i.e. $\partial_\mu A^{'\mu} = 0$ iff $\partial_\mu \partial^\mu \phi = 0$. This residual gauge symmetry can be fixed by imposing another condition such as the temporal gauge $A^0 = 0$. We have therefore 2 constraints imposed on the components of the gauge potential A^μ which means that only two of them are really independent. The underlying mechanism for the reduction of the number of degrees of freedom is actually more complicated than this simple counting.

We incorporate the Lorentz condition via a Lagrange multiplier ζ, i.e. we add to the Maxwell's Lagrangian density a term proportional to $(\partial^\mu A_\mu)^2$ in order to obtain a gauge-fixed Lagrangian density, viz

$$\mathcal{L}_\zeta = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \zeta (\partial^\mu A_\mu)^2 - \frac{1}{c} J_\mu A^\mu. \tag{9.90}$$

The added extra term is known as a gauge-fixing term. The conjugate fields are

$$\pi_0 = \frac{\delta \mathcal{L}_\zeta}{\delta \partial_t A_0} = -\frac{\zeta}{c} (\partial_0 A_0 - \partial_i A_i). \tag{9.91}$$

$$\pi_i = \frac{\delta \mathcal{L}_\zeta}{\delta \partial_t A_i} = \frac{1}{c} (\partial_0 A_i - \partial_i A_0). \tag{9.92}$$

We remark that in the limit $\zeta \longrightarrow 0$ the conjugate field π_0 vanishes and as a consequence canonical quantization becomes impossible. The source of the problem is gauge invariance which characterize the limit $\zeta \longrightarrow 0$. For $\zeta \neq 0$ canonical quantization (although a very involved exercise) can be carried out consistently. We will not do this exercise here but only quote the result for the 2−point function. The propagator of the photon field in a general gauge ζ is given by the formula (with $\hbar = c = 1$)

$$\begin{aligned} iD_F^{\mu\nu}(x-y) &= <0|T\bigg(\hat{A}^\mu_{\rm in}(x)\hat{A}^\nu_{\rm in}(y)\bigg)|0> \\ &= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 + i\epsilon}\bigg(- \eta^{\mu\nu} + (1 - \frac{1}{\zeta})\frac{p^\mu p^\nu}{p^2}\bigg) \exp(-ip(x-y)). \end{aligned} \tag{9.93}$$

In the following we will give a derivation of this fundamental result based on the path integral formalism.

9.2.3 Path Integral Quantization and the Faddeev-Popov Method

The starting point is to posit that the path integral of an Abelian vector field A^μ in the presence of a source J^μ is given by analogy with the scalar field by the functional integral (we set $\hbar = c = 1$)

$$\begin{aligned} Z[J] &= \int \prod_\mu \mathcal{D}A_\mu \exp iS[A] \\ &= \int \prod_\mu \mathcal{D}A_\mu \exp\bigg(-\frac{i}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} - i\int d^4x J_\mu A^\mu\bigg). \end{aligned} \tag{9.94}$$

This is the generating functional of all correlation functions of the field $A^\mu(x)$. This is clear from the result

$$\begin{aligned} < A^{\mu_1}(x_1)....A^{\mu_2}(x_2) > &\equiv \frac{\int \prod_\mu \mathcal{D}A_\mu A^{\mu_1}(x_1)....A^{\mu_2}(x_2) \exp iS_0[A]}{\int \prod_\mu \mathcal{D}A_\mu \exp iS_0[A]} \\ &= \bigg(\frac{i^n}{Z[J]}\frac{\delta^n Z[J]}{\delta J_{\mu_1}(x_1)...\delta J_{\mu_n}(x_n)}\bigg)_{J=0}. \end{aligned} \tag{9.95}$$

The Maxwell's action can be rewritten as

$$\begin{aligned} S_0[A] &= -\frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} \\ &= \frac{1}{2}\int d^4x A_\nu(\partial_\mu\partial^\mu\eta^{\nu\lambda} - \partial^\nu\partial^\lambda)A_\lambda. \end{aligned} \tag{9.96}$$

We Fourier transform $A_\mu(x)$ as

$$A_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\tilde{A}_\mu(k)e^{ikx}. \tag{9.97}$$

Then

$$S_0[A] = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\tilde{A}_\nu(k)(-k^2\eta^{\nu\lambda} + k^\nu k^\lambda)\tilde{A}_\lambda(-k). \tag{9.98}$$

We observe that the action is 0 for any configuration of the form $\tilde{A}_\mu(k) = k_\mu f(k)$. Thus we conclude that the so-called pure gauge configurations given by $A_\mu(x) = \partial_\mu\Lambda(x)$ are zero modes of the Laplacian which means in particular that the Laplacian can not be inverted. More importantly this means that in the path integral $Z[J]$ these zero modes (which are equivalent to $A_\mu = 0$) are not damped and thus the path integral is divergent. This happens for any other configuration A_μ. Indeed all gauge equivalent configurations $A_\mu^\Lambda = A_\mu + \partial_\mu\Lambda$ have the same probability amplitude and as a consequence the sum of their contributions to the path integral will be proportional to the divergent integral over the Abelian $U(1)$ gauge group which is here the integral over Λ. The problem lies therefore in gauge invariance which must be fixed in the path integral. This entails the selection of a single gauge configuration from each gauge orbit $A_\mu^\Lambda = A_\mu + \partial_\mu\Lambda$ as a representative and using it to compute the contribution of the orbit to the path integral.

In path integral quantization gauge fixing is done in an elegant and efficient way via the method of Faddeev and Popov. Let us say that we want to gauge fix by imposing the Lorentz condition $G(A) = \partial_\mu A^\mu - \omega = 0$. Clearly $G(A^\Lambda) = \partial_\mu A^\mu - \omega + \partial_\mu\partial^\mu\Lambda$ and thus

$$\int \mathcal{D}\Lambda\delta(G(A^\Lambda)) = \int \mathcal{D}\Lambda\delta(\partial_\mu A^\mu - \omega + \partial_\mu\partial_\mu\Lambda). \tag{9.99}$$

By performing the change of variables $\Lambda \longrightarrow \Lambda^{'} = \partial_\mu\partial^\mu\Lambda$ and using the fact that $\mathcal{D}\Lambda^{'} = |(\partial\Lambda^{'}/\partial\Lambda)|\mathcal{D}\Lambda = \det(\partial_\mu\partial^\mu)\mathcal{D}\Lambda$ we get

$$\int \mathcal{D}\Lambda\delta(G(A^\Lambda)) = \int \frac{\mathcal{D}\Lambda^{'}}{\det(\partial_\mu\partial^\mu)}\delta(\partial_\mu A^\mu - \omega + \Lambda^{'}) = \frac{1}{\det(\partial_\mu\partial^\mu)}. \quad (9.100)$$

This can be put in the form

$$\int \mathcal{D}\Lambda\delta\Big(G(A^\Lambda)\Big)\det\Big(\frac{\delta G(A^\Lambda)}{\delta\Lambda}\Big) = 1. \quad (9.101)$$

This is the generalization of

$$\int \prod_i da_i \delta^{(n)}\Big(\vec{g}(\vec{a})\Big)\det\Big(\frac{\partial g_i}{\partial a_j}\Big) = 1. \quad (9.102)$$

We insert 1 in the form (9.101) in the path integral as follows

$$\begin{aligned} Z[J] &= \int \prod_\mu \mathcal{D}A_\mu \int \mathcal{D}\Lambda\delta\Big(G(A^\Lambda)\Big)\det\Big(\frac{\delta G(A^\Lambda)}{\delta\Lambda}\Big)\exp iS[A] \\ &= \det(\partial_\mu\partial^\mu)\int \mathcal{D}\Lambda \int \prod_\mu \mathcal{D}A_\mu\delta\Big(G(A^\Lambda)\Big)\exp iS[A] \\ &= \det(\partial_\mu\partial^\mu)\int \mathcal{D}\Lambda \int \prod_\mu \mathcal{D}A^\Lambda_\mu\delta\Big(G(A^\Lambda)\Big)\exp iS[A^\Lambda]. \end{aligned} \quad (9.103)$$

Now we shift the integration varaible as $A^\Lambda_\mu \longrightarrow A_\mu$. We observe immediately that the integral over the $U(1)$ gauge group decouples, viz

$$\begin{aligned} Z[J] &= \det(\partial_\mu\partial^\mu)(\int \mathcal{D}\Lambda)\int \prod_\mu \mathcal{D}A_\mu\delta\Big(G(A)\Big)\exp iS[A] \\ &= \det(\partial_\mu\partial^\mu)(\int \mathcal{D}\Lambda)\int \prod_\mu \mathcal{D}A_\mu\delta\Big(\partial_\mu A^\mu - \omega\Big)\exp iS[A]. \end{aligned} \quad (9.104)$$

Next we want to set $\omega = 0$. We do this in a smooth way by integrating both sides of the above equation against a Gaussian weighting function centered around $\omega = 0$, viz

$$\begin{aligned} \int \mathcal{D}\omega \exp(-i\int d^4x\frac{\omega^2}{2\xi})Z[J] &= \det(\partial_\mu\partial^\mu)(\int \mathcal{D}\Lambda)\int \prod_\mu \mathcal{D}A_\mu \int \mathcal{D}\omega \exp(-i\int d^4x\frac{\omega^2}{2\xi})\delta\Big(\partial_\mu A^\mu - \omega\Big)\exp iS[A] \\ &= \det(\partial_\mu\partial^\mu)(\int \mathcal{D}\Lambda)\int \prod_\mu \mathcal{D}A_\mu \exp(-i\int d^4x\frac{(\partial_\mu A^\mu)^2}{2\xi})\exp iS[A]. \end{aligned} \quad (9.105)$$

Hence

$$\begin{aligned} Z[J] &= \mathcal{N}\int \prod_\mu \mathcal{D}A_\mu \exp(-i\int d^4x\frac{(\partial_\mu A^\mu)^2}{2\xi})\exp iS[A] \\ &= \mathcal{N}\int \prod_\mu \mathcal{D}A_\mu \exp\Big(-i\int d^4x\frac{(\partial_\mu A^\mu)^2}{2\xi} - \frac{i}{4}\int d^4xF_{\mu\nu}F^{\mu\nu} - i\int d^4xJ_\mu A^\mu\Big) \end{aligned} \quad (9.106)$$

Therefore the end result is the addition of a term proportional to $(\partial_\mu A^\mu)^2$ to the action which fixes gauge invariance to a sufficient degree.

9.2.4 The Photon Propagator

The above path integral can also be put in the form

$$Z[J] = \mathcal{N}\int\prod_\mu \mathcal{D}A_\mu \exp\bigg(\frac{i}{2}\int d^4x A_\nu\bigg(\partial_\mu\partial^\mu\eta^{\nu\lambda}+(\frac{1}{\xi}-1)\partial^\nu\partial^\lambda\bigg)A_\lambda - i\int d^4x J_\mu A^\mu\bigg). \tag{9.107}$$

We use the result

$$\begin{aligned}\int\prod_{i=1}^n dz_i e^{-z_iM_{ij}z_j-z_ij_i} &= e^{\frac{1}{4}j_iM^{-1}_{ij}j_j}\int\prod_{i=1}^n dz_i e^{-(z_i+\frac{1}{2}j_kM^{-1}_{ki})M_{ij}(z_j+\frac{1}{2}M^{-1}_{jk}j_k)}\\ &= e^{\frac{1}{4}j_iM^{-1}_{ij}j_j}\int\prod_{i=1}^n dy_i e^{-y_iM_{ij}y_j}\\ &= e^{\frac{1}{4}j_iM^{-1}_{ij}j_j}\int\prod_{i=1}^n dx_i e^{-x_im_ix_j}\\ &= e^{\frac{1}{4}j_iM^{-1}_{ij}j_j}\prod_{i=1}^n\sqrt{\frac{\pi}{m_i}}\\ &= e^{\frac{1}{4}j_iM^{-1}_{ij}j_j}\pi^{\frac{n}{2}}(\det M)^{-\frac{1}{2}}.\end{aligned} \tag{9.108}$$

By comparison we have

$$M_{ij}\longrightarrow -\frac{i}{2}\bigg(\partial_\mu\partial^\mu\eta^{\nu\lambda}+(\frac{1}{\xi}-1)\partial^\nu\partial^\lambda\bigg)\delta^4(x-y)\ ,\ j_i\longrightarrow iJ_\mu. \tag{9.109}$$

We define

$$\begin{aligned}M^{\nu\lambda}(x,y) &= -\frac{i}{2}\bigg(\partial_\mu\partial^\mu\eta^{\nu\lambda}+(\frac{1}{\xi}-1)\partial^\nu\partial^\lambda\bigg)\delta^4(x-y)\\ &= \frac{i}{2}\int\frac{d^4k}{(2\pi)^4}(k^2\eta^{\nu\lambda}+(\frac{1}{\xi}-1)k^\nu k^\lambda)e^{ik(x-y)}.\end{aligned} \tag{9.110}$$

Hence our path integral is actually given by

$$\begin{aligned}Z[J] &= \mathcal{N}\pi^{\frac{n}{2}}(\det M)^{-\frac{1}{2}}\exp\bigg(-\frac{1}{4}\int d^4x\int d^4y J^\mu(x)M^{-1}_{\mu\nu}(x,y)J^\nu(y)\bigg)\\ &= \mathcal{N}^{'}\exp\bigg(-\frac{1}{4}\int d^4x\int d^4y J^\mu(x)M^{-1}_{\mu\nu}(x,y)J^\nu(y)\bigg).\end{aligned} \tag{9.111}$$

The inverse of the Laplacian is defined by

$$\int d^4y M^{\nu\lambda}(x,y)M^{-1}_{\lambda\mu}(y,z) = \eta^\nu_\mu\delta^4(x-y). \tag{9.112}$$

For example the $2-$point function is given by

$$\begin{aligned}< A_\mu(x)A_\nu(y) > &= \bigg(\frac{i^2}{Z[J]}\frac{\delta^2Z[J]}{\delta J^\mu(x)\delta J^\nu(y)}\bigg)_{J=0}\\ &= \frac{1}{2}M^{-1}_{\mu\nu}(x,y).\end{aligned} \tag{9.113}$$

It is not difficult to check that the inverse is given by

$$M^{-1}_{\mu\nu}(x,y) = -2i\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i\epsilon}(\eta_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2})e^{ik(x-y)}. \tag{9.114}$$

Hence the propagator is

$$< A_\mu(x)A_\nu(y) > = \int\frac{d^4k}{(2\pi)^4}\frac{-i}{k^2+i\epsilon}(\eta_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2})e^{ik(x-y)}. \tag{9.115}$$

The most important gauges we will make use of are the Feynman gauge $\xi = 1$ and the Landau gauge $\xi = 0$.

9.3 Gauge Interactions

9.3.1 Spinor and Scalar Electrodynamics: Minimal Coupling

The actions of a free Dirac field and a free Abelian vector field in the presence of sources are given by (with $\hbar = c = 1$)

$$S[\psi,\bar{\psi}] = \int d^4x\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \int d^4x(\bar{\psi}\eta+\bar{\eta}\psi). \tag{9.116}$$

$$S[A] = -\frac{1}{4}\int d^4xF_{\mu\nu}F^{\mu\nu} - \int d^4xJ_\mu A^\mu. \tag{9.117}$$

The action $S[A]$ gives Maxwell's equations with a vector current source equal to the external vector current J^μ. As we have already discussed the Maxwell's action ($J^\mu = 0$) is invariant under the gauge symmetry transformations

$$A_\mu \longrightarrow A^\Lambda_\mu = A_\mu + \partial_\mu\Lambda. \tag{9.118}$$

The action $S[A]$ is also invariant under these gauge transformations provided the vector current J^μ is conserved, viz $\partial_\mu J^\mu = 0$.

The action describing the interaction of a photon which is described by the Abelian vector field A^μ and an electron described by the Dirac field ψ must be given by

$$S[\psi,\bar{\psi},A] = S[\psi,\bar{\psi}] + S[A] - \int d^4xj_\mu A^\mu. \tag{9.119}$$

The interaction term $-j_\mu A^\mu$ is dictated by the requirement that this action must also give Maxwell's equations with a vector current source equal now to the sum of the external vector current J^μ and the internal vector current j^μ. The internal vector current j^μ must clearly depend on the spinor fields ψ and $\bar{\psi}$ and furthermore it must be conserved.

In order to ensure that j^μ is conserved we will identify it with the Noether's current associated with the local symmetry transformations

$$\psi \longrightarrow \psi^\Lambda = \exp(-ie\Lambda)\psi\ ,\ \bar{\psi} \longrightarrow \bar{\psi}^\Lambda = \bar{\psi}\exp(ie\Lambda). \tag{9.120}$$

Indeed under these local transformations the Dirac action transforms as

$$S[\psi,\bar{\psi}] \longrightarrow S[\psi^\Lambda,\bar{\psi}^\Lambda] = S[\psi,\bar{\psi}] - e\int d^4x\Lambda\partial_\mu(\bar{\psi}\gamma^\mu\psi). \tag{9.121}$$

The internal current j^μ will be identified with $e\bar{\psi}\gamma^\mu\psi$, viz

$$j^\mu = e\bar{\psi}\gamma^\mu\psi. \tag{9.122}$$

This current is clearly invariant under the local transformations (9.120). By performing the local transformations (9.118) and (9.120) simultaneously, i.e. by considering the transformations (9.120) a part of gauge symmetry, we obtain the invariance of the action $S[\psi,\bar{\psi},A]$. The action remains invariant under the combined transformations (9.118) and (9.120) if we also include a conserved external vector current source J^μ and Grassmann spinor sources η and $\bar{\eta}$ which transform under gauge transformations as the dynamical Dirac spinors, viz $\eta \longrightarrow \eta^\Lambda = \exp(-ie\Lambda)\eta$ and $\bar{\eta} \longrightarrow \bar{\eta}^\Lambda = \bar{\eta}\exp(ie\Lambda)$. We write this result as (with $S_{\eta,\bar{\eta},J}[\psi,\bar{\psi},A] \equiv S[\psi,\bar{\psi},A]$)

$$S_{\eta,\bar{\eta},J}[\psi,\bar{\psi},A] \longrightarrow S_{\eta^\Lambda,\bar{\eta}^\Lambda,J}[\psi^\Lambda,\bar{\psi}^\Lambda,A^\Lambda] = S_{\eta,\bar{\eta},J}[\psi,\bar{\psi},A]. \tag{9.123}$$

The action $S[\psi,\bar{\psi},A]$ with the corresponding path integral define quantum spinor electrodynamics which is the simplest and most important gauge interaction. The action $S[\psi,\bar{\psi},A]$ can also be put in the form

$$S[\psi,\bar{\psi},A] \;=\; \int d^4x\bar{\psi}(i\gamma^\mu\nabla_\mu - m)\psi - \frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} + \int d^4x(\bar{\psi}\eta + \bar{\eta}\psi) - \int d^4x J_\mu A^\mu. \tag{9.124}$$

The derivative operator ∇_μ which is called the covariant derivative is given by

$$\nabla_\mu = \partial_\mu + ieA_\mu. \tag{9.125}$$

The action $S[\psi,\bar{\psi},A]$ could have been obtained from the free action $S[\psi,\bar{\psi}] + S[A]$ by making the simple replacement $\partial_\mu \longrightarrow \nabla_\mu$ which is known as the principle of minimal coupling. In flat Minkowski spacetime this prescription always works and it allows us to obtain the most minimal consistent interaction starting from a free theory.

As another example consider complex quartic scalar field given by the action

$$S[\phi,\phi^+] = \int d^4x\bigg(\partial_\mu\phi^+\partial^\mu\phi - m^2\phi^+\phi - \frac{g}{4}(\phi^+\phi)^2\bigg). \tag{9.126}$$

By applying the principle of minimal coupling we replace the ordinary ∂_μ by the covariant derivative $\nabla_\mu = \partial_\mu + ieA_\mu$ and then add the Maxwell's action. We get immediately the gauge invariant action

$$S[\phi,\phi^+,A] = \int d^4x\bigg(\nabla_\mu\phi^+\nabla^\mu\phi - m^2\phi^+\phi - \frac{g}{4}(\phi^+\phi)^2\bigg) - \frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu}. \tag{9.127}$$

This is indeed invariant under the local gauge symmetry transformations acting on A^μ, ϕ and ϕ^+ as

$$A_\mu \longrightarrow A_\mu^\Lambda = A_\mu + \partial_\mu\Lambda \;,\; \phi \longrightarrow \exp(-ie\Lambda)\phi \;,\; \phi^+ \longrightarrow \phi^+\exp(ie\Lambda). \tag{9.128}$$

It is not difficult to add vector and scalar sources to the action (9.126) without spoiling gauge invariance. The action (9.126) with the corresponding path integral define quantum scalar electrodynamics which describes the interaction of the photon A^μ with a charged scalar particle ϕ whose electric charge is $q = -e$.

9.3.2 The Geometry of $U(1)$ Gauge Invariance

The set of all gauge transformations which leave invariant the actions of spinor and scalar electrodynamics form a group called $U(1)$ and as a consequence spinor and scalar electrodynamics are said to be invariant under local $U(1)$ gauge symmetry. The group $U(1)$ is the group of 1×1 unitary matrices given by

$$U(1)=\{g=\exp(-ie\Lambda),\forall\Lambda\}. \tag{9.129}$$

In order to be able to generalize the local $U(1)$ gauge symmetry to local gauge symmetries based on other groups we will exhibit in this section the geometrical content of the gauge invariance of spinor electrodynamics. The starting point is the free Dirac action given by

$$S=\int d^4x\bar{\psi}(i\gamma^{\mu}\partial_{\mu}-m)\psi. \tag{9.130}$$

This is invariant under the global transformations

$$\psi\longrightarrow e^{-ie\Lambda}\psi\ ,\ \bar{\psi}\longrightarrow\bar{\psi}e^{ie\Lambda}. \tag{9.131}$$

We demand next that the theory must be invariant under the local transformations obtained by allowing Λ to be a function of x in the above equations, viz

$$\psi\longrightarrow\psi^g=g(x)\psi\ ,\ \bar{\psi}\longrightarrow\bar{\psi}^g=\bar{\psi}g^+(x). \tag{9.132}$$

The fermion mass term is trivially still invariant under these local $U(1)$ gauge transformations,i.e.

$$\bar{\psi}\psi\longrightarrow\bar{\psi}^g\psi^g=\bar{\psi}\psi. \tag{9.133}$$

The kinetic term is not so easy. The difficulty clearly lies with the derivative of the field which transforms under the local $U(1)$ gauge transformations in a complicated way. To appreciate more this difficulty let us consider the derivative of the field ψ in the direction defined by the vector n^{μ} which is given by

$$n^{\mu}\partial_{\mu}\psi=\lim\frac{\big[\psi(x+\epsilon n)-\psi(x)\big]}{\epsilon}\ ,\ \epsilon\longrightarrow 0. \tag{9.134}$$

The two fields $\psi(x+\epsilon n)$ and $\psi(x)$ transform under the local $U(1)$ symmetry with different phases given by $g(x+\epsilon n)$ and $g(x)$ respectively. The point is that the fields $\psi(x+\epsilon n)$ and $\psi(x)$ since they are evaluated at different spacetime points $x+\epsilon n$ and x they transform independently under the local $U(1)$ symmetry. As a consequence the derivative $n^{\mu}\partial_{\mu}\psi$ has no intrinsic geometrical meaning since it involves the comparison of fields at different spacetime points which transform independently of each other under $U(1)$.

In order to be able to compare fields $\psi(y)$ and $\psi(x)$ at different spacetime points y and x we need to introduce a new object which connects the two points y and x and which allows a meaningful comparison between $\psi(y)$ and $\psi(x)$. We introduce a comparator field $U(y,x)$ which connects the points y and x along a particular path with the properties:

- The comparator field $U(y,x)$ must be an element of the gauge group $U(1)$ and thus $U(y,x)$ is a pure phase, viz

$$U(y,x)=\exp(-ie\phi(y,x))\in U(1). \tag{9.135}$$

- Clearly we must have

$$U(x,x) = 1 \Leftrightarrow \phi(x,x) = 0. \tag{9.136}$$

- Under the $U(1)$ gauge transformations $\psi(x) \longrightarrow \psi^g(x) = g(x)\psi(x)$ and $\psi(y) \longrightarrow \psi^g(y) = g(y)\psi(y)$ the comparator field transforms as

$$U(y,x) \longrightarrow U^g(y,x) = g(y)U(y,x)g^+(x). \tag{9.137}$$

- We impose the restriction

$$U(y,x)^+ = U(x,y). \tag{9.138}$$

The third property means that the product $U(y,x)\psi(x)$ transforms as

$$U(y,x)\psi(x) \longrightarrow U^g(y,x)\psi^g(x) = g(y)U(y,x)g^+(x)g(x)\psi(x) = g(y)U(y,x)\psi(x). \tag{9.139}$$

Thus $U(y,x)\psi(x)$ transforms under the $U(1)$ gauge group with the same group element as the field $\psi(y)$. This means in particular that the comparison between $U(y,x)\psi(x)$ and $\psi(y)$ is meaningful. We are led therefore to define a new derivative of the field ψ in the direction defined by the vector n^μ by

$$n^\mu \nabla_\mu \psi = \lim \frac{\big[\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)\big]}{\epsilon} \ , \ \epsilon \longrightarrow 0. \tag{9.140}$$

This is known as the covariant derivative of ψ in the direction n^μ.

The second property $U(x,x) = 1$ allows us to conclude that if the point y is infinitesimally close to the point x then we can expand $U(y,x)$ around 1. We can write for $y = x + \epsilon n$ the expansion

$$U(x+\epsilon n, x) = 1 - ie\epsilon n_\mu A^\mu(x) + O(\epsilon^2). \tag{9.141}$$

The coefficient of the displacement vector $y_\mu - x_\mu = \epsilon n_\mu$ is a new vector field A^μ which is precisely, as we will see shortly, the electromagnetic vector potential. The coupling e will, on the other hand, play the role of the electric charge. We compute immediately

$$\nabla_\mu \psi = (\partial_\mu + ieA_\mu)\psi. \tag{9.142}$$

Thus ∇_μ is indeed the covariant derivative introduced in the previous section.

By using the language of differential geometry we say that the vector field A_μ is a connection on a $U(1)$ fiber bundle over spacetime which defines the parallel transport of the field ψ from x to y. The parallel transported field $\psi_{\|}$ is defined by

$$\psi_{\|}(y) = U(y,x)\psi(x). \tag{9.143}$$

The third property with a comparator $U(y,x)$ with y infinitesimally close to x, for example $y = x + \epsilon n$, reads explicitly

$$1 - ie\epsilon n^\mu A_\mu(x) \longrightarrow 1 - ie\epsilon n^\mu A^g_\mu(x) = g(y)\Big(1 - ie\epsilon n^\mu A_\mu(x)\Big)g^+(x). \tag{9.144}$$

Equivalently we have

$$A^g_\mu = gA_\mu g^+ + \frac{i}{e}\partial_\mu g.g^+ \Leftrightarrow A^g_\mu = A_\mu + \partial_\mu\Lambda. \tag{9.145}$$

Again we find the gauge field transformation law considered in the previous section. For completeness we find the transformation law of the covariant derivative of the field ψ. We have

$$\begin{aligned}\nabla_\mu\psi = (\partial_\mu + ieA_\mu)\psi \longrightarrow (\nabla_\mu\psi)^g &= (\partial_\mu + ieA^g_\mu)\psi^g\\ &= (\partial_\mu + ieA_\mu + ie\partial_\mu\Lambda)(e^{-ie\Lambda}\psi)\\ &= e^{-ie\Lambda}(\partial_\mu + ieA_\mu)\psi\\ &= g(x)\nabla_\mu\psi.\end{aligned} \tag{9.146}$$

Thus the covariant derivative of the field transforms exactly in the same way as the field itself. This means in particular that the combination $\bar{\psi}i\gamma^\mu\nabla_\mu\psi$ is gauge invariant. In summary given the free Dirac action we can obtain a gauge invariant Dirac action by the simple substitution $\partial_\mu \longrightarrow \nabla_\mu$. This is the principle of minimal coupling discussed in the previous section. The gauge invariant Dirac action is

$$S = \int d^4x\bar{\psi}(i\gamma^\mu\nabla_\mu - m)\psi. \tag{9.147}$$

We need finally to construct a gauge invariant action which provides a kinetic term for the vector field A^μ. This can be done by integrating the comparator $U(y,x)$ along a closed loop. For $y = x + \epsilon n$ we write $U(y,x)$ up to the order ϵ^2 as

$$U(y,x) = 1 - ie\epsilon n_\mu A^\mu + i\epsilon^2 X + O(\epsilon^3). \tag{9.148}$$

The fourth fundamental property of $U(y,x)$ restricts the comparator so that $U(y,x)^+ = U(x,y)$. This leads immediately to the solution $X = -en_\mu n_\nu\partial^\nu A^\mu/2$. Thus

$$\begin{aligned}U(y,x) &= 1 - ie\epsilon n_\mu A^\mu - \frac{ie}{2}\epsilon^2 n_\mu n_\nu\partial^\nu A^\mu + O(\epsilon^3)\\ &= 1 - ie\epsilon n_\mu A^\mu(x + \frac{\epsilon}{2}n) + O(\epsilon^3)\\ &= \exp(-ie\epsilon n_\mu A^\mu(x + \frac{\epsilon}{2}n)).\end{aligned} \tag{9.149}$$

We consider now the group element $U(x)$ given by the product of the four comparators associated with the four sides of a small square in the $(1,2)-$plane. This is given by

$$U(x) = \mathrm{tr}U(x, x+\epsilon\hat{1})U(x+\epsilon\hat{1}, x+\epsilon\hat{1}+\epsilon\hat{2})U(x+\epsilon\hat{1}+\epsilon\hat{2}, x+\epsilon\hat{2})U(x+\epsilon\hat{2}, x). \tag{9.150}$$

This is called the Wilson loop associated with the square in question. The trace tr is of course trivial for a $U(1)$ gauge group. The Wilson loop $U(x)$ is locally invariant under the gauge group $U(1)$, i.e. under $U(1)$ gauge transformations the Wilson loop $U(x)$ behaves as

$$U(x) \longrightarrow U^g(x) = U(x). \tag{9.151}$$

The Wilson loop is the phase accumulated if we parallel transport the spinor field ψ from the point x around the square and back to the point x. This phase can be computed explicitly. Indeed we have

$$\begin{aligned} U(x) &= \exp\left(ie\epsilon\Big[A^1(x+\frac{\epsilon}{2}\hat{1})+A^2(x+\epsilon\hat{1}+\frac{\epsilon}{2}\hat{2})-A^1(x+\frac{\epsilon}{2}\hat{1}+\epsilon\hat{2})-A^2(x+\frac{\epsilon}{2}\hat{2})\Big]\right) \\ &= \exp(-ie\epsilon^2F_{12}) \\ &= 1-ie\epsilon^2F_{12}-\frac{e^2\epsilon^4}{2}F_{12}^2+... \end{aligned} \tag{9.152}$$

In the above equation $F_{12}=\partial_1A_2-\partial_2A_1$. We conclude that the field strength tensor $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ is locally gauge invariant under $U(1)$ transformations. This is precisely the electromagnetic field strength tensor considered in the previous section.

The field strength tensor $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ can also be obtained from the commutator of the two covariant derivatives ∇_μ and ∇_ν acting on the spinor field ψ. Indeed we have

$$[\nabla_\mu,\nabla_\nu]\psi=ie(\partial_\mu A_\nu-\partial_\nu A_\mu)\psi. \tag{9.153}$$

Thus under $U(1)$ gauge transformations we have the behavior

$$[\nabla_\mu,\nabla_\nu]\psi\longrightarrow g(x)[\nabla_\mu,\nabla_\nu]\psi. \tag{9.154}$$

In other words $[\nabla_\nu,\nabla_\nu]$ is not a differential operator and furthermore it is locally invariant under $U(1)$ gauge transformations. This shows in a slightly different way that the field strength tensor $F_{\mu\nu}$ is the fundamental structure which is locally invariant under $U(1)$ gauge transformations. The field strength tensor $F_{\mu\nu}$ can be given by the expressions

$$F_{\mu\nu}=\frac{1}{ie}[\nabla_\mu,\nabla_\nu]=(\partial_\mu A_\nu-\partial_\nu A_\mu). \tag{9.155}$$

In summary we can conclude that any function of the vector field A^μ which depends on the vector field only through the field strength tensor $F_{\mu\nu}$ will be locally invariant under $U(1)$ gauge transformations and thus can serve as an action functional. By appealing to the requirement of renormalizability the only renormalizable $U(1)$ gauge action in four dimensions (which also preserves P and T symmetries) is Maxwell's action which is quadratic in $F_{\mu\nu}$ and also quadratic in A^μ. We get then the pure gauge action

$$S=-\frac{1}{4}\int d^4xF_{\mu\nu}F^{\mu\nu}. \tag{9.156}$$

The total action of spinor electrodynamics is therefore given by

$$S=\int d^4x\bar{\psi}(i\gamma^\mu\nabla_\mu-m)\psi-\frac{1}{4}\int d^4xF_{\mu\nu}F^{\mu\nu}. \tag{9.157}$$

9.3.3 Generalization: $SU(N)$ Yang-Mills Theory

We can now immediately generalize the previous construction by replacing the Abelian gauge group $U(1)$ by a different gauge group G which will generically be non-Abelian, i.e the generators of the corresponding Lie algebra will not commute. In this chapter we will be interested in the gauge groups $G=SU(N)$. Naturally we will start with the first non-trivial non-Abelian gauge group $G=SU(2)$ which is the case considered originally by Yang and Mills.

The group $SU(2)$ is the group of 2×2 unitary matrices which have determinant equal 1. This is given by

$$SU(2)=\{u_{ab},a,b=1,...,2:u^+u=1,\mathrm{det}u=1\}. \tag{9.158}$$

The generators of $SU(2)$ are given by Pauli matrices given by

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9.159}$$

Thus any element of $SU(2)$ can be rewritten as

$$u = \exp(-ig\Lambda) \ , \ \Lambda = \sum_A \Lambda^A \frac{\sigma^A}{2}. \tag{9.160}$$

The group $SU(2)$ has therefore 3 gauge parameters Λ^A in contrast with the group $U(1)$ which has only a single parameter. These 3 gauge parameters correspond to three orthogonal symmetry motions which do not commute with each other. Equivalently the generators of the Lie algebra $su(2)$ of $SU(2)$ (consisting of the Pauli matrices) do not commute which is the reason why we say that the group $SU(2)$ is non-Abelian. The Pauli matrices satisfy the commutation relations

$$[\frac{\sigma^A}{2}, \frac{\sigma^B}{2}] = if_{ABC}\frac{\sigma^C}{2} \ , \ f_{ABC} = \epsilon_{ABC}. \tag{9.161}$$

The $SU(2)$ group element u will act on the Dirac spinor field ψ. Since u is a 2×2 matrix the spinor ψ must necessarily be a doublet with components ψ^a, $a = 1, 2$. The extra label a will be called the color index. We write

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}. \tag{9.162}$$

We say that ψ is in the fundamental representation of the group $SU(2)$. The action of an element $u \in SU(2)$ is given by

$$\psi^a \longrightarrow (\psi^u)^a = \sum_B u^{ab}\psi^b. \tag{9.163}$$

We start from the free Dirac action

$$S = \int d^4x \sum_a \bar{\psi}^a(i\gamma^\mu\partial_\mu - m)\psi^a. \tag{9.164}$$

Clearly this is invariant under global $SU(2)$ transformations, i.e. transformations g which do not depend on x. Local $SU(2)$ gauge transformations are obtained by letting g depend on x. Under local $SU(2)$ gauge transformations the mass term remains invariant whereas the kinetic term transforms in a complicated fashion as in the case of local $U(1)$ gauge transformations. Hence as in the $U(1)$ case we appeal to the principle of minimal coupling and replace the ordinary derivative $n^\mu\partial_\mu$ with the covariant derivative $n^\mu\nabla_\mu$ which is defined by

$$n^\mu\nabla_\mu\psi = \lim\frac{\big[\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)\big]}{\epsilon} \ , \ \epsilon \longrightarrow 0. \tag{9.165}$$

Since the spinor field ψ is a $2-$component object the comparator $U(y, x)$ must be a 2×2 matrix which transforms under local $SU(2)$ gauge transformations as

$$U(y, x) \longrightarrow U^g(y, x) = u(y)U(y, x)u^+(x). \tag{9.166}$$

In fact $U(y,x)$ is an element of $SU(2)$. We must again impose the condition that $U(x,x)=1$. Hence for an infinitesimal separation $y-x=\epsilon n$ we can expand $U(y,x)$ as

$$U(x+\epsilon n,x)=1-ig\epsilon n^{\mu}A_{\mu}^{A}(x)\frac{\sigma^{A}}{2}+O(\epsilon^{2}). \tag{9.167}$$

In other words we have three vector fields $A_{\mu}^{A}(x)$. They can be unified in a single object $A_{\mu}(x)$ defined by

$$A_{\mu}(x)=A_{\mu}^{A}(x)\frac{\sigma^{A}}{2}. \tag{9.168}$$

We will call $A_{\mu}(x)$ the $SU(2)$ gauge field whereas we will refer to $A_{\mu}^{A}(x)$ as the components of the $SU(2)$ gauge field. Since $A^{\mu}(x)$ is 2×2 matrix it will carry two color indices a and b in an obvious way. The components of the $SU(2)$ gauge field in the fundamental representation of $SU(2)$ are given by $A_{ab}^{\mu}(x)$. The color index is called the $SU(2)$ fundamental index whereas the index A carried by the components $A_{\mu}^{A}(x)$ is called the $SU(2)$ adjoint index. In fact $A_{\mu}^{A}(x)$ are called the components of the $SU(2)$ gauge field in the adjoint representation of $SU(2)$.

First by inserting the expansion $U(x+\epsilon n,x)=1-ig\epsilon n^{\mu}A_{\mu}^{A}(x)\sigma^{A}/2+O(\epsilon^{2})$ in the definition of the covariant derivative we obtain the result

$$\nabla_{\mu}\psi=(\partial_{\mu}+igA_{\mu}^{A}\frac{\sigma^{A}}{2})\psi. \tag{9.169}$$

The spinor $U(x+\epsilon n,x)\psi(x)$ is the parallel transport of the spinor ψ from the point x to the point $x+\epsilon n$ and thus by construction it must transform under local $SU(2)$ gauge transformations in the same way as the spinor $\psi(x+\epsilon n)$. Hence under local $SU(2)$ gauge transformations the covariant derivative is indeed covariant, viz

$$\nabla_{\mu}\psi\longrightarrow u(x)\nabla_{\mu}\psi. \tag{9.170}$$

Next by inserting the expansion $U(x+\epsilon n,x)=1-ig\epsilon n^{\mu}A_{\mu}^{A}(x)\sigma^{A}/2+O(\epsilon^{2})$ in the transformation law $U(y,x)\longrightarrow U^{g}(y,x)=u(y)U(y,x)u^{+}(x)$ we obtain the transformation law

$$A_{\mu}\longrightarrow A_{\mu}^{u}=uA_{\mu}u^{+}+\frac{i}{g}\partial_{\mu}u.u^{+}. \tag{9.171}$$

For infinitesimal $SU(2)$ transformations we have $u=1-ig\Lambda$. We get

$$A_{\mu}\longrightarrow A_{\mu}^{u}=A_{\mu}+\partial_{\mu}\Lambda+ig[A_{\mu},\Lambda]. \tag{9.172}$$

In terms of components we have

$$\begin{aligned}A_{\mu}^{C}\frac{\sigma^{C}}{2}\longrightarrow A_{\mu}^{uC}\frac{\sigma^{C}}{2} &= A_{\mu}^{C}\frac{\sigma^{C}}{2}+\partial_{\mu}\Lambda^{C}\frac{\sigma^{C}}{2}+ig[A_{\mu}^{A}\frac{\sigma^{A}}{2},\Lambda^{B}\frac{\sigma^{B}}{2}]\\ &= \left(A_{\mu}^{C}+\partial_{\mu}\Lambda^{C}+igA_{\mu}^{A}\Lambda^{B}if_{ABC}\right)\frac{\sigma^{C}}{2}.\end{aligned} \tag{9.173}$$

In other words

$$A_{\mu}^{uC} = A_{\mu}^{C}+\partial_{\mu}\Lambda^{C}-gf_{ABC}A_{\mu}^{A}\Lambda^{B}. \tag{9.174}$$

The spinor field transforms under infinitesimal $SU(2)$ transformations as

$$\psi \longrightarrow \psi^u = \psi - ig\Lambda\psi. \tag{9.175}$$

We can now check explicitly that the covariant derivative is indeed covariant, viz

$$\nabla_\mu\psi \longrightarrow (\nabla_\mu\psi)^u = \nabla_\mu\psi - ig\Lambda\nabla_\mu\psi. \tag{9.176}$$

By applying the principle of minimal coupling to the free Dirac action (9.164) we replace the ordinary derivative $\partial_\mu\psi^a$ by the covariant derivative $(\nabla_\mu)_{ab}\psi^b$. We obtain the interacting action

$$S = \int d^4x \sum_{a,b} \bar{\psi}^a (i\gamma^\mu (\nabla_\mu)_{ab} - m\delta_{ab})\psi^b. \tag{9.177}$$

Clearly

$$(\nabla_\mu)_{ab} = \partial_\mu\delta_{ab} + igA_\mu^A (\frac{\sigma^A}{2})_{ab}. \tag{9.178}$$

This action is by construction invariant under local $SU(2)$ gauge transformations. It provides obviously the free term for the Dirac field ψ as well as the interaction term between the $SU(2)$ gauge field A^μ and the Dirac field ψ. There remains therefore to find an action which will provide the free term for the $SU(2)$ gauge field A^μ. As opposed to the $U(1)$ case the action which will provide a free term for the $SU(2)$ gauge field A^μ will also provide extra interaction terms (cubic and quartic) which involve only A^μ. This is another manifestation of the non-Abelian structure of the $SU(2)$ gauge group and it is generic to all other non-Abelian groups.

By analogy with the $U(1)$ case a gauge invariant action which depends only on A^μ can only depend on A^μ through the field strength tensor $F_{\mu\nu}$. This in turn can be constructed from the commutator of two covariant derivatives. We have then

$$\begin{aligned} F_{\mu\nu} &= \frac{1}{ig}[\nabla_\mu, \nabla_\nu] \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]. \end{aligned} \tag{9.179}$$

$F_{\mu\nu}$ is also a 2×2 matrix. In terms of components the above equation reads

$$\begin{aligned} F_{\mu\nu}^C \frac{\sigma^C}{2} &= \partial_\mu A_\nu^C \frac{\sigma^C}{2} - \partial_\nu A_\mu^C \frac{\sigma^C}{2} + ig[A_\mu^A \frac{\sigma^A}{2}, A_\nu^B \frac{\sigma^B}{2}] \\ &= \bigg(\partial_\mu A_\nu^C - \partial_\nu A_\mu^C + igA_\mu^A A_\nu^B . if_{ABC}\bigg)\frac{\sigma^C}{2}. \end{aligned} \tag{9.180}$$

Equivalently

$$F_{\mu\nu}^C = \partial_\mu A_\nu^C - \partial_\nu A_\mu^C - gf_{ABC}A_\mu^A A_\nu^B. \tag{9.181}$$

The last term in the above three formulas is of course absent in the case of $U(1)$ gauge theory. This is the term that will lead to novel cubic and quartic interaction vertices which involve only the gauge field A^μ. We remark also that although $F_{\mu\nu}$ is the commutator of two covariant derivatives it is not a differential operator. Since $\nabla_\mu\psi$ transforms as $\nabla_\mu\psi \longrightarrow u\nabla_\mu\psi$ we conclude that $\nabla_\mu\nabla_\nu\psi \longrightarrow u\nabla_\mu\nabla_\nu\psi$ and hence

$$F_{\mu\nu}\psi \longrightarrow uF_{\mu\nu}\psi. \tag{9.182}$$

This means in particular that

$$F_{\mu\nu} \longrightarrow F^u_{\mu\nu} = uF_{\mu\nu}u^+. \tag{9.183}$$

This can be verified explicitly by using the finite and infinitesimal transformation laws $A_\mu \longrightarrow uA_\mu u^+ + i\partial_\mu u.u^+/g$ and $A_\mu \longrightarrow A_\mu + \partial_\mu \Lambda + ig[A_\mu, \Lambda]$[4]. The infinitesimal form of the above transformation law is

$$F_{\mu\nu} \longrightarrow F^u_{\mu\nu} = F_{\mu\nu} + ig[F_{\mu\nu}, \Lambda]. \tag{9.184}$$

In terms of components this reads

$$F^C_{\mu\nu} \longrightarrow F^{uC}_{\mu\nu} = F^C_{\mu\nu} - gf_{ABC}F^A_{\mu\nu}\Lambda^B. \tag{9.185}$$

Although the field strength tensor $F_{\mu\nu}$ is not gauge invariant its gauge transformation $F_{\mu\nu} \longrightarrow uF_{\mu\nu}u^+$ is very simple. Any function of $F_{\mu\nu}$ will therefore transform in the same way as $F_{\mu\nu}$ and as a consequence its trace is gauge invariant under local $SU(2)$ transformations. For example $\mathrm{tr}F_{\mu\nu}F^{\mu\nu}$ is clearly gauge invariant. By appealing again to the requirement of renormalizability the only renormalizable $SU(2)$ gauge action in four dimensions (which also preserves P and T symmetries) must be quadratic in $F_{\mu\nu}$. The only candidate is $\mathrm{tr}F_{\mu\nu}F^{\mu\nu}$. We get then the pure gauge action

$$S = -\frac{1}{2}\int d^4x \mathrm{tr}F_{\mu\nu}F^{\mu\nu}. \tag{9.186}$$

We note that Pauli matrices satisfy

$$\mathrm{tr}\frac{\sigma^A}{2}\frac{\sigma^B}{2} = \frac{1}{2}\delta^{AB}. \tag{9.187}$$

Thus the above pure action becomes [5]

$$S = -\frac{1}{4}\int d^4x F^C_{\mu\nu}F^{\mu\nu C}. \tag{9.188}$$

This action provides as promised the free term for the $SU(2)$ gauge field A^μ but also it will provide extra cubic and quartic interaction vertices for the gauge field A^μ. In other words this action is not free in contrast with the $U(1)$ case. This interacting pure gauge theory is in fact highly non-trivial and strictly speaking this is what we should call Yang-Mills theory.

The total action is the sum of the gauge invariant Dirac action and the Yang-Mills action. This is given by

$$S = \int d^4x \sum_{a,b} \bar{\psi}^a (i\gamma^\mu(\nabla_\mu)_{ab} - m\delta_{ab})\psi^b - \frac{1}{4}\int d^4x F^C_{\mu\nu}F^{\mu\nu C}. \tag{9.189}$$

The final step is to generalize further to $SU(N)$ gauge theory which is quite straightforward. The group $SU(N)$ is the group of $N \times N$ unitary matrices which have determinant equal 1. This is given by

$$SU(N) = \{u_{ab}, a, b = 1, ..., N : u^+u = 1, \mathrm{det}u = 1\}. \tag{9.190}$$

[4]Exercise: Verify this.

[5]Exercise: Derive the equations of motion which follow from this action.

The generators of $SU(N)$ can be given by the so-called Gell-Mann matrices $t^A = \lambda^A/2$. They are traceless Hermitian matrices which generate the Lie algebra $su(N)$ of $SU(N)$. There are N^2-1 generators and hence $su(N)$ is an $(N^2-1)-$dimensional vector space. They satisfy the commutation relations

$$[t^A, t^B] = if_{ABC}t^C. \tag{9.191}$$

The non-trivial coefficients f_{ABC} are called the structure constants. The Gell-Mann generators t_a can be chosen such that

$$\mathrm{tr}t^At^B = \frac{1}{2}\delta^{AB}. \tag{9.192}$$

They also satisfy

$$t^At^B = \frac{1}{2N}\delta^{AB} + \frac{1}{2}(d_{ABC} + if_{ABC})t^C. \tag{9.193}$$

The coefficients d_{ABC} are symmetric in all indices. They can be given by $d_{ABC} = 2\mathrm{tr}t^A\{t^B, t^C\}$ and they satisfy for example

$$d_{ABC}d_{ABD} = \frac{N^2-4}{N}\delta_{CD}. \tag{9.194}$$

For example the group $SU(3)$ is generated by the 8 Gell-Mann 3×3 matrices $t^A = \lambda^A/2$ given by

$$\begin{aligned}
\lambda^1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\lambda^4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \ \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
\lambda^7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \ \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{aligned} \tag{9.195}$$

The structure constants f_{ABC} and the totally symmetric coefficients d_{ABC} are given in the case of the group $SU(3)$ by

$$f_{123} = 1\ ,\ f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = \frac{1}{2}\ ,\ f_{458} = f_{678} = \frac{\sqrt{3}}{2}. \tag{9.196}$$

$$\begin{aligned}
d_{118} &= d_{228} = d_{338} = -d_{888} = \frac{1}{\sqrt{3}} \\
d_{448} &= d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}} \\
d_{146} &= d_{157} = -d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2}.
\end{aligned} \tag{9.197}$$

Thus any finite element of the group $SU(N)$ can be rewritten in terms of the Gell-Mann matrices $t^A = \lambda^A/2$ as

$$u = \exp(-ig\Lambda)\ ,\ \Lambda = \sum_A \Lambda^A \frac{\lambda^A}{2}. \tag{9.198}$$

The spinor field ψ will be an $N-$component object. The $SU(N)$ group element u will act on the Dirac spinor field ψ in the obvious way $\psi \longrightarrow u\psi$. We say that the spinor field transforms in the fundamental representation of the $SU(N)$ gauge group. The covariant derivative will be defined by the same formula found in the $SU(2)$ case after making the replacement $\sigma^A \longrightarrow \lambda^A$, viz $(\nabla_\mu)_{ab} = \partial_\mu \delta_{ab} + igA_\mu^A (t^A)_{ab}$ (recall also that the range of the fundamental index a changes from 2 to N). The covariant derivative will transform covariantly under the $SU(N)$ gauge group. There are clearly N^2-1 components A_μ^A of the $SU(N)$ gauge field, i.e. $A_\mu = A_\mu^A t^A$. The transformation laws of A_μ and A_μ^A remain unchanged (only remember that the structure constants differ for different gauge groups). The field strength tensor $F_{\mu\nu}$ will be given, as before, by the commutator of two covariant derivatives. All results concerning $F_{\mu\nu}$ will remain intact with minimal changes involving the replacements $\sigma^A \longrightarrow \lambda^A$, $\epsilon_{ABC} \longrightarrow f_{ABC}$ (recall also that the range of the adjoint index changes from 3 to N^2-1). The total action will therefore be given by the same formula (9.189). We will refer to this theory as quantum chromodynamics (QCD) with $SU(N)$ gauge group whereas we will refer to the pure gauge action as $SU(N)$ Yang-Mills theory.

9.4 Quantization and Renormalization at 1−Loop

9.4.1 The Fadeev-Popov Gauge Fixing and Ghost Fields

We will be interested first in the $SU(N)$ Yang-Mills theory given by the action

$$\begin{aligned} S[A] &= -\frac{1}{2}\int d^4x \mathrm{tr} F_{\mu\nu}F^{\mu\nu} \\ &= -\frac{1}{4}\int d^4x F_{\mu\nu}^A F^{\mu\nu A}. \end{aligned} \tag{9.199}$$

The corresponding path integral is given by

$$Z = \int \prod_{\mu,A} \mathcal{D}A_\mu^A\ \exp(iS[A]) \tag{9.200}$$

This path integral is invariant under finite $SU(N)$ gauge transformations given explicitly by

$$A_\mu \longrightarrow A_\mu^u = uA_\mu u^+ + \frac{i}{g}\partial_\mu u.u^+. \tag{9.201}$$

Also it is invariant under infinitesimal $SU(N)$ gauge transformations given explicitly by

$$A_\mu \longrightarrow A_\mu^\Lambda = A_\mu + \partial_\mu \Lambda + ig[A_\mu,\Lambda] \equiv A_\mu + [\nabla_\mu,\Lambda]. \tag{9.202}$$

Equivalently

$$A_\mu^C \longrightarrow A_\mu^{\Lambda C} \ = \ A_\mu^C + \partial_\mu\Lambda^C - gf_{ABC}A_\mu^A\Lambda^B \equiv A_\mu^C + [\nabla_\mu,\Lambda]^C. \tag{9.203}$$

As in the case of electromagnetism we must fix the gauge before we can proceed any further since the path integral is ill defined as it stands. We want to gauge fix by imposing the Lorentz condition $G(A)=\partial_\mu A^\mu-\omega=0$. Clearly under infinitesimal $SU(N)$ gauge transformations we have $G(A^\Lambda)=\partial_\mu A^\mu-\omega+\partial_\mu[\nabla^\mu,\Lambda]$ and thus

$$\int\mathcal{D}\Lambda\delta(G(A^\Lambda))\det\bigg(\frac{\delta G(A^\Lambda)}{\delta\Lambda}\bigg) = \int\mathcal{D}\Lambda\delta(\partial_\mu A^\mu-\omega+\partial_\mu[\nabla^\mu,\Lambda])\det\partial_\mu[\nabla^\mu,..]. \tag{9.204}$$

By performing the change of variables $\Lambda\longrightarrow\Lambda^{'}=\partial_\mu[\nabla^\mu,\Lambda]$ and using the fact that $\mathcal{D}\Lambda^{'}=|(\partial\Lambda^{'}/\partial\Lambda)|\mathcal{D}\Lambda=\det(\partial_\mu[\nabla^\mu,..])\mathcal{D}\Lambda$ we get

$$\int\mathcal{D}\Lambda\delta(G(A^\Lambda))\det\bigg(\frac{\delta G(A^\Lambda)}{\delta\Lambda}\bigg) = \int\frac{\mathcal{D}\Lambda^{'}}{\det(\partial_\mu[\nabla^\mu,..])}\delta(\partial_\mu A^\mu-\omega+\Lambda^{'})\det(\partial_\mu[\nabla^\mu,..])=1. \tag{9.205}$$

This can also be put in the form (with u near the identity)

$$\int\mathcal{D}u\delta(G(A^u))\det\bigg(\frac{\delta G(A^u)}{\delta u}\bigg)=1\ ,\ \frac{\delta G(A^u)}{\delta u}=\partial_\mu[\nabla^\mu,..]. \tag{9.206}$$

For a given gauge configuration A^μ we define

$$\Delta^{-1}(A)=\int\mathcal{D}u\delta(G(A^u)). \tag{9.207}$$

Under a gauge transformation $A_\mu\longrightarrow A_\mu^v=vA_\mu v^+ + i\partial_\mu v.v^+/g$ we have $A_\mu^u\longrightarrow A_\mu^{uv}=uvA_\mu(uv)^+ + i\partial_\mu uv.(uv)^+/g$ and thus

$$\Delta^{-1}(A^v)=\int\mathcal{D}u\delta(G(A^{uv}))=\int\mathcal{D}(uv)\delta(G(A^{uv}))=\int\mathcal{D}u^{'}\delta(G(A^{u^{'}}))=\Delta^{-1}(A). \tag{9.208}$$

In other words Δ^{-1} is gauge invariant. Further we can write

$$1=\int\mathcal{D}u\delta(G(A^u))\Delta(A). \tag{9.209}$$

As we will see shortly we are interested in configurations A_μ which lie on the surface $G(A)=\partial^\mu A_\mu-\omega=0$. Thus only $SU(N)$ gauge transformations u which are near the identity are relevant in the above integral. Hence we conclude that (with u near the identity)

$$\Delta(A)=\det\bigg(\frac{\delta G(A^u)}{\delta u}\bigg). \tag{9.210}$$

The determinant $\det(\delta G(A^u)/\delta u)$ is gauge invariant and as a consequence is independent of u. The fact that this determinant is independent of u is also obvious from equation (9.206).

We insert 1 in the form (9.209) in the path integral as follows

$$\begin{aligned} Z &= \int\prod_{\mu,A}\mathcal{D}A_\mu^A\int\mathcal{D}u\delta(G(A^u))\Delta(A)\exp iS[A] \\ &= \int\mathcal{D}u\int\prod_{\mu,A}\mathcal{D}A_\mu^A\delta(G(A^u))\Delta(A)\exp iS[A] \\ &= \int\mathcal{D}u\int\prod_{\mu,A}\mathcal{D}A_\mu^{uA}\delta(G(A^u))\Delta(A^u)\exp iS[A^u]. \end{aligned} \tag{9.211}$$

Now we shift the integration varaible as $A_\mu^u \longrightarrow A_\mu$. The integral over the $SU(N)$ gauge group decouples an we end up with

$$Z = (\int \mathcal{D}u)\int \prod_{\mu,A}\mathcal{D}A_\mu^A \delta(G(A))\Delta(A)\exp iS[A]. \quad (9.212)$$

Because of the delta function we are interested in knowing $\Delta(A)$ only for configurations A^μ which lie on the surface $G(A) = 0$. This means in particular that the gauge transformations u appearing in (9.209) must be close to the identity so that we do not go far from the surface $G(A) = 0$. As a consequence $\Delta(A)$ can be equated with the determinant $\det(\delta G(A^u)/\delta u)$, viz

$$\Delta(A) = \det\bigg(\frac{\delta G(A^u)}{\delta u}\bigg) = \det\partial_\mu[\nabla^\mu, ..]. \quad (9.213)$$

In contrast with the case of $U(1)$ gauge theory, here the determinant $\det(\delta G(A^u)/\delta u)$ actually depends on the $SU(N)$ gauge field and hence it can not be taken out of the path integral. We have then the result

$$Z = (\int \mathcal{D}u)\int \prod_{\mu,A}\mathcal{D}A_\mu^A \delta(\partial_\mu A^\mu - \omega)\det\partial_\mu[\nabla^\mu, ..]\exp iS[A]. \quad (9.214)$$

Clearly ω must be an $N \times N$ matrix since A^μ is an $N \times N$ matrix. We want to set $\omega = 0$ by integrating both sides of the above equation against a Gaussian weighting function centered around $\omega = 0$, viz

$$\begin{aligned}\int \mathcal{D}\omega \exp(-i\int d^4x \mathrm{tr}\frac{\omega^2}{\xi})Z &= (\int \mathcal{D}u)\int \prod_{\mu,A}\mathcal{D}A_\mu^A \int \mathcal{D}\omega \exp(-i\int d^4x \mathrm{tr}\frac{\omega^2}{\xi})\delta(\partial_\mu A^\mu - \omega)\det\partial_\mu[\nabla^\mu, ..]\exp iS[A] \\ &= (\int \mathcal{D}u)\int \prod_{\mu,A}\mathcal{D}A_\mu^A \exp(-i\int d^4x \mathrm{tr}\frac{(\partial_\mu A^\mu)^2}{\xi})\det\partial_\mu[\nabla^\mu, ..]\exp iS[A]. \quad (9.215)\end{aligned}$$

The path integral of $SU(N)$ Yang-Mills theory is therefore given by

$$Z = \mathcal{N}\int \prod_{\mu,A}\mathcal{D}A_\mu^A \exp(-i\int d^4x \mathrm{tr}\frac{(\partial_\mu A^\mu)^2}{\xi})\det\partial_\mu[\nabla^\mu, ..]\exp iS[A]. \quad (9.216)$$

Let us recall that for Grassmann variables we have the identity

$$\det M = \int \prod_i d\theta_i^+ d\theta_i e^{-\theta_i^+ M_{ij}\theta_j}. \quad (9.217)$$

Thus we can express the determinant $\det\partial_\mu[\nabla^\mu, ..]$ as a path integral over Grassmann fields $\bar{c}$ and c as follows

$$\det\partial_\mu[\nabla^\mu, ..] = \int \mathcal{D}\bar{c}\mathcal{D}c\ \exp(-i\int d^4x\ \mathrm{tr}\bar{c}\partial_\mu[\nabla^\mu, c]). \quad (9.218)$$

The fields $\bar{c}$ and c are clearly scalar under Lorentz transformations (their spin is 0) but they are anti-commuting Grassmann-valued fields and hence they can not describe physical propagating particles (they simply have the wrong relation between spin and statistics). These fields are called Fadeev-Popov ghosts and they clearly carry two $SU(N)$ indices. More precisely since the covariant derivative is acting on them by commutator these fields must be $N \times N$ matrices and thus they can be rewritten as $c = c^A t^A$. We say that the ghost fields transform in the adjoint representation of the $SU(N)$ gauge group, i.e. as $c \longrightarrow ucu^+$ and $\bar{c} \longrightarrow u\bar{c}u^+$ which ensures global invariance. In terms of c^A the determinant reads

$$\det\partial_\mu[\nabla^\mu, ..] = \int \prod_A \mathcal{D}\bar{c}^A \mathcal{D}c^A \ \exp\bigg(i\int d^4x \ \bar{c}^A\bigg(- \partial_\mu\partial^\mu\delta^{AB} - gf_{ABC}\partial^\mu A_\mu^C\bigg)c^B\bigg). \quad (9.219)$$

The path integral of $SU(N)$ Yang-Mills theory becomes

$$\begin{aligned} Z &= \mathcal{N}\int \prod_{\mu,A} \mathcal{D}A_\mu^A \int \prod_A \mathcal{D}\bar{c}^A \mathcal{D}c^A \ \exp(-i\int d^4x \mathrm{tr}\frac{(\partial_\mu A^\mu)^2}{\xi})\exp(-i\int d^4x \ \mathrm{tr}\bar{c}\partial_\mu[\nabla^\mu, c])\exp iS[A] \\ &= \mathcal{N}\int \prod_{\mu,A} \mathcal{D}A_\mu^A \int \prod_A \mathcal{D}\bar{c}^A \mathcal{D}c^A \ \exp iS_{FP}[A, c, \bar{c}]. \end{aligned} \quad (9.220)$$

$$S_{FP}[A, c, \bar{c}] = S[A] - \int d^4x \mathrm{tr}\frac{(\partial_\mu A^\mu)^2}{\xi} - \int d^4x \ \mathrm{tr}\bar{c}\partial_\mu[\nabla^\mu, c]. \quad (9.221)$$

The second term is called the gauge fixing term whereas the third term is called the Faddev-Popov ghost term. We add sources to obtain the path integral

$$Z[J, b, \bar{b}] = \mathcal{N}\int \prod_{\mu,A} \mathcal{D}A_\mu^A \int \prod_A \mathcal{D}\bar{c}^A \mathcal{D}c^A \ \exp\bigg(iS_{FP}[A, c, \bar{c}] - i\int d^4x J_\mu^A A^{\mu A} + i\int d^4x(\bar{b}c + \bar{c}b)\bigg). \quad (9.222)$$

In order to compute propagators we drop all interactions terms. We end up with the partition function

$$\begin{aligned} Z[J, b, \bar{b}] &= \mathcal{N}\int \prod_{\mu,A} \mathcal{D}A_\mu^A \int \prod_A \mathcal{D}\bar{c}^A \mathcal{D}c^A \ \exp\bigg(\frac{i}{2}\int d^4x A_\nu^A\bigg(\partial_\mu\partial^\mu\eta^{\nu\lambda} + (\frac{1}{\xi} - 1)\partial^\nu\partial^\lambda\bigg)A_\lambda^A - i\int d^4x\bar{c}^A\partial_\mu\partial^\mu c^A \\ &- i\int d^4x J_\mu^A A^{\mu A} + i\int d^4x(\bar{b}c + \bar{c}b)\bigg). \end{aligned} \quad (9.223)$$

The free $SU(N)$ gauge part is $N^2 - 1$ copies of $U(1)$ gauge theory. Thus without any further computation the $SU(N)$ vector gauge field propagator is given by

$$< A_\mu^A(x)A_\nu^B(y) > = \int \frac{d^4k}{(2\pi)^4}\frac{-i\delta^{AB}}{k^2 + i\epsilon}(\eta_{\mu\nu} - (1 - \xi)\frac{k_\mu k_\nu}{k^2})e^{ik(x-y)}. \quad (9.224)$$

The propagator of the ghost field can be computed along the same lines used for the propagator of the Dirac field. We obtain [6]

$$< c^A(x)\bar{c}^B(y) > = \int \frac{d^4k}{(2\pi)^4}\frac{i\delta^{AB}}{k^2 + i\epsilon}e^{ik(x-y)}. \quad (9.225)$$

[6]Exercise: Perform this calculation explicitly.

9.4.2 Perturbative Renormalization and Feynman Rules

The QCD action with $SU(N)$ gauge group is given by

$$S_{QCD}[\psi,\bar{\psi},A,c,\bar{c}] \;=\; S_0[\psi,\bar{\psi},A,c,\bar{c}]+S_1[\psi,\bar{\psi},A,c,\bar{c}] \tag{9.226}$$

$$\begin{aligned} S_0[\psi,\bar{\psi},A,c,\bar{c}] &= \int d^4x \sum_a \bar{\psi}^a(i\gamma^\mu\partial_\mu - m)\psi^a - \frac{1}{4}\int d^4x(\partial_\mu A_\nu^A - \partial_\nu A_\mu^A)(\partial^\mu A^{\nu A} - \partial^\nu A^{\mu A}) \\ &- \frac{1}{2\xi}\int d^4x(\partial_\mu A^{\mu A})^2 - \int d^4x\ \bar{c}^A\partial_\mu\partial^\mu c^A. \end{aligned} \tag{9.227}$$

$$\begin{aligned} S_1[\psi,\bar{\psi},A,c,\bar{c}] &= -gt^A_{ab}\int d^4x \sum_{a,b}\bar{\psi}^a\gamma^\mu\psi^b A_\mu^A + gf_{ABC}\int d^4x\partial^\mu A^{\nu C}A_\mu^A A_\nu^B \\ &- \frac{g^2}{4}f_{ABC}f_{DEC}\int d^4x A_\mu^A A_\nu^B A^{\mu D}A^{\nu E} - gf_{ABC}\int d^4x(\bar{c}^A c^B\partial^\mu A_\mu^C + \bar{c}^A\partial^\mu c^B A_\mu^c). \end{aligned} \tag{9.228}$$

We introduce the renormalization constants Z_3, Z_2 and Z_2^c by introducing the renormalized fields A_R^μ, ψ_R and c_R which are defined in terms of the bare fields A^μ, ψ and c respectively by the equations

$$A_R^\mu = \frac{A^\mu}{\sqrt{Z_3}}\ ,\ \psi_R = \frac{\psi}{\sqrt{Z_2}}\ ,\ c_R = \frac{c}{\sqrt{Z_2^c}}. \tag{9.229}$$

The renormalization constants Z_3, Z_2 and Z_2^c can be expanded in terms of the counter terms δ_3, δ_2 and δ_2^c as

$$Z_3 = 1+\delta_3\ ,\ Z_2 = 1+\delta_2\ ,\ Z_2^c = 1+\delta_2^c. \tag{9.230}$$

Furthermore we relate the bare coupling constants g and m to the renormalized coupling g_R and m_R through the counter terms δ_1 and δ_m by

$$gZ_2\sqrt{Z_3} = g_R(1+\delta_1)\ ,\ Z_2 m = m_R + \delta_m. \tag{9.231}$$

Since we have also AAA, $AAAA$ and ccA vertices we need more counter terms δ_1^3, δ_1^4 and δ_1^c which we define by

$$gZ_3^{\frac{3}{2}} = g_R(1+\delta_1^3)\ ,\ g^2Z_3^2 = g_R^2(1+\delta_1^4)\ ,\ gZ_2^c\sqrt{Z_3} = g_R(1+\delta_1^c). \tag{9.232}$$

We will also define a "renormalized gauge fixing parameter" ξ_R by

$$\frac{1}{\xi_R} = \frac{Z_3}{\xi}. \tag{9.233}$$

As we will see shortly this is physically equivalent to imposing the gauge fixing condition on the renormalized gauge field A_R^μ instead of the bare gauge field A^μ.

The action divides therefore as

$$S = S_R + S_{\text{count ter}}. \tag{9.234}$$

The action S_R is given by the same formula as S with the replacement of all fields and coupling constants by the renormalized fields and renormalized coupling constants and also replacing ξ by ξ_R. The counter term action $S_{\rm count\ ter}$ is given explicitly by

$$\begin{aligned}
S_{\rm count\ ter} &= \delta_2\int d^4x\sum_a \bar{\psi}^a_R i\gamma^\mu\partial_\mu\psi^a_R-\delta_m\int d^4x\sum_a\bar{\psi}^a_R\psi^a_R-\frac{\delta_3}{4}\int d^4x(\partial_\mu A^A_{\nu R}-\partial_\nu A^A_{\mu R})(\partial^\mu A^{\nu A}_R-\partial^\nu A^{\mu A}_R)\\
&- \delta_2^c\int d^4x\ \bar{c}^A_R\partial_\mu\partial^\mu c^A_R-g_R\delta_1 t^A_{ab}\int d^4x\sum_{a,b}\bar{\psi}^a_R\gamma^\mu\psi^b_R A^A_{\mu R}+g_R\delta_1^3 f_{ABC}\int d^4x\partial^\mu A^{\nu C}_R A^A_{\mu R}A^B_{\nu R}\\
&- \frac{g_R^2\delta_1^4}{4}f_{ABC}f_{DEC}\int d^4x A^A_{\mu R}A^B_{\nu R}A^{\mu D}_R A^{\nu E}_R-g_R\delta_1^c f_{ABC}\int d^4x(\bar{c}^A_R c^B_R\partial^\mu A^C_{\mu R}+\bar{c}^A_R\partial^\mu c^B_R A^c_{\mu R}).
\end{aligned}\tag{9.235}$$

From the above discussion we see that we have eight counter terms δ_1, δ_2, δ_3, δ_2^c, δ_m, δ_1^c, δ_1^4 and δ_1^3 and five coupling constants g, m, Z_2, Z_3 and Z_2^c. The counter terms will be determined in terms of the coupling constants and hence there must be only five of them which are completely independent. The fact that only five counter terms are independent means that we need five renormalization conditions to fix them. This also means that the counter term must be related by three independent equations. It is not difficult to discover that these equations are

$$\frac{g_R}{g}\equiv\frac{Z_2\sqrt{Z_3}}{1+\delta_1}=\frac{Z_3^{\frac{3}{2}}}{1+\delta_1^3}.\tag{9.236}$$

$$\frac{g_R}{g}\equiv\frac{Z_2\sqrt{Z_3}}{1+\delta_1}=\frac{Z_3}{\sqrt{1+\delta_1^4}}.\tag{9.237}$$

$$\frac{g_R}{g}\equiv\frac{Z_2\sqrt{Z_3}}{1+\delta_1}=\frac{Z_2^c\sqrt{Z_3}}{1+\delta_1^c}.\tag{9.238}$$

At the one-loop order we can expand $Z_3=1+\delta_3$, $Z_2=1+\delta_2$ and $Z_2^c=1+\delta_2^c$ where δ_3, δ_2 and δ_2^c as well as δ_1, δ_1^3, δ_1^4 and δ_1^c are all of order $\hbar$ and hence the above equations become

$$\delta_1^3=\delta_3+\delta_1-\delta_2.\tag{9.239}$$

$$\delta_1^4=\delta_3+2\delta_1-2\delta_2.\tag{9.240}$$

$$\delta_1^c=\delta_2^c+\delta_1-\delta_2.\tag{9.241}$$

The independent counter terms are taken to be δ_1, δ_2, δ_3, δ_2^c, δ_m which correspond respectively to the coupling constants g, Z_2, Z_3, Z_2^c and m. The counter term δ_3 will be determined in the following from the gluon self-energy, the counter terms δ_2 and δ_m will be determined from the quark self-energy whereas the counter term δ_1 will be determined from the vertex. The counter term δ_2^c should be determined from the ghost self-energy [7].

For ease of writing we will drop in the following the subscript R on renormalized quantities and when we need to refer to the bare quantities we will use the subscript 0 to distinguish them from their renormalized counterparts.

We write next the corresponding Feynman rules in momentum space. These are shown in figure 11. In the next two sections we will derive these rules from first principle, i.e. starting from the formula (8.136). The Feynman rules corresponding to the bare action are summarized as follows:

[7]Exercise: Compute δ_2^c following the same steps taken for the other counter terms.

- The quark propagator is

$$< \psi^b_\beta(p)\bar{\psi}^a_\alpha(-p) >= \delta^{ab}\frac{(\gamma^\mu p_\mu + m)_{\beta\alpha}}{p^2 - m^2}. \quad (9.242)$$

- The gluon propagator is

$$< A^A_\mu(k)A^B_\nu(-k) >= \delta^{AB}\frac{1}{k^2}\bigg[\eta_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2}\bigg]. \quad (9.243)$$

- The ghost propagator is

$$< c^B(p)\bar{c}^A(-p) >= \delta^{AB}\frac{1}{p^2}. \quad (9.244)$$

- The quartic vertex is

$$\begin{aligned} < A^A_\mu A^B_\nu A^D_\rho A^E_\sigma > &= -g^2\bigg[f_{ABC}f_{DEC}(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}) + f_{BDC}f_{AEC}(\eta^{\sigma\rho}\eta^{\mu\nu} - \eta^{\rho\mu}\eta^{\sigma\nu}) \\ &+ f_{DAC}f_{BEC}(\eta^{\sigma\mu}\eta^{\rho\nu} - \eta^{\mu\nu}\eta^{\sigma\rho})\bigg]. \end{aligned} \quad (9.245)$$

- The cubic vertex is

$$< A^A_\mu(k)A^B_\nu(p)A^C_\rho(q) > = igf_{ABC}\bigg[(2p+k)^\mu\eta^{\rho\nu} - (p+2k)^\nu\eta^{\rho\mu} + (k-p)^\rho\eta^{\mu\nu}\bigg]\ ,\ q = -p-k. \quad (9.246)$$

- The quark-gluon vertex is

$$< A^A_\mu\bar{c}^A(k)c^B > = g(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}. \quad (9.247)$$

- The ghost-gluon vertex is

$$< A^C_\mu\bar{\psi}^a_\alpha\psi^b_\beta > = -igf_{ABC}k^\mu. \quad (9.248)$$

- Impose energy-momentum conservation at all vertices.

- Integrate over internal momenta.

- Symmetry factor. For example if the diagram is invariant under the permutation of two lines we should divide by 1/2.

- Each fermion line must be multiplied by -1.

- All one-loop diagrams should be multiplied by $\hbar/i$.

9.4.3 The Gluon Field Self-Energy at 1−Loop

We are interested in computing the proper $n-$point vertices of this theory which are connected $1-$particle irreducible $n-$point functions from which all external legs are amputated. The generating functional of the corresponding Feynman diagrams is of course the effective action. We recall the formal definition of the proper $n-$point vertices given by

$$\Gamma^{(n)}(x_1,...,x_n)=\Gamma_{,i_1...i_n}=\frac{\delta^n\Gamma[\phi_c]}{\delta\phi_c(x_1)...\delta\phi_c(x_n)}|_{\phi=0}. \tag{9.249}$$

The effective action upto the $1-$loop order is

$$\Gamma=S+\frac{1}{i}\Gamma_1\ ,\ \Gamma_1=\ln\det G_0\ ,\ G_0^{ik}=-S^{-1}_{,ik}. \tag{9.250}$$

As our first example we consider the proper $2-$point vertex of the non-Abelian vector field A_μ. This is defined by

$$\Gamma^{AB}_{\mu\nu}(x,y)\ \ =\ \ \frac{\delta^2\Gamma}{\delta A^{\mu A}(x)\delta A^{\nu B}(y)}|_{A,\psi,c=0}. \tag{9.251}$$

We use the powerful formula (8.136) which we copy here for convenience

$$\Gamma_1[\phi]_{,j_0k_0}\ \ =\ \ \frac{1}{2}G_0^{mn}S[\phi]_{,j_0k_0mn}+\frac{1}{2}G_0^{mm_0}G_0^{nn_0}S[\phi]_{,j_0mn}S[\phi]_{,k_0m_0n_0}. \tag{9.252}$$

We have then immediately four terms contributing to the gluon propagator at $1-$loop. These are given by (with $j_0=(x,\mu,A)$ and $k_0=(y,\nu,B)$)

$$\begin{aligned}\Gamma^{AB}_{\mu\nu}(x,y)\ \ &=\ \ \frac{\delta^2 S}{\delta A^{\mu A}(x)\delta A^{\nu B}(y)}|_{A,\psi,c=0}+\frac{1}{i}\bigg[\frac{1}{2}G_0^{A_mA_n}S_{,A_{j_0}A_{k_0}A_mA_n}+\frac{1}{2}G_0^{A_mA_{m_0}}G_0^{A_nA_{n_0}}S_{,A_{j_0}A_mA_n}S_{,A_{k_0}A_{m_0}A_{n_0}}\\ &+\ \ (-1)\times G_0^{\bar{c}_mc_{m_0}}G_0^{c_n\bar{c}_{n_0}}S_{,A_{j_0}\bar{c}_mc_n}S_{,A_{k_0}c_{m_0}\bar{c}_{n_0}}+(-1)\times G_0^{\bar{\psi}_m\psi_{m_0}}G_0^{\psi_n\bar{\psi}_{n_0}}S_{,A_{j_0}\bar{\psi}_m\psi_n}S_{,A_{k_0}\psi_{m_0}\bar{\psi}_{n_0}}\bigg].\end{aligned} \tag{9.253}$$

The corresponding Feynman diagrams are shown on figure 9. The minus signs in the last two diagrams are the famous fermion loops minus sign. To see how they actually originate we should go back to the derivation of (9.252) and see what happens if the fields are Grassmann valued. We start from the first derivative of the effective action Γ_1 which is given by the unambigous equation (8.112), viz

$$\Gamma_{1,j}\ \ =\ \ \frac{1}{2}G_0^{mn}S_{,jmn}. \tag{9.254}$$

Taking the second derivative we obtain

$$\Gamma_{1,ij}\ \ =\ \ \frac{1}{2}G_0^{mn}S_{,ijmn}+\frac{1}{2}\frac{\delta G_0^{mn}}{\delta\phi^i}S_{,jmn}. \tag{9.255}$$

The first term is correct. The second term can be computed using the identity $G_0^{mm_0}S_{,m_0n}=-\delta^m_n$ which can be rewritten as

$$\frac{\delta G_0^{mn}}{\delta\phi^i}=G_0^{mm_0}S_{,im_0n_0}G_0^{n_0n}. \tag{9.256}$$

We have then

$$\Gamma_{1,ij} = \frac{1}{2}G_0^{mn}S_{,ijmn} + \frac{1}{2}G_0^{mm_0}G_0^{n_0n}S_{,im_0n_0}S_{,jmn}$$
$$= \frac{1}{2}G_0^{mn}S_{,j_0k_0mn} + \frac{1}{2}G_0^{m_0m}G_0^{nn_0}S_{,j_0mn}S_{,k_0m_0n_0}. \quad (9.257)$$

Only the propagator $G_0^{m_0m}$ has reversed indices compared to (9.252) which is irrelevant for bosonic fields but reproduces a minus sign for fermionic fields.

The classical term in the gluon self-energy is given by

$$S_{,j_0k_0} = \frac{\delta^2 S}{\delta A^{\mu A}(x)\delta A^{\nu B}(y)}|_{A,\psi,c=0} = \bigg[\partial_\rho\partial^\rho\eta^{\mu\nu} + (\frac{1}{\xi}-1)\partial^\mu\partial^\nu\bigg]\delta^{AB}\delta^4(x-y)$$
$$= -\int\frac{d^4k}{(2\pi)^4}\bigg[k^2\eta^{\mu\nu} + (\frac{1}{\xi}-1)k^\mu k^\nu\bigg]\delta_{AB}e^{ik(x-y)}. \quad (9.258)$$

We compute

$$G_0^{j_0k_0} = \delta_{AB}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i\epsilon}\bigg[\eta^{\mu\nu} + (\xi-1)\frac{k^\mu k^\nu}{k^2}\bigg]e^{ik(x-y)}$$
$$= \delta_{AB}\int\frac{d^4k}{(2\pi)^4}G_0^{\mu\nu}(k)e^{ik(x-y)}$$
$$= \delta_{AB}G_0^{\mu\nu}(x,y). \quad (9.259)$$

The quartic vertex can be put into the fully symmetric form

$$-\frac{g^2}{4}f_{ABC}f_{DEC}\int d^4x A_\mu^A A_\nu^B A^{\mu D}A^{\nu E} = -\frac{g^2}{8}f_{ABC}f_{DEC}\int d^4x\int d^4y\int d^4z\int d^4w A_\mu^A(x)A_\nu^B(y)A_\rho^D(z)A_\sigma^E(w)$$
$$\times\ \delta^4(x-y)\delta^4(x-z)\delta^4(x-w)(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu})$$
$$= -\frac{g^2}{4!}\int d^4x\int d^4y\int d^4z\int d^4w A_\mu^A(x)A_\nu^B(y)A_\rho^D(z)A_\sigma^E(w)\delta^4(x-y)\delta^4(x-z)$$
$$\times\ \delta^4(x-w)\bigg[f_{ABC}f_{DEC}(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}) + f_{BDC}f_{AEC}(\eta^{\sigma\rho}\eta^{\mu\nu} - \eta^{\rho\mu}\eta^{\sigma\nu})$$
$$+\ f_{DAC}f_{BEC}(\eta^{\sigma\mu}\eta^{\rho\nu} - \eta^{\mu\nu}\eta^{\sigma\rho})\bigg]. \quad (9.260)$$

In other words (with $j_0 = (x,\mu,A)$, $k_0 = (y,\nu,B)$, $m = (z,\rho,D)$ and $n = (w,\sigma,E)$)

$$S_{,A_{j_0}A_{k_0}A_mA_n} = -g^2\delta^4(x-y)\delta^4(x-z)\delta^4(x-w)\bigg[f_{ABC}f_{DEC}(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}) + f_{BDC}f_{AEC}(\eta^{\sigma\rho}\eta^{\mu\nu} - \eta^{\rho\mu}\eta^{\sigma\nu})$$
$$+\ f_{DAC}f_{BEC}(\eta^{\sigma\mu}\eta^{\rho\nu} - \eta^{\mu\nu}\eta^{\sigma\rho})\bigg]. \quad (9.261)$$

We can now compute the first one-loop diagram as

$$\begin{aligned}
\frac{1}{2}G_0^{A_mA_n}S_{,A_{j_0}A_{k_0}A_mA_n} &= -\frac{g^2}{2}\delta^{DE}G_{0\rho\sigma}(x,x)\bigg[f_{ABC}f_{DEC}(\eta^{\rho\mu}\eta^{\sigma\nu}-\eta^{\sigma\mu}\eta^{\rho\nu})+f_{BDC}f_{AEC}(\eta^{\sigma\rho}\eta^{\mu\nu}-\eta^{\rho\mu}\eta^{\sigma\nu})\\
&+ f_{DAC}f_{BEC}(\eta^{\sigma\mu}\eta^{\rho\nu}-\eta^{\mu\nu}\eta^{\sigma\rho})\bigg]\delta^4(x-y)\\
&= -\frac{g^2}{2}\bigg[G_{0\rho}^{\rho}(x,x)\eta^{\mu\nu}-G_0^{\mu\nu}(x,x)\bigg]\bigg[f_{BDC}f_{ADC}-f_{DAC}f_{BDC}\bigg]\delta^4(x-y)\\
&= -g^2\bigg[G_{0\rho}^{\rho}(x,x)\eta^{\mu\nu}-G_0^{\mu\nu}(x,x)\bigg]f_{BDC}f_{ADC}\delta^4(x-y). \qquad (9.262)
\end{aligned}$$

The quantity $f_{BDC}f_{ADC}$ is actually the Casimir operator in the adjoint representation of the group. The adjoint representation of $SU(N)$ is $(N^2-1)-$dimensional. The generators in the adjoint representation can be given by $(t_G^A)_{BC}=if_{ABC}$. Indeed we can easily check that these matrices satisfy the fundamental commutation relations $[t_G^A,t_G^B]=if_{ABC}t_G^C$. We compute then $f_{BDC}f_{ADC}=(t_G^Ct_G^C)_{BA}=C_2(G)\delta_{BA}$. These generators must also satisfy $tr_Gt_G^At_G^B=C(G)\delta^{AB}$. For $SU(N)$ we have [8]

$$f_{BDC}f_{ADC}=C_2(G)\delta_{BA}=C(G)\delta_{BA}=N\delta_{BA}. \qquad (9.263)$$

Hence

$$\frac{1}{2}G_0^{A_mA_n}S_{,A_{j_0}A_{k_0}A_mA_n} = -g^2C_2(G)\delta_{AB}\bigg[G_{0\rho}^{\rho}(x,x)\eta^{\mu\nu}-G_0^{\mu\nu}(x,x)\bigg]\delta^4(x-y). \qquad (9.264)$$

In order to maintain gauge invariance we will use the powerful method of dimensional regularization. The above diagram takes now the form

$$\frac{1}{2}G_0^{A_mA_n}S_{,A_{j_0}A_{k_0}A_mA_n} = -g^2C_2(G)\delta_{AB}\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2}\bigg[(d+\xi-2)\eta^{\mu\nu}-(\xi-1)\frac{p^\mu p^\nu}{k^2}\bigg]\delta^4(x-y). \qquad (9.265)$$

This simplifies further in the Feynman gauge. Indeed for $\xi=1$ we get

$$\begin{aligned}
\frac{1}{2}G_0^{A_mA_n}S_{,A_{j_0}A_{k_0}A_mA_n} &= -g^2C_2(G)\delta_{AB}\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2}\bigg[(d-1)\eta^{\mu\nu}\bigg]\delta^4(x-y)\\
&= -g^2C_2(G)\delta_{AB}\int\frac{d^dp}{(2\pi)^d}\int\frac{d^dk}{(2\pi)^d}\frac{(p+k)^2}{p^2(p+k)^2}\bigg[(d-1)\eta^{\mu\nu}\bigg]e^{ik(x-y)}. \qquad (9.266)
\end{aligned}$$

We use now Feynman parameters, viz

$$\begin{aligned}
\frac{1}{(p+k)^2p^2} &= \int_0^1dx\int_0^1dy\delta(x+y-1)\frac{1}{\big[x(p+k)^2+yp^2\big]^2}\\
&= \int_0^1dx\frac{1}{(l^2-\Delta)^2}\ ,\ l=p+xk\ ,\ \Delta=-x(1-x)k^2. \qquad (9.267)
\end{aligned}$$

[8]Exercise: Derive this result.

We have then (using also rotational invariance)

$$\begin{aligned}
\frac{1}{2}G_0^{A_mA_n}S_{,A_{j_0}A_{k_0}A_mA_n} &= -g^2C_2(G)\delta_{AB}\int\frac{d^dk}{(2\pi)^d}\Big[(d-1)\eta^{\mu\nu}\Big]e^{ik(x-y)}\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{(l+(1-x)k)^2}{(l^2-\Delta)^2}\\
&= -g^2C_2(G)\delta_{AB}\int\frac{d^dk}{(2\pi)^d}\Big[(d-1)\eta^{\mu\nu}\Big]e^{ik(x-y)}\int_0^1 dx\Big\{\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{(l^2-\Delta)^2}\\
&+ (1-x)^2k^2\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2-\Delta)^2}\Big\}. \qquad (9.268)
\end{aligned}$$

The above two integrals over l are given by (after dimensional regularization and Wick rotation)

$$\begin{aligned}
\int\frac{d^dl}{(2\pi)^d}\frac{l^2}{(l^2-\Delta)^2} &= -i\int\frac{d^dl_E}{(2\pi)^d}\frac{1}{(l_E^2+\Delta)^2}\\
&= -i\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Delta^{1-\frac{d}{2}}}\frac{\Gamma(2-\frac{d}{2})}{\frac{2}{d}-1}. \qquad (9.269)
\end{aligned}$$

$$\begin{aligned}
\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2-\Delta)^2} &= i\int\frac{d^dl_E}{(2\pi)^d}\frac{l_E^2}{(l_E^2+\Delta)^2}\\
&= i\frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Delta^{2-\frac{d}{2}}}\Gamma(2-\frac{d}{2}). \qquad (9.270)
\end{aligned}$$

We get the final result

$$\begin{aligned}
\frac{1}{2}G_0^{A_mA_n}S_{,A_{j_0}A_{k_0}A_mA_n} &= \frac{i}{(4\pi)^{\frac{d}{2}}}g^2C_2(G)\delta_{AB}\int\frac{d^dk}{(2\pi)^d}\big[\eta^{\mu\nu}k^2\big]e^{ik(x-y)}\int_0^1\frac{dx}{\big(-x(1-x)k^2\big)^{2-\frac{d}{2}}}\Big(-\frac{1}{2}d(d-1)x(1-x)\\
&\times \Gamma(1-\frac{d}{2})-(d-1)(1-x)^2\Gamma(2-\frac{d}{2})\Big). \qquad (9.271)
\end{aligned}$$

We compute now the second diagram. First we write the pure gauge field cubic interaction in the totally symmetric form

$$\begin{aligned}
gf_{ABC}\int d^4x\partial^{\mu}A^{\nu C}A_{\mu}^{A}A_{\nu}^{B} &= \frac{gf_{ABC}}{3!}\int d^4x\int d^4y\int d^4zA_{\mu}^{A}(x)A_{\nu}^{B}(y)A_{\rho}^{C}(z)\Big[\eta^{\rho\nu}\Big(\partial_x^{\mu}\delta^4(x-z).\delta^4(x-y)\\
&- \partial_x^{\mu}\delta^4(x-y).\delta^4(x-z)\Big)-\eta^{\rho\mu}\Big(\partial_y^{\nu}\delta^4(y-z).\delta^4(x-y)-\partial_y^{\nu}\delta^4(y-x).\delta^4(z-y)\Big)\\
&- \eta^{\mu\nu}\Big(\partial_z^{\rho}\delta^4(z-x).\delta^4(z-y)-\partial_z^{\rho}\delta^4(z-y).\delta^4(x-z)\Big)\Big]. \qquad (9.272)
\end{aligned}$$

Thus we compute (with $j_0=(x,\mu,A)$, $k_0=(y,\nu,B)$ and $m=(z,\rho,C)$)

$$S_{,A_{j_0}A_{k_0}A_m} = igf_{ABC}S^{\mu\nu\rho}(x,y,z). \qquad (9.273)$$

$$\begin{aligned}
iS^{\mu\nu\rho}(x,y,z) &= \eta^{\rho\nu}\Big(\partial_x^{\mu}\delta^4(x-z).\delta^4(x-y)-\partial_x^{\mu}\delta^4(x-y).\delta^4(x-z)\Big)-\eta^{\rho\mu}\Big(\partial_y^{\nu}\delta^4(y-z).\delta^4(x-y)\\
&- \partial_y^{\nu}\delta^4(x-y).\delta^4(y-z)\Big)-\eta^{\mu\nu}\Big(\partial_z^{\rho}\delta^4(x-z).\delta^4(y-z)-\partial_z^{\rho}\delta^4(y-z).\delta^4(x-z)\Big)\\
&= i\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4l}{(2\pi)^4}S^{\mu\nu\rho}(k,p)(2\pi)^4\delta^4(p+k+l)\exp(ikx+ipy+ilz).
\end{aligned}$$
$$(9.274)$$

$$S^{\mu\nu\rho}(k,p)=(2p+k)^{\mu}\eta^{\rho\nu}-(p+2k)^{\nu}\eta^{\rho\mu}+(k-p)^{\rho}\eta^{\mu\nu}. \quad (9.275)$$

The second diagram is therefore given by

$$\begin{aligned}\frac{1}{2}G_0^{A_mA_{m_0}}G_0^{A_nA_{n_0}}S_{,A_{j_0}A_mA_n}S_{,A_{k_0}A_{m_0}A_{n_0}} &= -\frac{g^2C_2(G)\delta_{AB}}{2}\int d^4zd^4z_0d^4wd^4w_0G_{0\rho\rho_0}(z,z_0)G_{0\sigma\sigma_0}(w,w_0)S^{\mu\rho\sigma}(x,z,w)\\ &\times \quad S^{\nu\rho_0\sigma_0}(y,z_0,w_0)\\ &= -\frac{g^2C_2(G)\delta_{AB}}{2}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}G_{0\rho\rho_0}(p)G_{0\sigma\sigma_0}(k+p)S^{\mu\rho\sigma}(k,p)\\ &\times \quad S^{\nu\rho_0\sigma_0}(-k,-p)\exp ik(x-y). \quad (9.276)\end{aligned}$$

In the Feynman gauge this becomes

$$\begin{aligned}\frac{1}{2}G_0^{A_mA_{m_0}}G_0^{A_nA_{n_0}}S_{,A_{j_0}A_mA_n}S_{,A_{k_0}A_{m_0}A_{n_0}} &= -\frac{g^2C_2(G)\delta_{AB}}{2}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\frac{1}{p^2(k+p)^2}S^{\mu\rho\sigma}(k,p)\\ &\times \quad S^{\nu}{}_{\rho\sigma}(-k,-p)\exp ik(x-y). \quad (9.277)\end{aligned}$$

We use now Feynman parameters as before. We get

$$\begin{aligned}\frac{1}{2}G_0^{A_mA_{m_0}}G_0^{A_nA_{n_0}}S_{,A_{j_0}A_mA_n}S_{,A_{k_0}A_{m_0}A_{n_0}} &= -\frac{g^2C_2(G)\delta_{AB}}{2}\int\frac{d^4k}{(2\pi)^4}\exp ik(x-y)\int_0^1dx\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-\Delta)^2}\\ &\times \quad S^{\mu\rho\sigma}(k,l-xk)S^{\nu}{}_{\rho\sigma}(-k,-l+xk). \quad (9.278)\end{aligned}$$

Clearly by rotational symmetry only quadratic and constant terms in l^μ in the product $S^{\mu\rho\sigma}(k,l-xk)S^{\nu}{}_{\rho\sigma}(-k,-l+xk)$ give non-zero contribution to the integral over l. These are [9]

$$\begin{aligned}\frac{1}{2}G_0^{A_mA_{m_0}}G_0^{A_nA_{n_0}}S_{,A_{j_0}A_mA_n}S_{,A_{k_0}A_{m_0}A_{n_0}} &= -\frac{g^2C_2(G)\delta_{AB}}{2}\int\frac{d^4k}{(2\pi)^4}\exp ik(x-y)\int_0^1dx\bigg\{6(\frac{1}{d}-1)\eta^{\mu\nu}\\ &\times \quad \int\frac{d^4l}{(2\pi)^4}\frac{l^2}{(l^2-\Delta)^2}+\bigg((2-d)(1-2x)^2k^\mu k^\nu+2(1+x)(2-x)k^\mu k^\nu\\ &- \quad \eta^{\mu\nu}k^2(2-x)^2-\eta^{\mu\nu}k^2(1+x)^2\bigg)\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-\Delta)^2}\bigg\}. \quad (9.279)\end{aligned}$$

We now employ dimensional regularization and use the integrals (9.269) and (9.270). We obtain

$$\begin{aligned}\frac{1}{2}G_0^{A_mA_{m_0}}G_0^{A_nA_{n_0}}S_{,A_{j_0}A_mA_n}S_{,A_{k_0}A_{m_0}A_{n_0}} &= -\frac{ig^2C_2(G)\delta_{AB}}{2(4\pi)^{\frac{d}{2}}}\int\frac{d^dk}{(2\pi)^d}\exp ik(x-y)\int_0^1\frac{dx}{\big(-x(1-x)k^2\big)^{2-\frac{d}{2}}}\bigg\{\\ &- \quad 3(d-1)\eta^{\mu\nu}\Gamma(1-\frac{d}{2})x(1-x)k^2+\Gamma(2-\frac{d}{2})\bigg((2-d)(1-2x)^2k^\mu k^\nu\\ &+ \quad 2(1+x)(2-x)k^\mu k^\nu-\eta^{\mu\nu}k^2(2-x)^2-\eta^{\mu\nu}k^2(1+x)^2\bigg)\bigg\}. \quad (9.280)\end{aligned}$$

We go now to the third diagram which involves a ghost loop. We recall first the ghost field propagator

$$< c^A(x)\bar{c}^B(y) > \;=\; \int\frac{d^4k}{(2\pi)^4}\frac{i\delta^{AB}}{k^2+i\epsilon}e^{ik(x-y)}. \quad (9.281)$$

[9] Exercise: Derive explicitly these terms.

However we will need

$$G_0^{c^A(x)\bar{c}^B(y)} = \int \frac{d^4k}{(2\pi)^4}\frac{\delta^{AB}}{k^2+i\epsilon}e^{ik(x-y)}. \tag{9.282}$$

The interaction bteween the ghost and vector fileds is given by

$$-gf_{ABC}\int d^4x(\bar{c}^A c^B\partial^\mu A_\mu^C+\bar{c}^A\partial^\mu c^B A_\mu^c) = -gf_{ABC}\int d^4x\int d^4y\int d^4z\bar{c}^A(x)c^B(y)A_\mu^C(z)\partial_x^\mu(\delta^4(x-y)\delta^4(x-z)). \tag{9.283}$$

In other words (with $j_0=(z,C,\mu)$, $m=(x,A)$ and $n=(y,B)$)

$$\begin{aligned} S_{,A_{j_0}\bar{c}_m c_n} &= gf_{ABC}\partial_x^\mu(\delta^4(x-y)\delta^4(x-z))\\ &= -igf_{ABC}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4l}{(2\pi)^4}k^\mu(2\pi)^4\delta^4(p+k-l)\exp(-ikx-ipy+ilz). \end{aligned} \tag{9.284}$$

We compute the third diagram as follows. We have (with $j_0=(z,C,\mu)$, $k_0=(w,D,\nu)$, $m=(x,A)$, $n=(y,B)$ and $m_0=(x_0,A_0)$, $n_0=(y_0,B_0)$)

$$\begin{aligned} G_0^{\bar{c}_m c_{m_0}}G_0^{c_n\bar{c}_{n_0}}S_{,A_{j_0}\bar{c}_m c_n}S_{,A_{k_0}c_{m_0}\bar{c}_{n_0}} &= \sum_{A,A_0,B,B_0}\int d^4x\int d^4x_0\int d^4y\int d^4y_0\int\frac{d^4k}{(2\pi)^4}\frac{\delta^{A_0A}}{k^2}e^{ik(x_0-x)}\int\frac{d^4p}{(2\pi)^4}\frac{-\delta^{B_0B}}{p^2}\\ &\times e^{ip(y_0-y)}\big(gf_{ABC}\partial_x^\mu(\delta^4(x-y)\delta^4(x-z))\big)\big(-gf_{B_0A_0D}\partial_{y_0}^\nu(\delta^4(y_0-x_0)\delta^4(y_0-w))\big)\\ &= g^2f_{ABC}f_{ABD}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4p}{(2\pi)^4}\frac{(p+k)^\mu p^\nu}{(p+k)^2p^2}e^{ik(z-w)} \end{aligned} \tag{9.285}$$

We use Feynman parameters as before. Also we use rotational invariance to bring the above loop integral to the form

$$\begin{aligned} G_0^{\bar{c}_m c_{m_0}}G_0^{c_n\bar{c}_{n_0}}S_{,A_{j_0}\bar{c}_m c_n}S_{,A_{k_0}c_{m_0}\bar{c}_{n_0}} &= g^2C_2(G)\delta_{CD}\int\frac{d^4k}{(2\pi)^4}e^{ik(z-w)}\int_0^1 dx\int\frac{d^4l}{(2\pi)^4}\big(l^\mu l^\nu+x(x-1)k^\mu k^\nu\big)\frac{1}{(l^2-\Delta)^2}\\ &= g^2C_2(G)\delta_{CD}\int\frac{d^4k}{(2\pi)^4}e^{ik(z-w)}\int_0^1 dx\bigg\{\frac{1}{4}\eta^{\mu\nu}\int\frac{d^4l}{(2\pi)^4}\frac{l^2}{(l^2-\Delta)^2}\\ &+ x(x-1)k^\mu k^\nu\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-\Delta)^2}\bigg\}. \end{aligned} \tag{9.286}$$

Once more we employ dimensional regularization and use the integrals (9.269) and (9.270). Hence we get the loop integral (with $C\longrightarrow A$, $D\longrightarrow B$, $z\longrightarrow x$, $w\longrightarrow y$)

$$\begin{aligned} G_0^{\bar{c}_m c_{m_0}}G_0^{c_n\bar{c}_{n_0}}S_{,A_{j_0}\bar{c}_m c_n}S_{,A_{k_0}c_{m_0}\bar{c}_{n_0}} &= -g^2C_2(G)\frac{i}{(4\pi)^{\frac{d}{2}}}\delta_{AB}\int\frac{d^dk}{(2\pi)^d}e^{ik(x-y)}\bigg(-\frac{1}{2}\eta^{\mu\nu}k^2\Gamma(1-\frac{d}{2})+k^\mu k^\nu\Gamma(2-\frac{d}{2})\bigg)\\ &\times \int_0^1 dx\frac{x(1-x)}{\big(-x(1-x)k^2\big)^{2-\frac{d}{2}}}. \end{aligned} \tag{9.287}$$

By putting equations (9.271), (9.280) and (9.287) together we get

$$\begin{aligned}(9.271)+(9.280)-(9.287) &= g^2C_2(G)\frac{i}{(4\pi)^{\frac{d}{2}}}\delta_{AB}\int\frac{d^dk}{(2\pi)^d}e^{ik(x-y)}\int_0^1\frac{dx}{\big(-x(1-x)k^2\big)^{2-\frac{d}{2}}}\bigg\{\eta^{\mu\nu}k^2(d-2)\Gamma(2-\frac{d}{2})\\ &\times\ x(1-x)+\eta^{\mu\nu}k^2\Gamma(2-\frac{d}{2})\bigg(-(d-1)(1-x)^2+\frac{1}{2}(2-x)^2+\frac{1}{2}(1+x)^2\bigg)\\ &-\ k^\mu k^\nu\Gamma(2-\frac{d}{2})\bigg((1-\frac{d}{2})(1-2x)^2+2\bigg)\bigg\}.\end{aligned}\tag{9.288}$$

The pole at $d=2$ cancels exactly since the gamma function $\Gamma(1-d/2)$ is completely gone. There remains of course the pole at $d=4$. By using the symmetry of the integral over x under $x\longrightarrow 1-x$ we can rewrite the above integral as

$$\begin{aligned}(9.271)+(9.280)-(9.287) &= g^2C_2(G)\frac{i}{(4\pi)^{\frac{d}{2}}}\delta_{AB}\int\frac{d^dk}{(2\pi)^d}e^{ik(x-y)}\int_0^1\frac{dx}{\big(-x(1-x)k^2\big)^{2-\frac{d}{2}}}\bigg\{\eta^{\mu\nu}k^2(1-\frac{d}{2})\Gamma(2-\frac{d}{2})\\ &\times\ \bigg((1-2x)^2+(1-2x)\bigg)+\eta^{\mu\nu}k^2\Gamma(2-\frac{d}{2}).4x-k^\mu k^\nu\Gamma(2-\frac{d}{2})\bigg((1-\frac{d}{2})(1-2x)^2+2\bigg)\bigg\}.\end{aligned}\tag{9.289}$$

Again by the symmetry $x\longrightarrow 1-x$ we can replace x in every linear term in x by $1/2$[10]. We obtain the final result

$$\begin{aligned}(9.271)+(9.280)-(9.287) &= g^2C_2(G)\frac{i\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\delta_{AB}\int\frac{d^dk}{(2\pi)^d}e^{ik(x-y)}\bigg(\eta^{\mu\nu}k^2-k^\mu k^\nu\bigg)(k^2)^{\frac{d}{2}-2}\int_0^1\frac{dx}{\big(-x(1-x)\big)^{2-\frac{d}{2}}}\\ &\times\ \bigg((1-\frac{d}{2})(1-2x)^2+2\bigg)\\ &= g^2C_2(G)\frac{i\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\delta_{AB}\int\frac{d^dk}{(2\pi)^d}e^{ik(x-y)}\bigg(\eta^{\mu\nu}k^2-k^\mu k^\nu\bigg)(k^2)^{\frac{d}{2}-2}\bigg(\frac{5}{3}+\text{regular terms}\bigg).\end{aligned}\tag{9.290}$$

The gluon field is therefore transverse as it should be for any vector field with an underlying gauge symmetry. Indeed the exhibited the tensor structure $\eta^{\mu\nu}k^2-k^\mu k^\nu$ is consistent with Ward identity. This result does not depend on the gauge fixing parameter although the proportionality factor actually does[11].

There remains the fourth and final diagram which as it turns out is the only diagram which is independent of the gauge fixing parameter. We recall the Dirac field propagator

$$<\psi^a_\alpha(x)\bar{\psi}^b_\beta(y)> \ = \ i\delta^{ab}\int\frac{d^4p}{(2\pi)^4}\frac{(\gamma^\mu p_\mu+m)_{\alpha\beta}}{p^2-m^2+i\epsilon}e^{-ip(x-y)}.\tag{9.291}$$

However we will need something a little different. We have

$$\begin{aligned}S_{,\psi^a_\alpha(x)\bar{\psi}^b_\beta(y)}\equiv\frac{\delta^2S}{\delta\psi^a_\alpha(x)\delta\bar{\psi}^b_\beta(y)}|_{A,\psi,c=0} &= (i\gamma^\mu\partial^y_\mu-m)_{\beta\alpha}\delta^4(y-x)\delta^{ab}\\ &= \int\frac{d^4k}{(2\pi)^4}(\gamma^\mu k_\mu-m)_{\beta\alpha}e^{ik(x-y)}\delta^{ab}.\end{aligned}\tag{9.292}$$

[10] Exercise: Why.

[11] Exercise: Determine the corresponding factor for an arbitrary value of the gauge fixing parameter ξ.

Thus we must have

$$G_0^{\psi^a_\alpha(x)\bar{\psi}^b_\beta(y)} = \delta^{ab}\int\frac{d^4k}{(2\pi)^4}\frac{(\gamma^\mu k_\mu+m)_{\alpha\beta}}{k^2-m^2+i\epsilon}e^{-ik(x-y)}. \quad (9.293)$$

Indeed we can check

$$\int d^4y\sum_{b,\beta}S_{,\psi^a_\alpha(x)\bar{\psi}^b_\beta(y)}G_0^{\psi^{a_0}_{\alpha_0}(x_0)\bar{\psi}^b_\beta(y)} = \delta^{aa_0}\delta_{\alpha\alpha_0}\delta^4(x-x_0). \quad (9.294)$$

The interaction bteween the Dirac and vector fileds is given by

$$-gt^A_{ab}\int d^4x\sum_{a,b}\bar{\psi}^a\gamma^\mu\psi^bA^A_\mu = -gt^A_{ab}(\gamma^\mu)_{\alpha\beta}\int d^4x\int d^4y\int d^4z\bar{\psi}^a_\alpha(x)\psi^b_\beta(y)A^A_\mu(z)\delta^4(x-y)\delta^4(x-z). \quad (9.295)$$

In other words (with $j_0=(z,A,\mu)$, $m=(x,a,\alpha)$ and $n=(y,b,\beta)$)

$$S_{,A_{j_0}\bar{\psi}_m\psi_n} = gt^A_{ab}(\gamma^\mu)_{\alpha\beta}\delta^4(x-y)\delta^4(x-z). \quad (9.296)$$

By using these results we compute the fourth diagram is given by (with $j_0=(z,A,\mu)$, $k_0=(w,B,\nu)$, $m=(x,a,\alpha)$, $n=(y,b,\beta)$, $m_0=(x_0,a_0,\alpha_0)$, $n_0=(y_0,b_0,\beta_0)$ and $\mathrm{tr}\gamma^\mu=0$, $\mathrm{tr}\gamma^\mu\gamma^\nu=4\eta^{\mu\nu}$, $\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho=0$, $\mathrm{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma=4(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho})$)

$$\begin{aligned}G_0^{\bar{\psi}_m\psi_{m_0}}G_0^{\psi_n\bar{\psi}_{n_0}}S_{,A_{j_0}\bar{\psi}_m\psi_n}S_{,A_{k_0}\psi_{m_0}\bar{\psi}_{n_0}} &= g^2\mathrm{tr}t^At^B\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{tr}(\gamma^\rho p_\rho+m)\gamma^\mu(\gamma^\rho(p+k)_\rho+m)\gamma^\nu}{(p^2-m^2)((p+k)^2-m^2)}e^{-ik(z-w)}\\ &= 4g^2\mathrm{tr}t^At^B\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\frac{p^\mu(p+k)^\nu+p^\nu(p+k)^\mu-\eta^{\mu\nu}(p^2+pk-m^2)}{(p^2-m^2)((p+k)^2-m^2)}\\ &\times\ e^{-ik(z-w)}.\end{aligned} \quad (9.297)$$

We use now Feynman parameters in the form

$$\begin{aligned}\frac{1}{(p^2-m^2)((p+k)^2-m^2)} &= \int_0^1dx\int_0^1dy\delta(x+y-1)\frac{1}{\Big[x(p^2-m^2)+y((p+k)^2-m^2)\Big]^2}\\ &= \int_0^1dx\frac{1}{(l^2-\Delta)^2}\ ,\ l=p+(1-x)k\ ,\ \Delta=m^2-x(1-x)k^2.\end{aligned} \quad (9.298)$$

By using also rotational invariance we can bring the integral to the form

$$\begin{aligned}
G_0^{\bar{\psi}_m\psi_{m_0}}G_0^{\psi_n\bar{\psi}_{n_0}}S_{,A_{j_0}\bar{\psi}_m\psi_n}S_{,A_{k_0}\psi_{m_0}\bar{\psi}_{n_0}} &= 4g^2\mathrm{tr}t^At^B\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}e^{-ik(z-w)}\\
&\times \int\frac{d^4l}{(2\pi)^4}\bigg[2l^\mu l^\nu-2x(1-x)k^\mu k^\nu-\eta^{\mu\nu}\big(l^2-x(1-x)k^2-m^2\big)\bigg]\frac{1}{(l^2-\Delta)^2}\\
&= 4g^2\mathrm{tr}t^At^B\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}e^{-ik(z-w)}\\
&\times \int\frac{d^4l}{(2\pi)^4}\bigg[\frac{1}{2}l^2\eta^{\mu\nu}-2x(1-x)k^\mu k^\nu-\eta^{\mu\nu}\big(l^2-x(1-x)k^2-m^2\big)\bigg]\frac{1}{(l^2-\Delta)^2}\\
&= 4g^2\mathrm{tr}t^At^B\int_0^1 dx\int\frac{d^4k}{(2\pi)^4}e^{-ik(z-w)}\bigg\{\bigg[x(1-x)(k^2\eta^{\mu\nu}-2k^\mu k^\nu)+m^2\eta^{\mu\nu}\bigg]\\
&\times \int\frac{d^4l}{(2\pi)^4}\frac{1}{(l^2-\Delta)^2}-\frac{1}{2}\eta^{\mu\nu}\int\frac{d^4l}{(2\pi)^4}\frac{l^2}{(l^2-\Delta)^2}\bigg\}
\end{aligned}\tag{9.299}$$

After using the integrals (9.269) and (9.270), the fourth diagram becomes (with $z\longrightarrow x$, $w\longrightarrow y$)

$$\begin{aligned}
G_0^{\bar{\psi}_m\psi_{m_0}}G_0^{\psi_n\bar{\psi}_{n_0}}S_{,A_{j_0}\bar{\psi}_m\psi_n}S_{,A_{k_0}\psi_{m_0}\bar{\psi}_{n_0}} &= 8ig^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\mathrm{tr}t^At^B\int\frac{d^dk}{(2\pi)^d}(k^2\eta^{\mu\nu}-k^\mu k^\nu)e^{-ik(x-y)}\\
&\times (k^2)^{\frac{d}{2}-2}\int_0^1 dx\frac{x(1-x)}{\big(\frac{m^2}{k^2}-x(1-x)\big)^{2-\frac{d}{2}}}\\
&= \frac{4}{3}g^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}C(N)\delta_{AB}\int\frac{d^dk}{(2\pi)^d}i(k^2\eta^{\mu\nu}-k^\mu k^\nu)e^{-ik(x-y)}(k^2)^{\frac{d}{2}-2}\bigg(1\\
&+ \text{regular terms}\bigg).
\end{aligned}\tag{9.300}$$

For n_f flavors (instead of a single flavor) of fermions in the representation t_r^a (instead of the fundamental representation t_a) we obtain (with also a change $k\longrightarrow -k$)

$$\begin{aligned}
G_0^{\bar{\psi}_m\psi_{m_0}}G_0^{\psi_n\bar{\psi}_{n_0}}S_{,A_{j_0}\bar{\psi}_m\psi_n}S_{,A_{k_0}\psi_{m_0}\bar{\psi}_{n_0}} &= \frac{4}{3}n_fg^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}C(r)\delta_{AB}\int\frac{d^dk}{(2\pi)^d}i(k^2\eta^{\mu\nu}-k^\mu k^\nu)e^{ik(x-y)}(k^2)^{\frac{d}{2}-2}\bigg(1\\
&+ \text{regular terms}\bigg).
\end{aligned}\tag{9.301}$$

By putting (9.290) and (9.301) together we get the final result

$$\begin{aligned}
\Gamma_{\mu\nu}^{AB}(x,y) &= (9.290)-(9.301)\\
&- \int\frac{d^4k}{(2\pi)^4}\bigg(k^2\eta^{\mu\nu}+(\frac{1}{\xi}-1)k^\mu k^\nu\bigg)\delta_{AB}e^{ik(x-y)}\\
&+ g^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\bigg(\frac{5}{3}C_2(G)-\frac{4}{3}n_fC(r)\bigg)\int\frac{d^dk}{(2\pi)^d}(k^2\eta^{\mu\nu}-k^\mu k^\nu)\delta_{AB}e^{ik(x-y)}(k^2)^{\frac{d}{2}-2}\bigg(1\\
&+ \text{regular terms}\bigg).
\end{aligned}\tag{9.302}$$

The final step is to add the contribution of the counter terms. This leads to the one-loop result in the Feynman-t'Hooft gauge given by

$$\begin{aligned}
\Gamma^{AB}_{\mu\nu}(x,y) &= -\int\frac{d^dk}{(2\pi)^d}\bigg(k^2\eta^{\mu\nu}+(\frac{1}{\xi}-1)k^\mu k^\nu\bigg)\delta_{AB}e^{ik(x-y)}\\
&+ g^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\bigg(\frac{5}{3}C_2(G)-\frac{4}{3}n_fC(r)\bigg)\int\frac{d^dk}{(2\pi)^d}(k^2\eta^{\mu\nu}-k^\mu k^\nu)\delta_{AB}e^{ik(x-y)}(k^2)^{\frac{d}{2}-2}\bigg(1\\
&+ \text{regular terms}\bigg)-\delta_3\int\frac{d^dk}{(2\pi)^d}\bigg(k^2\eta^{\mu\nu}+(\frac{1}{\xi}-1)k^\mu k^\nu\bigg)\delta_{AB}e^{ik(x-y)}. \qquad (9.303)
\end{aligned}$$

Equivalently

$$\begin{aligned}
\Gamma^{AB}_{\mu\nu}(k) &= -\bigg(k^2\eta^{\mu\nu}+(\frac{1}{\xi}-1)k^\mu k^\nu\bigg)\delta_{AB}\\
&+ g^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\bigg(\frac{5}{3}C_2(G)-\frac{4}{3}n_fC(r)\bigg)(k^2\eta^{\mu\nu}-k^\mu k^\nu)\delta_{AB}(k^2)^{\frac{d}{2}-2}\bigg(1\\
&+ \text{regular terms}\bigg)-\delta_3\bigg(k^2\eta^{\mu\nu}-k^\mu k^\nu\bigg)\delta_{AB}. \qquad (9.304)
\end{aligned}$$

Remark that the $1/\xi$ term in the classical contribution (the first term) can be removed by undoing the gauge fixing procedure. In 4 dimensions the coupling constant g^2 is dimensionless.

In dimension $d=4-\epsilon$ the coupling constant g is in fact not dimensionless but has dimension of $1/\text{mass}^{(d/2-2)}$. The dimensionless coupling constant $\hat{g}$ can therefore be given in terms of an arbitrary mass scale μ by the formula

$$\hat{g}=g\mu^{\frac{d}{2}-2}\Leftrightarrow g^2=\hat{g}^2\mu^\epsilon. \qquad (9.305)$$

We get then

$$\begin{aligned}
\Gamma^{AB}_{\mu\nu}(k) &= -\bigg(k^2\eta^{\mu\nu}+(\frac{1}{\xi}-1)k^\mu k^\nu\bigg)\delta_{AB}+\frac{\hat{g}^2}{16\pi^2}\Gamma(\frac{\epsilon}{2})\bigg(\frac{4\pi\mu^2}{k^2}\bigg)^{\frac{\epsilon}{2}}\bigg(\frac{5}{3}C_2(G)-\frac{4}{3}n_fC(r)\bigg)(k^2\eta^{\mu\nu}-k^\mu k^\nu)\delta_{AB}\bigg(1\\
&+ \text{regular terms}\bigg)-\delta_3\bigg(k^2\eta^{\mu\nu}-k^\mu k^\nu\bigg)\delta_{AB}\\
&= -\bigg(k^2\eta^{\mu\nu}+(\frac{1}{\xi}-1)k^\mu k^\nu\bigg)\delta_{AB}+\frac{\hat{g}^2}{16\pi^2}\bigg(\frac{2}{\epsilon}+\ln 4\pi-\gamma-\ln\frac{k^2}{\mu^2}\bigg)\bigg(\frac{5}{3}C_2(G)+\frac{4}{3}n_fC(r)\bigg)(k^2\eta^{\mu\nu}-k^\mu k^\nu)\delta_{AB}\\
&\times \bigg(1+\text{regular terms}\bigg)-\delta_3\bigg(k^2\eta^{\mu\nu}-k^\mu k^\nu\bigg)\delta_{AB}
\end{aligned}$$
$$(9.306)$$

It is now clear that in order to eliminate the divergent term we need, in the spirit of minimal subtraction, only subtract the logarithmic divergence exhibited here by the the term $2/\epsilon$ which has a pole at $\epsilon=0$. In other words the counter term δ_3 is chosen such that

$$\delta_3=\frac{\hat{g}^2}{16\pi^2}\bigg(\frac{2}{\epsilon}\bigg)\bigg(\frac{5}{3}C_2(G)-\frac{4}{3}n_fC(r)\bigg). \qquad (9.307)$$

9.4.4 The Quark Field Self-Energy at 1−Loop

This is defined

$$\Gamma^{ab}_{\alpha\beta}(x,y) = \frac{\delta^2\Gamma}{\delta\psi^a_\alpha(x)\delta\bar{\psi}^b_\beta(y)}|_{A,\psi,c=0}$$
$$= \frac{\delta^2 S}{\delta\psi^a_\alpha(x)\delta\bar{\psi}^b_\beta(y)}|_{A,\psi,c=0} + \frac{1}{i}\frac{\delta^2\Gamma_1}{\delta\psi^a_\alpha(x)\delta\bar{\psi}^b_\beta(y)}|_{A,\psi,c=0}. \quad (9.308)$$

The first term is given by

$$\frac{\delta^2 S}{\delta\psi^a_\alpha(x)\delta\bar{\psi}^b_\beta(y)}|_{A,\psi,c=0} = (i\gamma^\mu\partial^y_\mu - m)_{\beta\alpha}\delta^4(y-x)\delta^{ab}$$
$$= \int\frac{d^4k}{(2\pi)^4}(\gamma^\mu k_\mu - m)_{\beta\alpha}e^{ik(x-y)}\delta^{ab}. \quad (9.309)$$

Again by using the elegant formula (9.257) we obtain (with $j_0=(x,\alpha,a)$ and $k_0=(y,\beta,b)$)

$$\Gamma_{1,j_0k_0} = -G_0^{\bar{\psi}_m\psi_{m_0}}G_0^{A_nA_{n_0}}S_{,\psi_{j_0}\bar{\psi}_mA_n}S_{,\bar{\psi}_{k_0}\psi_{m_0}A_{n_0}}. \quad (9.310)$$

We recall the results

$$G_0^{A^{\mu A}(x)A^{\nu B}(y)} = \delta_{AB}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+i\epsilon}\Big[\eta^{\mu\nu}+(\xi-1)\frac{k^\mu k^\nu}{k^2}\Big]e^{ik(x-y)}. \quad (9.311)$$

$$G_0^{\psi^a_\alpha(x)\bar{\psi}^b_\beta(y)} = \delta^{ab}\int\frac{d^4p}{(2\pi)^4}\frac{(\gamma^\mu p_\mu+m)_{\alpha\beta}}{p^2-m^2+i\epsilon}e^{-ip(x-y)}. \quad (9.312)$$

$$S_{,A^{\mu A}(z)\bar{\psi}^a_\alpha(x)\psi^b_\beta(y)} = gt^A_{ab}(\gamma^\mu)_{\alpha\beta}\delta^4(x-y)\delta^4(x-z). \quad (9.313)$$

We compute then

$$\Gamma_{1,j_0k_0} = -g^2(t^At^A)_{ba}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\big(\gamma^\mu(\gamma^\rho p_\rho+m)\gamma^\nu\big)_{\beta\alpha}\frac{1}{k^2(p^2-m^2)}\Big(\eta_{\mu\nu}+(\xi-1)\frac{k_\mu k_\nu}{k^2}\Big)e^{i(k+p)(x-y)}. \quad (9.314)$$

This is given by the second diagram on figure 10. In the Feynman- t'Hooft gauge this reduces to (also using $\gamma^\mu\gamma^\rho\gamma_\mu=-(2-\epsilon)\gamma^\rho$, $\gamma^\mu\gamma_\mu=d$ and $(t^At^A)_{ab}=C_2(r)\delta_{ab}$ where $C_2(r)$ is the Casimir in the representation r)

$$\Gamma_{1,j_0k_0} = -g^2C_2(r)\delta_{ba}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\big(\gamma^\mu(\gamma^\rho(p+k)_\rho+m)\gamma_\mu\big)_{\beta\alpha}\frac{1}{k^2((p+k)^2-m^2)}e^{ip(x-y)}$$
$$= -g^2C_2(r)\delta_{ba}\int\frac{d^4p}{(2\pi)^4}\int\frac{d^4k}{(2\pi)^4}\big(-(2-\epsilon)\gamma^\rho(p+k)_\rho+md\big)_{\beta\alpha}\frac{1}{k^2((p+k)^2-m^2)}e^{ip(x-y)}. \quad (9.315)$$

We employ Feynman parameters in the form

$$\frac{1}{k^2((p+k)^2-m^2)} = \int_0^1 dx\frac{1}{\big(l^2-\Delta\big)^2}\ ,\ l=k+(1-x)p\ ,\ \Delta=-x(1-x)p^2+(1-x)m^2. \quad (9.316)$$

We obtain

$$\begin{aligned}\Gamma_{1,j_0k_0} &= -g^2C_2(r)\delta_{ba}\int\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\int_0^1dx\int\frac{d^4l}{(2\pi)^4}\frac{1}{\big(l^2-\Delta\big)^2}\big(-(2-\epsilon)\gamma^\rho(l+xp)_\rho+md\big)_{\beta\alpha}\\ &= -g^2C_2(r)\delta_{ba}\int\frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\int_0^1dx\int\frac{d^4l}{(2\pi)^4}\frac{1}{\big(l^2-\Delta\big)^2}\big(-(2-\epsilon)x\gamma^\rho p_\rho+md\big)_{\beta\alpha}.\end{aligned} \tag{9.317}$$

After Wick rotation and dimensional regularization we can use the integral (9.269). We get

$$\begin{aligned}\Gamma_{1,j_0k_0} &= -g^2C_2(r)\frac{i\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\delta_{ba}\int\frac{d^4p}{(2\pi)^4}\big(-\frac{1}{2}(2-\epsilon)\gamma^\rho p_\rho+md\big)_{\beta\alpha}e^{ip(x-y)}(p^2)^{-\frac{\epsilon}{2}}\int_0^1\frac{dx}{\big(-x(1-x)+(1-x)\frac{m^2}{p^2}\big)^{2-\frac{d}{2}}}\\ &= -g^2C_2(r)\frac{i\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\delta_{ba}\int\frac{d^4p}{(2\pi)^4}\big(-\frac{1}{2}(2-\epsilon)\gamma^\rho p_\rho+md\big)_{\beta\alpha}e^{ip(x-y)}(p^2)^{-\frac{\epsilon}{2}}\bigg(1+\text{regular terms}\bigg).\end{aligned} \tag{9.318}$$

The quark field self-energy at 1−loop is therefore given by

$$\begin{aligned}\Gamma^{ab}_{\alpha\beta}(x,y) &= \int\frac{d^4p}{(2\pi)^4}(\gamma^\mu p_\mu-m)_{\beta\alpha}e^{ip(x-y)}\delta^{ab}\\ &- g^2C_2(r)\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\delta_{ba}\int\frac{d^4p}{(2\pi)^4}\big(-\frac{1}{2}(2-\epsilon)\gamma^\rho p_\rho+md\big)_{\beta\alpha}e^{ip(x-y)}(p^2)^{-\frac{\epsilon}{2}}\bigg(1+\text{regular terms}\bigg).\end{aligned} \tag{9.319}$$

We add the contribution of the counter terms. We obtain

$$\begin{aligned}\Gamma^{ab}_{\alpha\beta}(x,y) &= \int\frac{d^4p}{(2\pi)^4}(\gamma^\mu p_\mu-m)_{\beta\alpha}e^{ip(x-y)}\delta^{ab}\\ &- \frac{\hat{g}^2}{16\pi^2}C_2(r)\delta_{ba}\int\frac{d^4p}{(2\pi)^4}\bigg(\frac{2}{\epsilon}+\ln 4\pi-\gamma-\ln\frac{p^2}{\mu^2}\bigg)\big(-\gamma^\rho p_\rho+md\big)_{\beta\alpha}e^{ip(x-y)}\bigg(1+\text{regular terms}\bigg)\\ &+ \int\frac{d^4p}{(2\pi)^4}(\delta_2\gamma^\mu p_\mu-\delta_m)_{\beta\alpha}e^{ip(x-y)}\delta^{ab}.\end{aligned} \tag{9.320}$$

In order to cancel the divergence we must choose the counter terms δ_2 and δ_m to be

$$\delta_2=-\frac{\hat{g}^2}{16\pi^2}C_2(r)\bigg(\frac{2}{\epsilon}\bigg). \tag{9.321}$$

$$\delta_m=-\frac{\hat{g}^2}{16\pi^2}C_2(r)\bigg(\frac{8m}{\epsilon}\bigg). \tag{9.322}$$

These two counter terms allow us to determine the renormalized mass m in terms of the bare mass up to the one-loop order.

9.4.5 The Vertex at $1-$Loop

The quark-gluon vertex at one-loop is given by

$$\begin{aligned}
\Gamma^{abA}_{\alpha\beta\mu}(x,y,z) &= \frac{\delta^3\Gamma}{\delta\bar{\psi}^a_\alpha(x)\delta\psi^b_\beta(y)\delta A^A_\mu(z)}|_{A,\psi,c=0}\\
&= \frac{\delta^3 S}{\delta\bar{\psi}^a_\alpha(x)\delta\psi^b_\beta(y)\delta A^A_\mu(z)}|_{A,\psi,c=0}+\frac{1}{i}\frac{\delta^3\Gamma_1}{\delta\bar{\psi}^a_\alpha(x)\delta\psi^b_\beta(y)\delta A^A_\mu(z)}|_{A,\psi,c=0}\\
&= g(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}\delta^4(x-y)\delta^4(x-z)+\frac{1}{i}\frac{\delta^3\Gamma_1}{\delta\bar{\psi}^a_\alpha(x)\delta\psi^b_\beta(y)\delta A^A_\mu(z)}|_{A,\psi,c=0}.
\end{aligned} \tag{9.323}$$

In this section we compute the one-loop correction using Feynman rules directly. We write

$$\begin{aligned}
\int d^4x\int d^4y\int d^4z e^{-ikx-ipy-ilz}\Gamma^{abA}_{\alpha\beta\mu}(x,y,z) &= g(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}(2\pi)^4\delta^4(k+p+l)+\frac{1}{i}\int d^4x\int d^4y\int d^4z e^{-ikx-ipy-ilz}\\
&\times \frac{\delta^3\Gamma_1}{\delta\bar{\psi}^a_\alpha(x)\delta\psi^b_\beta(y)\delta A^A_\mu(z)}|_{A,\psi,c=0}\\
&= \bigg[g(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}+\frac{1}{i}\bigg(\text{Feynman diagrams}\bigg)\bigg](2\pi)^4\delta^4(k+p+l).
\end{aligned} \tag{9.324}$$

It is not difficult to convince ourselves that there are only two possible Feynman diagrams contributing to this $3-$point proper vertex which we will only evaluate their leading divergent part in the Feynman-'t Hooft gauge. The first diagram on figure 12 is given explicitly by

$$\begin{aligned}
12a &= -ig^3 f_{CDA}(t^Dt^C)_{ab}\int\frac{d^dk}{(2\pi)^d}\bigg[(-k+p_1-2p_2)^\rho\eta^{\lambda\mu}-(k+2p_1-p_2)^\lambda\eta^{\rho\mu}+(2k+p_1+p_2)^\mu\eta^{\lambda\rho}\bigg]\\
&\times \frac{\big(\gamma_\lambda(\gamma.k+m)\gamma_\rho\big)_{\alpha\beta}}{(k^2-m^2)(k+p_1)^2(k+p_2)^2}\\
&= -\frac{g^3C_2(G)}{2}(t^A)_{ab}\int\frac{d^dk}{(2\pi)^d}\bigg[(-k+p_1-2p_2)^\rho\eta^{\lambda\mu}-(k+2p_1-p_2)^\lambda\eta^{\rho\mu}+(2k+p_1+p_2)^\mu\eta^{\lambda\rho}\bigg]\\
&\times \frac{\big(\gamma_\lambda(\gamma.k+m)\gamma_\rho\big)_{\alpha\beta}}{(k^2-m^2)(k+p_1)^2(k+p_2)^2}.
\end{aligned} \tag{9.325}$$

In the second line we have used the fact that $f_{CDA}t^Dt^C=f_{CDA}[t^D,t^C]/2=if_{CDA}f_{DCE}t^E/2$. We make now the approximation of neglecting the quark mass and all external momenta since the divergence is actually independent of both [12]. The result reduces to

$$\begin{aligned}
12a &= -\frac{g^3C_2(G)}{2}(t^A)_{ab}\int\frac{d^dk}{(2\pi)^d}\bigg[-k^\rho\eta^{\lambda\mu}-k^\lambda\eta^{\rho\mu}+2k^\mu\eta^{\lambda\rho}\bigg]\\
&\times \frac{\big(\gamma_\lambda(\gamma.k)\gamma_\rho\big)_{\alpha\beta}}{(k^2)^3}\\
&= -\frac{g^3C_2(G)}{2}(t^A)_{ab}\int\frac{d^dk}{(2\pi)^d}\bigg[-2(\gamma^\mu)_{\alpha\beta}k^2-2(2-\epsilon)k^\mu k^\nu(\gamma_\nu)_{\alpha\beta}\bigg]\frac{1}{(k^2)^3}\\
&= -\frac{g^3C_2(G)}{2}(t^A)_{ab}\int\frac{d^dk}{(2\pi)^d}\bigg[-2(\gamma^\mu)_{\alpha\beta}k^2-\frac{2(2-\epsilon)}{d}k^2(\gamma^\mu)_{\alpha\beta}\bigg]\frac{1}{(k^2)^3}\\
&= \frac{g^3C_2(G)}{2}(t^A)_{ab}\frac{4(d-1)}{d}(\gamma^\mu)_{\alpha\beta}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^2}\\
&= \frac{3ig^3C_2(G)}{2(4\pi)^2}(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}\Gamma(2-\frac{d}{2}).
\end{aligned} \tag{9.326}$$

The second diagram on figure 12 is given explicitly by

$$12b = g^3(t^Ct^At^C)_{ab}\int\frac{d^dk}{(2\pi)^d}\frac{\big(\gamma_\lambda(-\gamma.(k+p_2)+m)\gamma^\mu(-\gamma.(k+p_1)+m)\gamma^\lambda\big)_{\alpha\beta}}{k^2((k+p_1)^2-m^2)((k+p_2)^2-m^2)}. \tag{9.327}$$

We compute

$$\begin{aligned}
t^Ct^At^C &= t^Ct^Ct^A+t^C[t^A,t^C]\\
&= C_2(N)t^A+if_{ACB}t^Ct^B\\
&= C_2(N)t^A+\frac{i}{2}f_{ACB}[t^C,t^B]\\
&= \bigg[C_2(N)-\frac{1}{2}C_2(G)\bigg]t^A.
\end{aligned} \tag{9.328}$$

We get then

$$12b = g^3\big(C_2(N)-\frac{1}{2}C_2(G)\big)(t^A)_{ab}\int\frac{d^dk}{(2\pi)^d}\frac{\big(\gamma_\lambda(-\gamma.(k+p_2)+m)\gamma^\mu(-\gamma.(k+p_1)+m)\gamma^\lambda\big)_{\alpha\beta}}{k^2((k+p_1)^2-m^2)((k+p_2)^2-m^2)}. \tag{9.329}$$

Again as before we are only interested at this stage in the leading divergent part and thus we can make the approximation of dropping the quark mass and all external momenta[13]. We obtain

[12]Exercise: Compute this integral without making these approximations and show that the divergence is indeed independent of the quark mass and all external momenta.

[13]Exercise: Compute this integral without making these approximations.

thus

$$\begin{aligned}
12b &= g^3\big(C_2(N)-\frac{1}{2}C_2(G)\big)(t^A)_{ab}\int\frac{d^dk}{(2\pi)^d}\frac{\big(\gamma_\lambda(-\gamma.k)\gamma^\mu(-\gamma.k)\gamma^\lambda\big)_{\alpha\beta}}{(k^2)^3}\\
&= g^3\big(C_2(N)-\frac{1}{2}C_2(G)\big)(t^A)_{ab}\int\frac{d^dk}{(2\pi)^d}\frac{\big(\gamma_\lambda\gamma_\rho\gamma^\mu\gamma_\sigma\gamma^\lambda\big)_{\alpha\beta}k^\rho k^\sigma}{(k^2)^3}\\
&= g^3\big(C_2(N)-\frac{1}{2}C_2(G)\big)(t^A)_{ab}\frac{1}{d}\int\frac{d^dk}{(2\pi)^d}\frac{\big(\gamma_\lambda\gamma_\rho\gamma^\mu\gamma^\rho\gamma^\lambda\big)_{\alpha\beta}}{(k^2)^2}\\
&= g^3\big(C_2(N)-\frac{1}{2}C_2(G)\big)(t^A)_{ab}\frac{(2-\epsilon)^2}{d}(\gamma^\mu)_{\alpha\beta}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2)^2}\\
&= \frac{ig^3}{(4\pi)^2}\big(C_2(N)-\frac{1}{2}C_2(G)\big)(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}\Gamma(2-\frac{d}{2}).
\end{aligned}\tag{9.330}$$

By putting the two results $12a$ and $12b$ together we obtain

$$12a+12b = \frac{ig^3}{(4\pi)^2}\big(C_2(N)+C_2(G)\big)(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}\Gamma(2-\frac{d}{2}).\tag{9.331}$$

Again if the quarks are in the representation t^a_r instead of the fundamental representation t^a we would have obtained

$$12a+12b = \frac{ig^3}{(4\pi)^2}\big(C_2(r)+C_2(G)\big)(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}\Gamma(2-\frac{d}{2}).\tag{9.332}$$

The dressed quark-gluon vertex at one-loop is therefore given by

$$\begin{aligned}
\int d^4x\int d^4y\int d^4z e^{-ikx-ipy-ilz}\Gamma^{abA}_{\alpha\beta\mu}(x,y,z) &= \Big[g(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}+\frac{g^3}{(4\pi)^2}\big(C_2(r)+C_2(G)\big)(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}\Gamma(2-\frac{d}{2})\Big]\\
&\times\ (2\pi)^4\delta^4(k+p+l)\\
&= \Big[g(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}+\frac{g^3}{(4\pi)^2}\big(C_2(r)+C_2(G)\big)(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}\Big(\frac{2}{\epsilon}+...\Big)\Big]\\
&\times\ (2\pi)^4\delta^4(k+p+l)
\end{aligned}\tag{9.333}$$

Adding the contribution of the counter terms is trivial since the relevant counter term is of the same form as the bare vertex. We get

$$\begin{aligned}
\int d^4x\int d^4y\int d^4z e^{-ikx-ipy-ilz}\Gamma^{abA}_{\alpha\beta\mu}(x,y,z) &= \Big[g(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}+\frac{g^3}{(4\pi)^2}\big(C_2(r)+C_2(G)\big)(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}\Big(\frac{2}{\epsilon}+...\Big)\\
&+\ \delta_1 g(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}\Big](2\pi)^4\delta^4(k+p+l).
\end{aligned}\tag{9.334}$$

We conclude that, in order to subtract the logarithmic divergence in the vertex, the counter term δ_1 must be chosen such that

$$\delta_1 = -\frac{g^2}{(4\pi)^2}\big(C_2(r)+C_2(G)\big)\Big(\frac{2}{\epsilon}\Big).\tag{9.335}$$

In a more careful treatment we should get [14]

$$\delta_1 = -\frac{\hat{g}^2}{(4\pi)^2}\big(C_2(r) + C_2(G)\big)\left(\frac{2}{\epsilon}\right). \tag{9.336}$$

We recall that the renormalized coupling g is related to the bare coupling g_0 by the relation

$$\begin{aligned} \frac{g}{g_0} &= \frac{Z_2\sqrt{Z_3}}{1+\delta_1} \\ &= 1 - \delta_1 + \delta_2 + \frac{1}{2}\delta_3 \\ &= 1 + \frac{\hat{g}^2}{16\pi^2}\frac{1}{\epsilon}\left[\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)\right] \\ &= 1 + \mu^{-\epsilon}\frac{g^2}{16\pi^2}\frac{1}{\epsilon}\left[\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)\right]. \end{aligned} \tag{9.337}$$

This is equivalent to

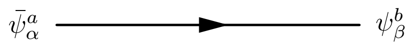
$$g = g_0 + \mu^{-\epsilon}\frac{g_0^3}{16\pi^2}\frac{1}{\epsilon}\left[\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)\right]. \tag{9.338}$$

We compute then

$$\begin{aligned} \mu\frac{\partial g}{\partial \mu} &= -\mu^{-\epsilon}\frac{g_0^3}{16\pi^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)\right] \\ &= -\frac{g^3}{16\pi^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}n_f C(r)\right]. \end{aligned} \tag{9.339}$$

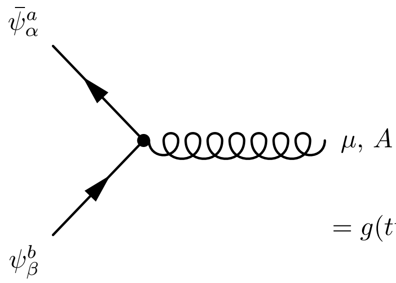
[14] This should become apparent if you solve the previous two exercises.

$$= \frac{\delta^{AB}}{k^2}\bigg[\eta_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2}\bigg].$$

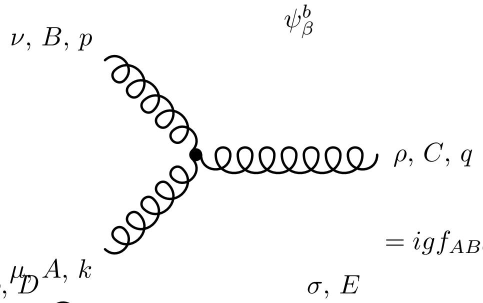


$$= \frac{\delta^{ab}(\gamma.p + m)_{\beta\alpha}}{p^2 - m^2}.$$

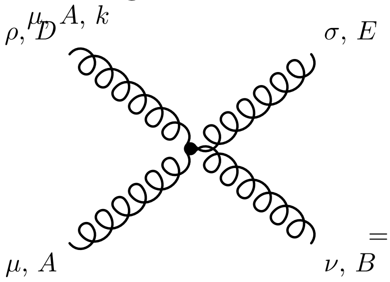
$$= \frac{\delta^{AB}}{p^2}.$$



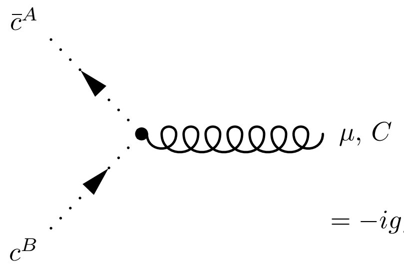
$$= g(t^A)_{ab}(\gamma^\mu)_{\alpha\beta}.$$



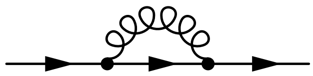
$$= igf_{ABC}\bigg[(k + 2p)^\mu \eta^{\rho\nu} - (p + 2k)^\nu \eta^{\rho\mu} + (k - p)^\rho \eta^{\mu\nu}\bigg].$$



$$\begin{aligned} = &\quad -g^2\bigg[f_{ABC}f_{DEC}(\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu}) + f_{BDC}f_{AEC}(\eta^{\sigma\rho}\eta^{\mu\nu} - \eta^{\rho\mu}\eta^{\sigma\nu}) \\ + &\quad f_{DAC}f_{BEC}(\eta^{\sigma\mu}\eta^{\rho\nu} - \eta^{\mu\nu}\eta^{\sigma\rho})\bigg]. \end{aligned}$$



$$= -igf_{ABC}k^{\mu}.$$



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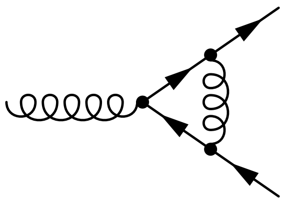

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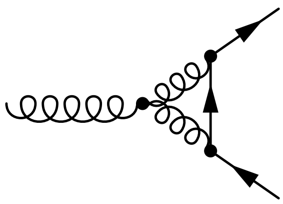

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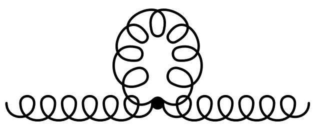

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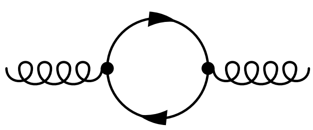

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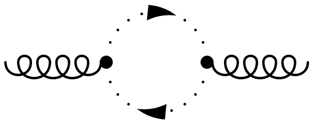

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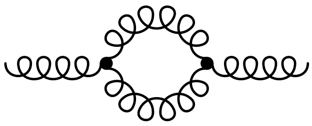

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10
The Renormalization Group

10.1 Critical Phenomena and The ϕ^4 Theory

10.1.1 Critical Line and Continuum Limit

We are interested in the critical properties of systems which are ergodic at finite volume, i.e. they can access all regions of their phase space with non zero probability. In the infinite volume limit these systems may become non ergodic and as a consequence the phase space decomposes into disjoint sets corresponding to different phases. The thermodynamical limit is related to the largest eigenvalue of the so-called transfer matrix. If the system remains ergodic then the largest eigenvalue of the transfer matrix is non degenerate while it becomes degenerate if the system becomes non ergodic.

The boundary between the different phases is demarcated by a critical line or a second order phase transition which is defined by the requirement that the correlation length, which is the inverse of the smallest decay rate of correlation functions or equivalent the smallest physical mass, diverges at the transition point.

The properties of these systems near the transition line are universal and are described by the renormalization group equations of Euclidean scalar field theory. The requirement of locality in field theory is equivalent to short range forces in second order phase transitions. The property of universality is intimately related to the property of renormalizability of the field theory. More precisely universality in second order phase transitions emerges in the regime in which the correlation length is much larger than the macroscopic scale which corresponds, on the field theory side, to the fact that renormalizable local field theory is insensitive to short distance physics in the sense that we obtain a unique renormalized Lagrangian in the limit in which all masses and momenta are much smaller than the UV cutoff Λ.

The Euclidean $O(N)$ ϕ^4 action is given by (with some change of notation compared to previous chapters and sections)

$$S[\phi] = -\int d^dx\bigg(\frac{1}{2}(\partial_\mu\phi^i)^2 + \frac{1}{2}m^2\phi^i\phi^i + \frac{\lambda}{4}(\phi^i\phi^i)^2\bigg). \tag{10.1}$$

We will employ lattice regularization in which $x = an$, $\int d^dx = a^d \sum_n$, $\phi^i(x) = \phi^i_n$ and $\partial_\mu \phi^i = (\phi^i_{n+\hat{\mu}} - \phi^i_n)/a$. The lattice action reads

$$\begin{aligned} S[\phi] &= \sum_n \bigg(a^{d-2} \sum_\mu \phi^i_n \phi^i_{n+\hat{\mu}} - \frac{a^{d-2}}{2}(m^2a^2 + 2d)\phi^i_n\phi^i_n - \frac{a^d\lambda}{4}(\phi^i_n\phi^i_n)^2 \bigg) \\ &= \sum_n \bigg(2\kappa \sum_\mu \Phi^i_n \Phi^i_{n+\hat{\mu}} - \Phi^i_n\Phi^i_n - g(\Phi^i_n\Phi^i_n - 1)^2 \bigg). \end{aligned} \tag{10.2}$$

The mass parameter m^2 is replaced by the so-called hopping parameter κ and the coupling constant λ is replaced by the coupling constant g where

$$m^2a^2 = \frac{1-2g}{\kappa} - 2d \ , \ \frac{\lambda}{a^{d-4}} = \frac{g}{\kappa^2}. \tag{10.3}$$

The fields ϕ^i_n and Φ^i_n are related by

$$\phi^i_n = \sqrt{\frac{2\kappa}{a^{d-2}}}\Phi^i_n. \tag{10.4}$$

$$m^2a^2 = \frac{1-2g}{\kappa} - 2d \ , \ \frac{\lambda}{a^{d-4}} = \frac{g}{\kappa^2}. \tag{10.5}$$

The partition function is given by

$$\begin{aligned} Z &= \int \prod_{n,i} d\Phi^i_n \ e^{S[\phi]} \\ &= \int d\mu(\Phi) \ e^{2\kappa \sum_n \sum_\mu \Phi^i_n \Phi^i_{n+\hat{\mu}}}. \end{aligned} \tag{10.6}$$

The measure $d\mu(\phi)$ is given by

$$\begin{aligned} d\mu(\Phi) &= \prod_{n,i} d\Phi^i_n \ e^{-\sum_n \left(\Phi^i_n\Phi^i_n + g(\Phi^i_n\Phi^i_n - 1)^2\right)} \\ &= \prod_n \bigg(d^N\vec{\Phi}_n \ e^{-\vec{\Phi}^2_n - g(\vec{\Phi}^2_n - 1)^2} \bigg) \\ &\equiv \prod_n d\mu(\Phi_n). \end{aligned} \tag{10.7}$$

This is a generalized Ising model. Indeed in the limit $g \longrightarrow \infty$ the dominant configurations are such that $\Phi_1^2 + ... + \Phi_N^2 = 1$, i.e. points on the sphere S^{N-1}. Hence

$$\frac{\int d\mu(\Phi_n) f(\vec{\Phi}_n)}{\int d\mu(\Phi_n)} = \frac{\int d\Omega_{N-1} f(\vec{\Phi}_n)}{\int d\Omega_{N-1}} \ , \ g \longrightarrow \infty. \tag{10.8}$$

For $N = 1$ we obtain

$$\frac{\int d\mu(\Phi_n) f(\vec{\Phi}_n)}{\int d\mu(\Phi_n)} = \frac{1}{2}(f(+1) + f(-1)) \ , \ g \longrightarrow \infty. \tag{10.9}$$

Thus the limit $g \longrightarrow \infty$ of the $O(1)$ model is precisely the Ising model in d dimensions. The limit $g \longrightarrow \infty$ of the $O(3)$ model corresponds to the Heisenberg model in d dimensions. The $O(N)$ models on the lattice are thus intimately related to spin models.

There are two phases in this model. A disordered (paramagnetic) phase characterized by $<\Phi_n^i>=0$ and an ordered (ferromagnetic) phase characterized by $<\Phi_n^i>=v_i \neq 0$. This can be seen in various ways. The easiest way is to look for the minima of the classical potential

$$V[\phi]=-\int d^dx\bigg(\frac{1}{2}m^2\phi^i\phi^i+\frac{\lambda}{4}(\phi^i\phi^i)^2\bigg). \tag{10.10}$$

The equation of motion reads

$$[m^2+\frac{\lambda}{2}\phi^j\phi^j]\phi^i=0. \tag{10.11}$$

For $m^2>0$ there is a unique solution $\phi^i=0$ whereas for $m^2<0$ there is a second solution given by $\phi^j\phi^j=-2m^2/\lambda$.

A more precise calculation is as follows. Let us compute the expectation value $<\Phi_n^i>$ on the lattice which is defined by

$$\begin{aligned} <\phi_n^i> &= \frac{\int d\mu(\Phi)\ \Phi_n^i e^{2\kappa\sum_n\sum_\mu \Phi_n^i\Phi_{n+\hat{\mu}}^i}}{\int d\mu(\Phi)\ e^{2\kappa\sum_n\sum_\mu \Phi_n^i\Phi_{n+\hat{\mu}}^i}}\\ &= \frac{\int d\mu(\Phi)\ \Phi_n^i e^{\kappa\sum_n \Phi_n^i\sum_\mu(\Phi_{n+\hat{\mu}}^i+\Phi_{n-\hat{\mu}}^i)}}{\int d\mu(\Phi)\ e^{\kappa\sum_n \Phi_n^i\sum_n\sum_\mu(\Phi_{n+\hat{\mu}}^i+\Phi_{n-\hat{\mu}}^i)}}. \end{aligned} \tag{10.12}$$

Now we approximate the spins Φ_n^i at the $2d$ nearest neighbors of each spin Φ_n^i by the average $v^i=<\Phi_n^i>$, viz

$$\frac{\sum_\mu(\Phi_{n+\hat{\mu}}^i+\Phi_{n-\hat{\mu}}^i)}{2d}=v^i. \tag{10.13}$$

This is a crude form of the mean field approximation. Equation (10.12) becomes

$$\begin{aligned} v^i &= \frac{\int d\mu(\Phi)\ \Phi_n^i e^{4\kappa d\sum_n \Phi_n^i v^i}}{\int d\mu(\Phi)\ e^{4\kappa d\sum_n \Phi_n^i v^i}}\\ &= \frac{\int d\mu(\Phi_n)\ \Phi_n^i e^{4\kappa d \Phi_n^i v^i}}{\int d\mu(\Phi_n^i)\ e^{4\kappa d\Phi_n^i v^i}}. \end{aligned} \tag{10.14}$$

The extra factor of 2 in the exponents comes from the fact the coupling between any two nearest neighbor spins on the lattice occurs twice. We write the above equation as

$$v^i=\frac{\partial}{\partial J^i}\ln Z[J]|_{J^i=4\kappa d v^i}. \tag{10.15}$$

$$\begin{aligned} Z[J] &= \int d\mu(\Phi_n)\ e^{\Phi_n^i J^i}\\ &= \int d^N\Phi_n^i\ e^{-\Phi_n^i\Phi_n^i-g(\Phi_n^i\Phi_n^i-1)^2+\Phi_n^i J^i}. \end{aligned} \tag{10.16}$$

The limit $g \longrightarrow 0$: In this case we have

$$Z[J] = \int d^N\Phi_n^i \; e^{-\Phi_n^i\Phi_n^i+\Phi_n^i J^i} = Z[0]\; e^{\frac{J^iJ^i}{4}}. \tag{10.17}$$

In other words

$$v^i = 2\kappa_c dv^i \Rightarrow \kappa_c = \frac{1}{2d}. \tag{10.18}$$

The limit $g \longrightarrow \infty$: In this case we have

$$\begin{aligned} Z[J] &= \mathcal{N}\int d^N\Phi_n^i \;\delta(\Phi_n^i\Phi_n^i-1)\; e^{\Phi_n^iJ^i} \\ &= \mathcal{N}\int d^N\Phi_n^i \;\delta(\Phi_n^i\Phi_n^i-1)\;\bigg[1+\Phi_n^iJ^i+\frac{1}{2}\Phi_n^i\Phi_n^jJ^iJ^j+...\bigg]. \end{aligned} \tag{10.19}$$

By using rotational invariance in N dimensions we obtain

$$\int d^N\Phi_n^i \;\delta(\Phi_n^i\Phi_n^i-1)\;\Phi_n^i = 0. \tag{10.20}$$

$$\int d^N\Phi_n^i \;\delta(\Phi_n^i\Phi_n^i-1)\;\Phi_n^i\Phi_n^j = \frac{\delta^{ij}}{N}\int d^N\Phi_n^i \;\delta(\Phi_n^i\Phi_n^i-1)\;\Phi_n^k\Phi_n^k = \frac{\delta^{ij}}{N}\frac{Z[0]}{\mathcal{N}}. \tag{10.21}$$

Hence

$$Z[J] = Z[0]\bigg[1+\frac{J^iJ^i}{2N}+...\bigg]. \tag{10.22}$$

Thus

$$v^i = \frac{J^i}{N} = \frac{4\kappa_c dv^i}{N} \Rightarrow \kappa_c = \frac{N}{4d}. \tag{10.23}$$

The limit of The Ising Model: In this case we have

$$N=1\;,\; g\longrightarrow\infty. \tag{10.24}$$

We compute then

$$\begin{aligned} Z[J] &= \mathcal{N}\int d\Phi_n \;\delta(\Phi_n^2-1)\; e^{\Phi_nJ} \\ &= Z[0]\cosh J. \end{aligned} \tag{10.25}$$

Thus

$$v = \tanh 4\kappa dv. \tag{10.26}$$

A graphical sketch of the solutions of this equation is shown on figure 17. Clearly for κ near κ_c the solution v is near 0 and thus we can expand the above equation as

$$v = 4\kappa dv - \frac{1}{3}(4\kappa d)^3v^2+.... \tag{10.27}$$

The solution is

$$\frac{1}{3}(4d)^2\kappa^3 v^2 = \kappa - \kappa_c. \tag{10.28}$$

Thus only for $\kappa > \kappa_c$ there is a non zero solution.

In summary we have the two phases

$$\kappa > \kappa_c \ : \ \text{broken, ordered, ferromagnetic} \tag{10.29}$$

$$\kappa < \kappa_c \ : \ \text{symmetric, disordered, paramagnetic.} \tag{10.30}$$

The critical line $\kappa_c = \kappa_c(g)$ interpolates in the $\kappa - g$ plane between the two lines given by

$$\kappa_c = \frac{N}{4d} \ , \ g \longrightarrow \infty. \tag{10.31}$$

$$\kappa_c = \frac{1}{2d} \ , \ g \longrightarrow 0. \tag{10.32}$$

See figure 18.

For $d = 4$ the critical value at $g = 0$ is $\kappa_c = 1/8$ for all N. This critical value can be derived in a different way as follows. From equation (8.172) we know that the renormalized mass at one-loop order in the continuum ϕ^4 with $O(N)$ symmetry is given by the equation (with $\lambda \longrightarrow 6\lambda$)

$$\begin{aligned} m_R^2 &= m^2 + (N+2)\lambda I(m^2, \Lambda) \\ &= m^2 + \frac{(N+2)\lambda}{16\pi^2}\Lambda^2 + \frac{(N+2)\lambda}{16\pi^2} m^2 \ln\frac{m^2}{\Lambda^2} + \frac{(N+2)\lambda}{16\pi^2} m^2 \mathbf{C} + \text{finite terms.} \end{aligned} \tag{10.33}$$

This equation reads in terms of dimensionless quantities as follows

$$a^2 m_R^2 = am^2 + \frac{(N+2)\lambda}{16\pi^2} + \frac{(N+2)\lambda}{16\pi^2} a^2 m^2 \ln a^2 m^2 + \frac{(N+2)\lambda}{16\pi^2} a^2 m^2 \mathbf{C} + a^2 \times \text{finite terms.} \tag{10.34}$$

The lattice space a is formally identified with the inverse cut off $1/\Lambda$, viz

$$a = \frac{1}{\Lambda}. \tag{10.35}$$

Thus we obtain in the continuum limit $a \longrightarrow 0$ the result

$$a^2 m^2 \longrightarrow -\frac{(N+2)\lambda}{16\pi^2} + \frac{(N+2)\lambda}{16\pi^2} a^2 m^2 \ln a^2 m^2 + \frac{(N+2)\lambda}{16\pi^2} a^2 m^2 \mathbf{C} + a^2 \times \text{finite terms.} \tag{10.36}$$

In other words (with $r_0 = (N+2)/8\pi^2$)

$$a^2 m^2 \longrightarrow a^2 m_c^2 = -\frac{r_0}{2}\lambda + O(\lambda^2). \tag{10.37}$$

This is the critical line for small values of the coupling constant as we will now show. Expressing this equation in terms of κ and g we obtain

$$\frac{1-2g}{\kappa} - 8 \longrightarrow -\frac{r_0}{2}\frac{g}{\kappa^2} + O(\lambda^2). \tag{10.38}$$

This can be brought to the form

$$\left[\kappa - \frac{1}{16}(1-2g)\right]^2 \longrightarrow \frac{1}{256}\left[1 + 16r_0 g - 4g\right] + O(g^2/\kappa^2). \tag{10.39}$$

We get the result

$$\kappa \longrightarrow \kappa_c = \frac{1}{8} + (\frac{r_0}{2} - \frac{1}{4})g + O(g^2). \tag{10.40}$$

This result is of fundamental importance. The continuum limit $a \longrightarrow 0$ corresponds precisely to the limit in which the mass approaches its critical value. This happens for every value of the coupling constant and hence the continuum limit $a \longrightarrow 0$ is the limit in which we approach the critical line. The continuum limit is therefore a second order phase transition.

10.1.2 Mean Field Theory

We start from the partition function of an $O(1)$ model given by

$$Z(J) \quad = \quad \int \prod_n d\mu(\Phi_n) \; e^{\sum_{n,m} \Phi_n V_{nm} \Phi_m + \sum_n J_n \Phi_n}. \tag{10.41}$$

The positive matrix V_{nm} (for the case of ferromagnetic interactions with $\kappa > 0$) is defined by

$$V_{nm} = \kappa \sum_{\hat{\mu}} \big(\delta_{m,n+\hat{\mu}} + \delta_{m,n-\hat{\mu}}\big). \tag{10.42}$$

The measure is defined by

$$d\mu(\Phi_n) = d\Phi_n \; e^{-\Phi_n^2 - g(\Phi_n^2-1)^2}. \tag{10.43}$$

We introduce the Hubbard transformation

$$\int \prod_n dX_n \; e^{-\frac{1}{4}\sum_{n,m} X_n V^{-1}_{nm} X_m + \sum_n \Phi_n X_n} = K e^{\sum_{n,m} \Phi_n V_{nm} \Phi_m}. \tag{10.44}$$

We obtain

$$\begin{aligned} Z(J) \quad &= \quad \frac{1}{K} \int \prod_n dX_n \; e^{-\frac{1}{4}\sum_{n,m} X_n V^{-1}_{nm} X_m} \int \prod_n d\mu(\Phi_n) \; e^{\sum_n (X_n+J_n)\Phi_n} \\ &= \quad \frac{1}{K} \int \prod_n dX_n \; e^{-\frac{1}{4}\sum_{n,m} X_n V^{-1}_{nm} X_m - \sum_n A(X_n+J_n)}. \end{aligned} \tag{10.45}$$

The function A is defined by

$$A(X_n + J_n) = -\ln z(X_n+J_n) \; , \; z(X_n + J_n) = \int d\mu(\Phi_n) \; e^{(X_n+J_n)\Phi_n}. \tag{10.46}$$

In the case of the Ising model we have explicitly

$$z(X_n + J_n) = \int d\mu(\Phi_n) \; e^{(X_n+J_n)\Phi_n} = K' \frac{1}{2}\big(e^{(X_n+J_n)} + e^{-(X_n+J_n)}\big) = K' \cosh(X_n + J_n). \tag{10.47}$$

We introduce a new variable ϕ_n as follows

$$\phi_n = X_n + J_n. \tag{10.48}$$

The partition function becomes (using also the fact that V and V^{-1} are symmetric matrices)

$$Z(J) = \frac{1}{K}\int \prod_n d\phi_n \; e^{-\frac{1}{4}\sum_{n,m}\phi_n V^{-1}_{nm}\phi_m + \frac{1}{2}\sum_{n,m}\phi_n V^{-1}_{nm}J_m - \frac{1}{4}\sum_{n,m}J_n V^{-1}_{nm}J_m - \sum_n A(\phi_n)}. \tag{10.49}$$

We replace V_{ij} by $W_{ij} = V_{ij}/L$ and we replace every spin Φ_n by $\hat{\Phi}_n = \sum_{l=1}^{L}\Phi_n^l$, i.e. by the sum of L spins Φ_n^l which are assumed to be distributed with the same probability $d\mu(\Phi_n^l)$. We get the partition function

$$\begin{aligned}
Z(J) &= \int \prod_{n,l} d\mu(\Phi_n^l)\; e^{\sum_{n,m}\hat{\Phi}_n W_{nm}\hat{\Phi}_m + \sum_n J_n\hat{\Phi}_n} \\
&= \frac{1}{K}\int \prod_n dX_n \; e^{-\frac{1}{4}\sum_{n,m}X_n W^{-1}_{nm}X_m}\Big(\int \prod_n d\mu(\Phi_n)\; e^{\sum_n (X_n+J_n)\Phi_n}\Big)^L \\
&= \frac{1}{K}\int \prod_n d\phi_n \; e^{-L\left[\frac{1}{4}\sum_{n,m}\phi_n V^{-1}_{nm}\phi_m - \frac{1}{2}\sum_{n,m}\phi_n V^{-1}_{nm}J_m + \frac{1}{4}\sum_{n,m}J_n V^{-1}_{nm}J_m + \sum_n A(\phi_n)\right]} \\
&\equiv \frac{1}{K}\int \prod_n d\phi_n \; e^{-LV(\phi_n)}.
\end{aligned} \tag{10.50}$$

In the limit $L \longrightarrow \infty$ we can apply the saddle point method. The partition function is dominated by the configuration which solves the equation of motion

$$\frac{dV}{d\phi_n} = 0 \Leftrightarrow \phi_n - J_n + 2\sum_m V_{nm}\frac{dA}{d\phi_m} = 0. \tag{10.51}$$

In other words we replace the field at each site by the best equivalent magnetic field. This approximation performs better at higher dimensions. Clearly steepest descent allows an expansion in powers of $1/L$. We see that mean field is the tree level approximation of the field theory obtained from (10.50) by neglecting the quadratic term in J_n and redefining the current J_n as $J_n^{\rm redefined} = \sum_m V^{-1}_{nm}J_m/2$.

The partition function becomes (up to a multiplicative constant factor)

$$Z(J) = e^{-L\left[\frac{1}{4}\sum_{n,m}\phi_n V^{-1}_{nm}\phi_m - \frac{1}{2}\sum_{n,m}\phi_n V^{-1}_{nm}J_m + \frac{1}{4}\sum_{n,m}J_n V^{-1}_{nm}J_m + \sum_n A(\phi_n)\right]}|_{\text{saddle point}}. \tag{10.52}$$

The vacuum energy (which plays the role of the thermodynamic free energy) is then given by

$$\begin{aligned}
W(J) &= \frac{1}{L}\ln Z[J] \\
&= -\Big[\frac{1}{4}\sum_{n,m}\phi_n V^{-1}_{nm}\phi_m - \frac{1}{2}\sum_{n,m}\phi_n V^{-1}_{nm}J_m + \frac{1}{4}\sum_{n,m}J_n V^{-1}_{nm}J_m + \sum_n A(\phi_n)\Big]|_{\text{saddle point}}.
\end{aligned} \tag{10.53}$$

The order parameter is the magnetization which is conjugate to the magnetic field J_n. It is defined by

$$\begin{aligned} M_m &= \frac{\partial W}{\partial J_m} \\ &= \frac{1}{2}\sum_n (\phi_n - J_n)V^{-1}_{nm} \\ &= -\frac{dA}{d\phi_m}. \end{aligned} \tag{10.54}$$

The effective action (which plays the role of the thermodynamic energy) is the Legendre transform of $W(J)$ defined by

$$\begin{aligned} \Gamma(M) &= \sum_n M_n J_n - W(J) \\ &= \sum_n M_n J_n + \sum_{n,m} M_n V_{nm} M_m + \sum_n A(\phi_n) \\ &= -\sum_{n,m} M_n V_{nm} M_m + \sum_n B(M_n). \end{aligned} \tag{10.55}$$

The function $B(M_n)$ is the Legendre transform of $A(\phi_n)$ given by

$$B(M_n) = M_n\phi_n + A(\phi_n). \tag{10.56}$$

For the Ising model we compute (up to an additive constant)

$$\begin{aligned} A(\phi_n) &= -\ln\cosh\phi_n \\ &= -\phi_n - \ln\frac{1+e^{-2\phi_n}}{2}. \end{aligned} \tag{10.57}$$

The magnetization in the Ising model is given by

$$M_n = \frac{1-e^{-2\phi_n}}{1+e^{-2\phi_n}} \Leftrightarrow \phi_n = \frac{1}{2}\ln(1+M_n) - \frac{1}{2}\ln(1-M_n). \tag{10.58}$$

Thus

$$A(\phi_n) = \frac{1}{2}\ln(1+M_n) + \frac{1}{2}\ln(1-M_n). \tag{10.59}$$

$$B(M_n) = \frac{1}{2}(1+M_n)\ln(1+M_n) + \frac{1}{2}(1-M_n)\ln(1-M_n). \tag{10.60}$$

From the definition of the effective potential we get the equation of motion

$$\begin{aligned} \frac{\partial\Gamma}{\partial M_n} &= -2\sum_m V_{nm}M_m + \frac{\partial B}{\partial M_n} \\ &= (J_n - \phi_n) + \phi_n \\ &= J_n. \end{aligned} \tag{10.61}$$

Thus for zero magnetic field the magnetization is given by an extremum of the effective potential. On the other hand the partition function for zero magnetic field is given by $Z = \exp(-L\Gamma)$ and hence the saddle point configurations which dominate the partition function correspond to extrema of the effective potential.

In systems where translation is a symmetry of the physics we can assume that the magnetization is uniform, i.e. $M_n = M =$ constant and as a consequence the effective potential per degree of freedom is given by

$$\frac{\Gamma(M)}{\mathcal{N}} = -vM^2 + B(M). \tag{10.62}$$

The number $\mathcal{N}$ is the total number of degrees of freedom, viz $\mathcal{N} = \sum_n 1$. The positive parameter v is finite for short range forces and plays the role of the inverse temperature $\beta = 1/T$. It is given explicitly by

$$v = \frac{\sum_{n,m} V_{nm}}{\mathcal{N}}. \tag{10.63}$$

It is a famous exact result of statistical mechanics that the effective potential $\Gamma(M)$ is a convex function of M, i.e. for M, M_1 and M_2 such that $M = xM_1 + (1-x)M_2$ with $0 < x < 1$ we must have

$$\Gamma(M) \leq x\Gamma(M_1) + (1-x)\Gamma(M_2). \tag{10.64}$$

This means that a linear interpolation is always greater than the potential which means that $\Gamma(M)$ is an increasing function of M for $|M| \longrightarrow \infty$. This can be made more precise as follows. First we compute

$$\frac{d^2A}{d\phi^2} = - < (\Phi - < \Phi >)^2 > . \tag{10.65}$$

Thus $-d^2A/d\phi^2 > 0$ and as a consequence A is a convex function of ϕ [1]. From the definition of the partition function $z(\phi)$ and the explicit form of the measure $d\mu(\Phi)$ we can see that $\Phi \longrightarrow 0$ for $\phi \longrightarrow \pm\infty$ and hence we obtain the condition

$$\frac{d^2A}{d\phi^2} \longrightarrow 0 \ , \ \phi \longrightarrow \infty. \tag{10.66}$$

Since $M =< \Phi >$ this condition also means that $M^2 - < \Phi^2 > \longrightarrow 0$ for $\phi \longrightarrow \pm\infty$. Now by differentiating $M_n = \partial W/\partial J_n$ with respect to M_n and using the result $\partial J_n/\partial M_n = \partial^2\Gamma/\partial M_n^2$ we obtain

$$1 = \frac{\partial^2 W}{\partial J_n^2}\frac{\partial^2\Gamma}{\partial M_n^2}. \tag{10.67}$$

We compute (using $V_{nn} = 0$) the result $\partial^2\Gamma/\partial M_n^2 = d^2B/dM_n^2$. By recalling that $\phi_n = X_n + J_n$ we also compute (using $V_{nn}^{-1} = 0$) the result $\partial^2 W/\partial J_n^2 = -d^2A/d\phi_n^2$. Hence we obtain

$$-1 = \frac{d^2B}{dM_n^2}\frac{d^2A}{d\phi_n^2}. \tag{10.68}$$

[1] Exercise: Verify this explicitly.

Thus the function B is also convex in the variable M. Furthermore the condition (10.65) leads to the condition that the function B goes to infinity faster than M^2 for $M \longrightarrow \pm\infty$[2] (or else that $|M|$ is bounded as in the case of the Ising model).

The last important remark is to note that the functions $A(\phi)$ and $B(M)$ are both even in their respective variables.

There are two possible scenario we now consider:

- **First Order Phase Transition:** For high temperature (small value of v) the effective action is dominated by the second term $B(M)$ which is a convex function. The minimum of $\Gamma(M)$ is $M = 0$. We start decreasing the temperature by increasing v. At some $T = T_c$ (equivalently $v = v_c$) new minima of $\Gamma(M)$ appear which are degenerate with $M = 0$. For $T < T_c$ the new minima become absolute minima and as a consequence the magnetization jumps discontinuously from 0 to a finite value corresponding to these new minima. See figure 14.

  In this case the second derivative of the effective potential at the minimum $\Gamma^{''}(0)$ is always strictly positive and as a consequence the correlation length, which is inversely proportional to the square root of $\Gamma^{''}()$, is always finite.

- **Second Order Phase Transition:** The more interesting possibility occurs when the minimum at the origin $M = 0$ becomes at some critical temperature $T = T_c$ a maximum and simultaneously new minima appear which start moving away from the origin as we decreasing the temperature. The critical temperature T_c is defined by the condition $\Gamma^{''}(0) = 0$ or equivalently

$$2v_c = B^{''}(0). \tag{10.69}$$

  Above T_c we have only the solution $M = 0$ whereas below T_c we have two minima moving continuously away from the origin. In this case the magnetization remains continuous at $v = v_c$ and as a consequence the transition is also termed continuous. Clearly the correlation length diverges at $T = T_c$. See figure 14.

10.1.3 Critical Exponents in Mean Field

In the following we will only consider the second scenario. Thus we assume that we have a second order phase transition at some temperature $T = T_c$ (equivalently $v = v_c$). We are interested in the thermodynamic of the system for temperatures T near T_c. The transition is continuous and thus we can assume that the magnetization M is small near $T = T_c$ and as a consequence we can expand the effective action (thermodynamic energy) in powers of M. We write then

$$\begin{aligned}
\Gamma(M) &= -\sum_{n,m} M_n V_{nm} M_m + \sum_n B(M_n) \\
&= -\sum_{n,m} M_n V_{nm} M_m + \sum_n \left[\frac{a}{2!}M_n^2 + \frac{b}{4!}M_n^4 + ...\right].
\end{aligned} \tag{10.70}$$

The function $B(M_n)$ is the Legendre transform of $A(\phi_n)$, i.e.

$$B(M_n) = M_n\phi_n + A(\phi_n). \tag{10.71}$$

[2]Exercise: Verify this explicitly.

We expand $A(\phi_n)$ in powers of ϕ_n as

$$A(\phi_n) = \frac{a^{'}}{2!}\phi_n^2 + \frac{b^{'}}{4!}\phi_n^4 + ... \tag{10.72}$$

Thus

$$M_n = -\frac{dA}{d\phi_n} = -a^{'}\phi_n - \frac{b^{'}}{6}\phi_n^3 + ... \tag{10.73}$$

We compute

$$\begin{aligned} \frac{d^2A}{d\phi_n^2} &= -[\frac{d^2B}{dM_n^2}]^{-1} \\ &= -\frac{1}{a} + \frac{b}{2a^2}M_n^2 + ... \\ &= -\frac{1}{a} + \frac{b}{2a^2}a^{'2}\phi_n^2 + ... \end{aligned} \tag{10.74}$$

By integration this equation we obtain

$$A = -\frac{1}{2a}\phi_n^2 + \frac{b}{4!a^2}a^{'2}\phi_n^4 + ... \tag{10.75}$$

Hence

$$a^{'} = -\frac{1}{a} \ , \ b^{'} = \frac{b}{a^4}. \tag{10.76}$$

The critical temperature is given by the condition $\Gamma^{''}(0) = 0$ (where Γ here denotes the effective potential $\Gamma(M) = \mathcal{N}(-vM^2 + B(M)))$. This is equivalent to the condition $B^{''}(0) = 2v_c$ which gives the value (recall that the coefficient a is positive since B is convex)

$$v_c = \frac{a}{2}. \tag{10.77}$$

The equation of motion $\Gamma^{'}(0) = 0$ gives the condition $B^{'}(M) = 2vM$. For $v < v_c$ we have no spontaneous magnetization whereas for $v > v_c$ we have a non zero spontaneous magnetization given by

$$M = \sqrt{\frac{12}{b}}(v - v_c)^{1/2}. \tag{10.78}$$

The magnetization is associated with the critical exponent β defined for T near T_c from below by

$$M \sim (T_c - T)^{\beta}. \tag{10.79}$$

We have clearly

$$\beta = \frac{1}{2}. \tag{10.80}$$

The inverse of the (magnetic) susceptibility is defined by (with J being the magnetic field)

$$\begin{aligned}
\chi^{-1} &= \frac{\partial M}{\partial J} \\
&= \frac{\delta^2\Gamma}{\delta M^2} \\
&= \mathcal{N}(-2v+a+\frac{b}{2}M^2). \quad (10.81)
\end{aligned}$$

We have the 2−cases

$$\begin{aligned}
&v<v_c \ , \ M=0 \Rightarrow \chi^{-1}=2(v_c-v) \\
&v>v_c \ , \ M=\sqrt{\frac{12}{b}}(v-v_c)^{1/2} \Rightarrow \chi^{-1}=4(v-v_c). \quad (10.82)
\end{aligned}$$

The susceptibility is associated with the critical exponent γ defined by

$$\chi \sim |T-T_c|^{-\gamma}. \quad (10.83)$$

Clearly we have

$$\gamma=1. \quad (10.84)$$

The quantum equation of motion (equation of state) relates the source (external magnetic field), the temperature and the spontaneous magnetization. It is given by

$$\begin{aligned}
J &= \frac{\partial\Gamma}{\partial M} \\
&= \mathcal{N}(2(v_c-v)M+\frac{b}{6}M^3) \\
&= \frac{\mathcal{N}b}{3}M^3. \quad (10.85)
\end{aligned}$$

The equation of state is associated with the critical exponent δ defined by

$$J \sim M^\delta. \quad (10.86)$$

Clearly we have

$$\delta=3. \quad (10.87)$$

Let us derive the 2−point correlation function given by

$$\begin{aligned}
G^{(2)}_{nm} &= \left[\frac{\delta^2\Gamma}{\delta M_n\delta M_m}\right]^{-1} \\
&= \left[-2V_{nm}+a\delta_{nm}+\frac{b}{2}M_n^2\delta_{nm}\right]^{-1}. \quad (10.88)
\end{aligned}$$

Define

$$\Gamma^{(2)}_{nm} = -2V_{nm}+a\delta_{nm}+\frac{b}{2}M_n^2\delta_{nm}. \quad (10.89)$$

The two functions $G^{(2)}_{nm}$ and $\Gamma^{(2)}_{nm}$ can only depend on the difference $n-m$ due to invariance under translation. Thus Fourier transform is and its inverse are defined by

$$K_{nm}=\int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\tilde{K}(k)\ e^{ik(n-m)}\ ,\ \tilde{K}(k)=\sum_n K_{nm}e^{-ik(n-m)}. \tag{10.90}$$

For simplicity we assume a uniform magnetization, viz $M=M_n$. Thus

$$\begin{aligned}\tilde{\Gamma}^{(2)}(k) &= \sum_n \Gamma^{(2)}_{nm}e^{-ik(n-m)}\\ &= -2\tilde{V}(k)+a+\frac{b}{2}M^2.\end{aligned} \tag{10.91}$$

Hence

$$G^{(2)}_{nm} = \int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\frac{1}{-2\tilde{V}(k)+a+\frac{b}{2}M^2}\ e^{ik(n-m)}. \tag{10.92}$$

The function $\tilde{V}(k)$ is given explicitly by

$$\tilde{V}(k)=\sum_n V_{nm}e^{-ik(n-m)}. \tag{10.93}$$

We assume a short range interaction which means that the potential V_{nm} decays exponentially with the distance $|n-m|$. In other words we must have

$$V_{nm}<Me^{-\kappa|n-m|}\ ,\ \kappa>0. \tag{10.94}$$

This condition implies that the Fourier transform $\tilde{V}(k)$ is analytic for $|\mathrm{Im}\ k|<\kappa$ [3]. Furthermore positivity of the potential V_{nm} and its invariance under translation gives the requirement

$$|\tilde{V}(k)|\leq\sum_n V_{nm}=\tilde{V}(0)=v. \tag{10.95}$$

For small momenta k we can then expand $\tilde{V}(k)$ as

$$\tilde{V}(k)=v(1-\rho^2k^2+O(k^4)). \tag{10.96}$$

The 2−point function admits therefore the expansion

$$G^{(2)}_{nm} = \int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\frac{\tilde{G}^{(2)}(0)}{1+\xi^2k^2+O(k^4)}\ e^{ik(n-m)}. \tag{10.97}$$

$$\tilde{G}^{(2)}(0)=\frac{1}{2(v_c-v)+\frac{b}{2}M^2}. \tag{10.98}$$

$$\xi^2=\frac{2v\rho^2}{2(v_c-v)+\frac{b}{2}M^2}. \tag{10.99}$$

[3]Exercise: Construct an explicit argument.

The length scale ξ is precisely the so-called correlation length which measures the exponential decay of the 2−point function. Indeed we can write the 2−point function as

$$G^{(2)}_{nm} = \int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\tilde{G}^{(2)}(0)e^{-\xi^2k^2}\ e^{ik(n-m)}. \quad (10.100)$$

More generally it is not difficult to show that the denominator $-2\tilde{V}(k)+a+bM^2/2$ is strictly positive for $v>v_c$ and hence the 2−point function decays exponentially which indicates that the correlation length is finite.

We have the two cases

$$v<v_c\ :\ M=0\Rightarrow \xi^2=\frac{v\rho^2}{v_c-v}$$
$$v>v_c\ :\ M=\sqrt{\frac{12}{b}}(v-v_c)^{1/2}\Rightarrow \xi^2=\frac{v\rho^2}{2(v-v_c)}. \quad (10.101)$$

The correlation length ξ is associated with the critical exponent ν defined by

$$\xi\sim|T-T_c|^{-\nu}. \quad (10.102)$$

Clearly we have

$$\nu=\frac{1}{2}. \quad (10.103)$$

The correlation length thus diverges at the critical temperature $T=T_c$.

A more robust calculation which shows this fundamental result is easily done in the continuum. In the continuum limit the 2−point function (10.97) becomes

$$G^{(2)}(x,y) = \int\frac{d^dk}{(2\pi)^d}\frac{1}{m^2+k^2}\ e^{ik(x-y)}. \quad (10.104)$$

The squared mass parameter is given by

$$m^2=\frac{1}{\xi^2}=\frac{2(v_c-v)+\frac{b}{2}M^2}{2v\rho^2}\sim|v-v_c|\sim[T-T_c|. \quad (10.105)$$

We compute [4]

$$G^{(2)}(x,y) = \frac{2}{(4\pi)^{d/2}}(\frac{2m}{r})^{d/2-1}K_{1-d/2}(mr). \quad (10.106)$$

For large distances we obtain [5]

$$G^{(2)}(x,y) = \frac{1}{2m}(\frac{m}{2\pi})^{(d-1)/2}\frac{e^{-mr}}{r^{(d-1)/2}}\ ,\ r\longrightarrow\infty. \quad (10.107)$$

The last crucial critical exponent is the anomalous dimension η. This is related to the behavior of the 2−point function at $T=T_c$. At $T=T_c$ we have $v=v_c$ and $M=0$ and hence the 2−point function becomes

$$G^{(2)}_{nm} = \int_{-\pi}^{\pi}\frac{d^dk}{(2\pi)^d}\frac{1}{2v_c(\rho^2k^2-O(k^4))}\ e^{ik(n-m)}. \quad (10.108)$$

[4]Exercise: Do this important integral.
[5]Exercise: Check this limit

Thus the denominator vanishes only at $k=0$ which is consistent with the fact that the correlation length is infinite at $T=T_c$. This also leads to algebraic decay. This can be checked more easily in the continuum limit where the 2−point function becomes

$$G^{(2)}(x,y) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2}\, e^{ik(x-y)}. \tag{10.109}$$

We compute [6]

$$G^{(2)}(x,y) = \frac{2^{d-2}}{(4\pi)^{d/2}}\Gamma(d/2-1)\frac{1}{r^{d-2}}. \tag{10.110}$$

The critical exponent η is defined by the behavior

$$G^{(2)}(x,y) \sim \frac{1}{r^{d-2+\eta}}. \tag{10.111}$$

The mean field prediction is therefore given by

$$\eta = 0. \tag{10.112}$$

In this section we have not used any particular form for the potential V_{mn}. It will be an interesting exercise to compute directly all the critical exponents β, γ, δ, ν and η for the case of the $O(1)$ model corresponding to the nearest-neighbor interaction (10.42) [7]. This of course includes the Ising model as a special case.

10.2 The Callan-Symanzik Renormalization Group Equation

10.2.1 Power Counting Theorems

We consider a ϕ^r theory in d dimensions given by the action

$$S[\phi] = \int d^dx\Big[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\mu^2}{2}\phi^2 - \frac{g}{4}\phi^r\Big]. \tag{10.113}$$

The case of interest is of course $d=4$ and $r=4$. In natural units where $\hbar = c = 1$ the action is dimensionless, viz $[S]=1$. In these units time and length has the same dimension whereas mass, energy and momentum has the same dimension. We take the fundamental dimension to be that of length or equivalently that of mass. We have clearly (for example from Heisenberg uncertainty principle)

$$L = \frac{1}{M}. \tag{10.114}$$

$$[t] = [x] = L = M^{-1}\ ,\ [m] = [E] = [p] = M. \tag{10.115}$$

It is clear that the Lagrangian density is of mass dimension M^d and as a consequence the field is of mass dimension $M^{(d-2)/2}$ and the coupling constant g is of mass dimension $M^{d-rd/2+r}$ (use the fact that $[\partial]=M$). We write

$$[\phi] = M^{\frac{d-2}{2}}. \tag{10.116}$$

[6] Exercise: Do this important integral.
[7] Exercise: Compute the exponents β, γ, δ, ν and η for the potential (10.42).

$$[g] = M^{d-r\frac{d-2}{2}} \equiv M^{\delta_r} \ , \ \delta_r = d - r\frac{d-2}{2}. \tag{10.117}$$

The main result of power counting states that ϕ^r theory is renormalizable only in d_c dimension where d_c is given by the condition

$$\delta_r = 0 \Leftrightarrow d_c = \frac{2r}{r-2}. \tag{10.118}$$

The effective action is given by

$$\Gamma[\phi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 ... \int d^d x_n \Gamma^{(n)}(x_1, ..., x_n) \phi_c(x_1) ... \phi_c(x_n). \tag{10.119}$$

Since the effective action is dimensionless the $n-$point proper vertices $\Gamma^{(n)}(x_1, ..., x_n)$ have mass dimension such that

$$1 = \frac{1}{M^{nd}} [\Gamma^{(n)}(x_1, ..., x_n)] M^{n\frac{d-2}{2}} \Leftrightarrow [\Gamma^{(n)}(x_1, ..., x_n)] = M^{\frac{nd}{2}+n}. \tag{10.120}$$

The Fourier transform is defined as usual by

$$\int d^d x_1 ... \int d^d x_n \Gamma^{(n)}(x_1, ..., x_n) \ e^{ip_1 x_1 + ... + ip_n x_n} = (2\pi)^d \delta^d(p_1 + ... + p_n) \tilde{\Gamma}^{(n)}(p_1, ..., p_n). \tag{10.121}$$

From the fact that $\int d^d p \delta^d(p) = 1$ we conclude that $[\delta^d(p)] = M^{-d}$ and hence

$$[\tilde{\Gamma}^{(n)}(p_1, ..., p_n)] = M^{d-n(\frac{d}{2}-1)}. \tag{10.122}$$

The $n-$point function $G^{(n)}(x_1, ..., x_n)$ is the expectation value of the product of n fields and hence it has mass dimension

$$[G^{(n)}(x_1, ..., x_n)] = M^{n\frac{d-2}{2}}. \tag{10.123}$$

The Fourier transform is defined by

$$\int d^d x_1 ... \int d^d x_n G^{(n)}(x_1, ..., x_n) \ e^{ip_1 x_1 + ... + ip_n x_n} = (2\pi)^d \delta^d(p_1 + ... + p_n) \tilde{G}^{(n)}(p_1, ..., p_n). \tag{10.124}$$

Hence

$$[\tilde{G}^{(n)}(p_1, ..., p_n)] = M^{d-n(\frac{d}{2}+1)}. \tag{10.125}$$

We consider now an arbitrary Feynman diagram in a ϕ^r theory in d dimensions. This diagram is contributing to some $n-$point proper vertex $\tilde{\Gamma}^{(n)}(p_1, ..., p_n)$ and it can be characterized by the following:

- L=number of loops.
- V=number of vertices.
- P=number of propagators (internal lines).
- n=number of external lines (not to be considered propagators).

We remark that each propagator is associated with a momentum variable. In other words we have P momenta which must be constrained by the V delta functions associated with the V vertices and hence there can only be $P-V$ momentum integrals in this diagram. However, only one delta function (which enforces energy-momentum conservation) survives after integration and thus only $V-1$ delta functions are actually used. The number of loops L must be therefore given by

$$L = P - (V-1). \tag{10.126}$$

Since we have r lines coming into a vertex the total number of lines coming to V vertices is rV. Some of these lines are propagators and some are external lines. Clearly among the rV lines we have precisely n external lines. Since each propagator connects two vertices it must be counted twice. We have then

$$rV = n + 2P. \tag{10.127}$$

It is clear that $\tilde{\Gamma}^{(n)}(p_1,...,p_n)$ must be proportional to g^V, viz

$$\tilde{\Gamma}^{(n)}(p_1,...,p_n) = g^V f(p_1,...,p_n). \tag{10.128}$$

We have clearly

$$[f(p_1,...,p_n)] = M^{\delta}\ ,\ \delta = -V\delta_r + d - n(\frac{d}{2}-1). \tag{10.129}$$

The index δ is called the superficial degree of divergence of the Feynman graph. The physical significance of δ can be unraveled as follows. Schematically the function f is of the form

$$f(p_1,...,p_n) \sim \int_0^{\Lambda} d^dk_1 ... \int_0^{\Lambda} d^dk_P \frac{1}{k_1^2-\mu^2} ... \frac{1}{k_P^2-\mu^2}\big[\delta^d(\sum p - \sum k)\big]^{V-1}. \tag{10.130}$$

If we neglect, in a first step,the delta functions than we can see immediately that the asymptotic behavior of the integral $f(p_1,...,p_n)$ is $\Lambda^{P(d-2)}$. This can be found by factoring out the dependence of f on Λ via the rescaling $k \longrightarrow \Lambda k$. By taking the delta functions into considerations we see immediately that the number of independent variables reduces and hence the asymptotic behavior of $f(p_1,...,p_n)$ becomes

$$f(p_1,...,p_n) \sim \Lambda^{P(d-2)-d(V-1)}. \tag{10.131}$$

By using $P = (rV-n)/2$ we arrive at the result

$$\begin{aligned} f(p_1,...,p_n) &\sim \Lambda^{-V\delta_r + d - n(\frac{d}{2}-1)} \\ &\sim \Lambda^{\delta}. \end{aligned} \tag{10.132}$$

The index δ controls therefore the ultraviolet behavior of the graph. From the last two equations it is obvious that δ is the difference between the power of k in the numerator and the power of k in the denominator, viz

$$\delta = (\text{power of } k \text{ in numerator}) - (\text{power of } k \text{ in denominator}) \tag{10.133}$$

Clearly a negative index δ corresponds to convergence whereas a positive index δ corresponds to divergence. Since δ is only a superficial degree of divergence there are exceptions to this simple rule. More precisely we have the following first power counting theorem:

- For $\delta > 0$ the diagram diverges as Λ^d. However symmetries (if present) can reduce/eliminate divergences in this case.
- For $\delta = 0$ the diagram diverges as $\ln \Lambda$. An exception is the trivial diagram ($P = L = 0$).
- For $\delta < 0$ the diagram converges absolutely if it contains no divergent subdiagrams. In other words a diagram with $\delta < 0$ which contains divergent subdiagrams is generically divergent.

As an example let us consider ϕ^4 in 4 dimensions. In this case

$$\delta = 4 - n. \tag{10.134}$$

Clearly only the 2−point and the 4−point proper vertices are superficially divergent, i.e. they have $\delta \geq 0$. In particular for $n = 4$ we have $\delta = 0$ indicating possible logarithmic divergence which is what we had already observed in actual calculations. For $n = 6$ we observe that $\delta = -2 < 0$ which indicates that the 6−point proper vertex is superficially convergent. In other words the diagrams contributing to the 6−point proper vertex may or may not be convergent depending on whether or not they contain divergent subdiagrams. For example the one-loop diagram on figure 13 is convergent whereas the two-loop diagrams are divergent.

The third rule of the first power counting theorem can be restated as follows:

- A Feynman diagram is absolutely convergent if and only if it has a negative superficial degree of divergence and all its subdiagrams have negative superficial degree of divergence.

The ϕ^4 theory in $d = 4$ is an example of a renormalizable field theory. In a renormalizable field theory only a finite number of amplitudes are superficially divergent. As we have already seen, the divergent amplitudes in the case of the ϕ^4 theory in $d = 4$ theory, are the 2−point and the 4−point amplitudes. All other amplitudes may diverge only if they contain divergent subdiagrams corresponding to the 2−point and the 4−point amplitudes.

Another class of field theories is non-renormalizable field theories. An example is ϕ^4 in $D = 6$. In this case

$$\delta_r = -2 \ , \ \delta = 2V + 6 - 2n. \tag{10.135}$$

The formula for δ depends now on the order of perturbation theory as opposed to what happens in the case of $D = 4$. Thus for a fixed n the superficial degree of divergence increases by increasing the order of perturbation theory, i.e. by increasing V. In other words at a sufficiently high order of perturbation theory all amplitudes are divergent.

In a renormalizable field theory divergences occur generally at each order in perturbation theory. For ϕ^4 theory in $d = 4$ all divergences can be removed order by order in perturbation theory by redefining the mass, the coupling constant and the wave function renormalization. This can be achieved by imposing three renormalization conditions on $\tilde{\Gamma}^{(2)}(p)$, $d\tilde{\Gamma}^{(2)}(p)/dp^2$ and $\tilde{\Gamma}^{(4)}(p_1, ..., p_4)$ at 0 external momenta corresponding to three distinct experiments.

In contrast we will require an infinite number of renormalization conditions in order to remove the divergences occurring at a sufficiently high order in a non-renormalizable field theory since all amplitudes are divergent in this case. This corresponds to an infinite number of distinct experiments and as a consequence the theory has no predictive power.

From the formula for the superficial degree of divergence $\delta = -\delta_r V + d - n(d/2 - 1)$ we see that δ_r, the mass dimension of the coupling constant, plays a central role. For $\delta_r = 0$ (such as ϕ^4 in $d = 4$ and ϕ^3 in $d = 6$ we see that the index δ is independent of the order of perturbation theory which is a special behavior of renormalizable theory. For $\delta_r < 0$ (such as ϕ^4 in $d > 4$)

we see that δ depends on V in such a way that it increases as V increases and hence we obtain more divergencies at each higher order of perturbation theory. Thus $\delta_r < 0$ defines the class of non-renormalizable field theories as $\delta_r = 0$ defines the class of renormalizable field theories.

Another class of field theories is super-renormalizable field theories for which $\delta_r > 0$ (such as ϕ^3 in $D = 4$). In this case the superficial degree of divergence δ decreases with increasing order of perturbation theory and as a consequence only a finite number of Feynman diagrams are superficially divergent. In this case no amplitude diverges.

The second (main) power counting theorem can be summarized as follows:

- Super-Renormalizable Theories: The coupling constant g has positive mass dimension. There are no divergent amplitudes and only a finite number of Feynman diagrams superficially diverge.

- Renormalizable Theories: The coupling constant g is dimensionless. There is a finite number of superficially divergent amplitudes. However since divergences occur at each order in perturbation theory there is an infinite number of Feynman diagrams which are superficially divergent.

- Non-Renormalizable Theories: The coupling constant g has negative mass dimension. All amplitudes are superficially divergent at a sufficiently high order in perturbation theory.

10.2.2 Renormalization Constants and Renormalization Conditions

We write the ϕ^4 action in $d = 4$ as

$$S[\phi] = \int d^dx\bigg[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}(\phi^2)^2\bigg]. \tag{10.136}$$

The bare field ϕ, the bare coupling constant λ and the bare mass m^2 are given in terms of the renormalized field ϕ_R, the renormalized coupling constant λ_R and the renormalized mass m_R^2 respectively by the relations

$$\phi = \sqrt{Z}\phi_R. \tag{10.137}$$

$$\lambda = Z_g/Z^2\lambda_R. \tag{10.138}$$

$$m^2 = (m_R^2 + \delta_m)/Z. \tag{10.139}$$

The renormalization constant Z is called wave function renormalization constant (or equivalently field amplitude renormalization constant) whereas Z_g/Z^2 is the coupling constant renormalization constant.

The action S given by equation (10.136) can be split as follows

$$S = S_R + \delta S. \tag{10.140}$$

The renormalized action S_R is given by

$$S_R[\phi_R] = \int d^dx\bigg[\frac{1}{2}\partial_\mu\phi_R\partial^\mu\phi_R - \frac{1}{2}m_R^2\phi_R^2 - \frac{\lambda_R}{4!}(\phi_R^2)^2\bigg]. \tag{10.141}$$

The counter-term action δS is given by

$$\delta S[\phi_R] = \int d^4x\bigg[\frac{\delta_Z}{2}\partial_\mu\phi_R\partial^\mu\phi_R - \frac{1}{2}\delta_m\phi_R^2 - \frac{\delta_\lambda}{4!}(\phi_R^2)^2\bigg]. \quad (10.142)$$

The counterterms δ_Z, δ_m and δ_λ are given by

$$\delta_Z = Z - 1 \ , \ \delta_m = Zm^2 - m_R^2 \ , \ \delta_\lambda = \lambda Z^2 - \lambda_R = (Z_g - 1)\lambda_R. \quad (10.143)$$

The renormalized $n-$point proper vertex $\Gamma_R^{(n)}$ is given in terms of the bare $n-$point proper vertex $\Gamma^{(n)}$ by

$$\Gamma_R^{(n)}(x_1,...,x_n) = Z^{\frac{n}{2}}\Gamma^{(n)}(x_1,...,x_n). \quad (10.144)$$

The effective action is given by (where ϕ denotes here the classical field)

$$\Gamma_R[\phi_R] = \sum_{n=0}\frac{1}{n!}\Gamma_R^{(n)}(x_1,...,x_n)\phi_R(x_1)...\phi_R(x_n). \quad (10.145)$$

We assume a momentum cutoff regularization. The renormalization constants Z and Z_g and the counterterm δ_m are expected to be of the form

$$\delta_m = a_1(\Lambda)\lambda_r + a_2(\Lambda)\lambda_r^2 + ...$$
$$Z = 1 + b_1(\Lambda)\lambda_r + b_2(\Lambda)\lambda_r^2 + ...$$
$$Z_g = 1 + c_1(\Lambda)\lambda_r + c_2(\Lambda)\lambda_r^2 + ... \quad (10.146)$$

All other quantities can be determined in terms of Z and Z_g and the counterterm δ_m. We can state our third theorem as follows:

- Renormalizability of the ϕ^4 theory in $d = 4$ means precisely that we can choose the constants a_i, b_i and c_i such that all correlation functions have a finite limit order by order in λ_R when $\Lambda \longrightarrow \infty$.

We can eliminate the divergences by imposing appropriate renormalization conditions at zero external momentum. For example we can choose to impose conditions consistent with the tree level action, i.e.

$$\tilde{\Gamma}_R^{(2)}(p)|_{p^2=0} = m_R^2$$
$$\frac{d}{dp^2}\tilde{\Gamma}^{(2)}(p)|_{p^2=0} = 1$$
$$\tilde{\Gamma}^{(4)}(p_1,...,p_4)|_{p_i^2=0} = \lambda_R. \quad (10.147)$$

This will determine the superficially divergent amplitudes completely and removes divergences at all orders in perturbation theory [8].

[8] Exercise:

- Show that the loopwise expansion is equivalent to an expansion in powers of λ.
- Write down the one-loop effective action of the ϕ^4 theory in $d = 4$. Use a Gaussian cutoff.
- Compute a_1, b_1 and c_1 at the one-loop order of perturbation theory.
- Consider one-loop renormalization of ϕ^3 in $d = 6$.

It is well established that a far superior regularization method,than the simple cutoff used above, is dimensional regularization in which case we use, instead of renormalization conditions, the so-called minimal subtraction (MS) and modified minimal subtraction (MMS) schemes to renormalize the theory. In minimal subtraction scheme we subtract only the pole term and nothing else.

In dimension $d \neq 4$ the coupling constant λ is not dimensionless. The dimensionless coupling constant in this case is given by g defined by

$$g = \mu^{-\epsilon}\lambda \ , \ \epsilon = 4 - d. \tag{10.148}$$

The bare action can then be put in the form

$$S = \int d^dx \bigg[\frac{Z}{2}\partial_\mu\phi_R\partial^\mu\phi_R - \frac{Z_m m_R^2}{2}\phi_R^2 - \frac{\mu^\epsilon g_R Z_g}{4!}(\phi_R^2)^2\bigg]. \tag{10.149}$$

The new renormalization condition Z_m is defined through the equation

$$m^2 = m_R^2\frac{Z_m}{Z}. \tag{10.150}$$

The mass μ^2 is an arbitrary mass scale parameter which plays a central role in dimensional regularization and minimal subtraction. The mass μ^2 will define the subtraction point. In other words the mass scale at which we impose renormalization conditions in the form

$$\begin{aligned}
&\tilde{\Gamma}_R^{(2)}(p)|_{p^2=0} = m_R^2\\
&\frac{d}{dp^2}\tilde{\Gamma}^{(2)}(p)|_{p^2=\mu^2} = 1\\
&\tilde{\Gamma}^{(4)}(p_1,...,p_4)|_{\rm SP} = \mu^\epsilon g_R.
\end{aligned} \tag{10.151}$$

The symmetric point SP is defined by $p_i.p_j = \mu^2(4\delta_{ij}-1)/3$. For massive theories we can simply choose $\mu = m_R$. According to Weinberg's theorem (and other considerations) the only correlation functions of massless ϕ^4 which admit a zero momentum limit is the $2-$point function. This means in particular that the second and third renormalization conditions (10.147) do not make sense in the massless limit $m_R^2 \longrightarrow 0$ and should be replaced by the second and third renormalization conditions (10.151). This is also the reason why we have kept the first renormalization condition unchanged. The renormalization conditions (10.151) are therefore better behaved.

As pointed above the renormalization prescription known as minimal subtraction is far superior than the above prescription of imposing renormalization conditions since it is intimately tied to dimensional regularization. In this prescription the mass scale μ^2 appears only via (10.148). We will keep calling μ^2 the subtraction point since minimal subtraction must be physically equivalent to imposing the renormalization conditions (10.151) although the technical detail is generically different in the two cases.

The renormalized proper vertices $\tilde{\Gamma}_R^{(n)}$ depend on the momenta p_1,...,p_n but also on the renormalized mass m_R^2, the renormalized coupling constant g_R and the cutoff Λ. In the case of dimensional regularization the cutoff is $\epsilon = 4-d$ whereas in the case of lattice regularization the cutoff is the inverse lattice spacing. The proper vertices $\tilde{\Gamma}_R^{(n)}$ will also depend on the mass scale μ^2 explicitly and implicitly through m_R^2 and g_R. The renormalized proper vertices are related to the bare proper vertices as

$$\tilde{\Gamma}_R^{(n)}(p_i,\mu^2;m_R^2,g_R,\Lambda) = Z^{\frac{n}{2}}\tilde{\Gamma}^{(n)}(p_i;m^2,g,\Lambda). \tag{10.152}$$

The renormalization constant Z (and also other renormalization constants Z_g, Z_m and counterterms δ_Z, δ_m and δ_λ) will only depend on the dimensionless parameters g_R, m_R^2/μ^2, Λ^2/μ^2, m_R^2/Λ^2 as well as on Λ, viz

$$Z = Z(g_R, \frac{m_R^2}{\mu^2}, \frac{\Lambda^2}{\mu^2}, \frac{m_R^2}{\Lambda^2}, \Lambda). \tag{10.153}$$

In dimensional regularization we have

$$\tilde{\Gamma}_R^{(n)}(p_i, \mu^2; m_R^2, g_R, \epsilon) = Z^{\frac{n}{2}}(g_R, \frac{m_R^2}{\mu^2}, \epsilon)\tilde{\Gamma}^{(n)}(p_i; m^2, g, \epsilon). \tag{10.154}$$

Renormalizability of the ϕ^4 theory in $d = 4$ via renormalization conditions (the fourth theorem) can be stated as follows:

- The renormalized proper vertices $\tilde{\Gamma}_R^{(n)}(p_i, \mu^2; m_R^2, g_R, \Lambda)$ at fixed p_i, μ^2, g_R, m_R^2 have a large cut-off limit $\tilde{\Gamma}_R^{(n)}(p_i, \mu^2; m_R^2, g_R)$ which are precisely the physical proper vertices, viz

$$\tilde{\Gamma}_R^{(n)}(p_i, \mu^2; m_R^2, g_R, \Lambda) = \tilde{\Gamma}_R^{(n)}(p_i, \mu^2; m_R^2, g_R) + O\big(\frac{(\ln\Lambda)^L}{\Lambda^2}\big). \tag{10.155}$$

  The renormalized physical proper vertices $\tilde{\Gamma}_R^{(n)}(p_i, \mu^2; m_R^2, g_R)$ are universal in the sense that they do not depend on the specific cut-off procedure as long as the renormalization conditions (10.151) are kept unchanged. In the above equation L is the number of loops.

10.2.3 Renormalization Group Functions and Minimal Subtraction

The bare mass m^2 and the bare coupling constant λ are related to the renormalized mass m_R^2 and renormalized coupling constant λ_R by the relations

$$m^2 = m_R^2\frac{Z_m}{Z}. \tag{10.156}$$

$$\lambda = \frac{Z_g}{Z^2}\lambda_R = \frac{Z_g}{Z^2}\mu^\epsilon g_R. \tag{10.157}$$

In dimensional regularization the renormalization constants will only depend on the dimensionless parameters g_R, m_R^2/μ^2 as well as on ϵ. We may choose the subtraction mass scale $\mu^2 = m_R^2$. Clearly the bare quantities m^2 and λ are independent of the mass scale μ. Thus by differentiating both sides of the above second equation with respect to μ^2 keeping m^2 and λ fixed we obtain

$$0 = \left(\mu\frac{\partial\lambda}{\partial\mu}\right)_{\lambda, m^2} \Rightarrow \beta = -\epsilon g_R - \left(\mu\frac{\partial}{\partial\mu}\ln Z_g/Z^2\right)_{\lambda, m^2} g_R. \tag{10.158}$$

The so-called renormalization group beta function β (also called the Gell-Mann Law function) is defined by

$$\beta = \beta(g_R, \frac{m_R^2}{\mu^2}) = \left(\mu\frac{\partial g_R}{\partial\mu}\right)_{\lambda, m^2} \tag{10.159}$$

Let us define the new dimensionless coupling constant

$$G = \frac{Z_g}{Z^2}g_R. \tag{10.160}$$

Alternatively by differentiating both sides of equation (10.157) with respect to μ keeping m^2 and λ fixed we obtain

$$0=\left(\mu\frac{\partial\lambda}{\partial\mu}\right)_{\lambda,m^2}\Rightarrow 0=\epsilon G+\beta\frac{\partial}{\partial g_R}G+\left(\mu\frac{\partial}{\partial\mu}m_R\right)_{\lambda,m^2}\frac{\partial}{\partial m_R}G. \tag{10.161}$$

The last term is absent when $\mu=m_R$.

Next by differentiating both sides of equation (10.156) with respect to μ keeping m^2 and λ fixed we obtain

$$0=\left(\mu\frac{\partial m^2}{\partial\mu}\right)_{\lambda,m^2}\Rightarrow 0=\left(\mu\frac{\partial m_R^2}{\partial\mu}\right)_{\lambda,m^2}+m_R^2\left(\mu\frac{\partial}{\partial\mu}\ln Z_m/Z\right)_{\lambda,m^2}. \tag{10.162}$$

We define the renormalization group function γ by

$$\begin{aligned}\gamma_m=\gamma_m(g_R,\frac{m_R^2}{\mu^2}) &= \left(\mu\frac{\partial}{\partial\mu}\ln m_R^2\right)_{\lambda,m^2}\\ &= -\left(\mu\frac{\partial}{\partial\mu}\ln Z_m/Z\right)_{\lambda,m^2}.\end{aligned} \tag{10.163}$$

In the minimal subtraction scheme the renormalization constants will only depend on the dimensionless parameters g_R and as a consequence the renormalization group functions will only depend on g_R. In this case we find

$$\begin{aligned}\beta(g_R) &= -\epsilon g_R\left[1+g_R\frac{d}{dg_R}\ln\frac{Z_g}{Z}\right]^{-1}\\ &= -\epsilon\left[\frac{d}{dg_R}\ln G(g_R)\right]^{-1}.\end{aligned} \tag{10.164}$$

$$\gamma_m(g_R) = -\beta(g_R)\frac{d}{dg_R}\ln\frac{Z_m}{Z}. \tag{10.165}$$

We go back to the renormalized proper vertices (in minimal subtraction) given by

$$\tilde{\Gamma}_R^{(n)}(p_i,\mu^2;m_R^2,g_R,\epsilon)=Z^{n/2}(g_R,\epsilon)\tilde{\Gamma}^{(n)}(p_i;m^2,g,\epsilon). \tag{10.166}$$

Again the bare proper vertices must be independent of the subtraction mass scale, viz

$$0=\left(\mu\frac{\partial}{\partial\mu}\tilde{\Gamma}^{(n)}\right)_{\lambda,m^2}. \tag{10.167}$$

By differentiating both sides of equation (10.166) with respect to μ keeping m^2 and λ fixed we obtain

$$\left(\mu\frac{\partial}{\partial\mu}\tilde{\Gamma}_R^{(n)}\right)_{\lambda,m^2}=\frac{n}{2}\left(\mu\frac{\partial}{\partial\mu}\ln Z\right)_{\lambda,m^2}\tilde{\Gamma}_R^{(n)}. \tag{10.168}$$

Equivalently we have

$$\left(\mu\frac{\partial}{\partial\mu}+\mu\frac{\partial m_R^2}{\partial\mu}\frac{\partial}{\partial m_R^2}+\mu\frac{\partial g_R}{\partial\mu}\frac{\partial}{\partial g_R}\right)_{\lambda,m^2}\tilde{\Gamma}_R^{(n)}=\frac{n}{2}\left(\mu\frac{\partial}{\partial\mu}\ln Z\right)_{\lambda,m^2}\tilde{\Gamma}_R^{(n)}. \tag{10.169}$$

We get finally

$$\Big(\mu\frac{\partial}{\partial\mu}+m_R^2\gamma_m\frac{\partial}{\partial m_R^2}+\beta\frac{\partial}{\partial g_R}-\frac{n}{2}\eta\Big)\tilde{\Gamma}_R^{(n)}=0. \tag{10.170}$$

This is our first renormalization group equation. The new renormalization group function η (also called the anomalous dimension of the field operator) is defined by

$$\begin{aligned}\eta(g_R) &= \Big(\mu\frac{\partial}{\partial\mu}\ln Z\Big)_{\lambda,m^2}\\ &= \beta(g_R)\frac{d}{dg_R}\ln Z.\end{aligned} \tag{10.171}$$

Renormalizability of the ϕ^4 theory in $d=4$ via minimal subtraction (the fifth theorem) can be stated as follows:

- The renormalized proper vertices $\tilde{\Gamma}_R^{(n)}(p_i,\mu^2;m_R^2,g_R,\epsilon)$ and the renormalization group functions $\beta(g_R)$, $\gamma(g_R)$ and $\eta(g_R)$ have a finite limit when $\epsilon\longrightarrow 0$.

By using the above the theorem and the fact that $G(g_R)=g_R+....$ we conclude that the beta function must be of the form

$$\beta(g_R)=-\epsilon g_R+\beta_2(\epsilon)g_R^2+\beta_3(\epsilon)g_R^3+... \tag{10.172}$$

The functions $\beta_i(\epsilon)$ are regular in the limit $\epsilon\longrightarrow 0$. By using the result (10.164) we find

$$\begin{aligned}g_R\frac{G'}{G} &= -\frac{\epsilon g_R}{\beta}\\ &= 1+\frac{\beta_2(\epsilon)}{\epsilon}g_R+(\frac{\beta_2^2(\epsilon)}{\epsilon^2}+\frac{\beta_3(\epsilon)}{\epsilon})g_R^2+...\end{aligned} \tag{10.173}$$

The most singular term in ϵ is captured by the function $\beta_2(\epsilon)$. By integrating this equation we obtain

$$G(g_R)=g_R\Big[1-\frac{\beta_2(0)}{\epsilon}g_R\Big]^{-1}+\text{less singular terms}. \tag{10.174}$$

The function $G(g_R)$ can then be expanded as

$$G(g_R)=g_R+\sum_{n=2}g_R^n\tilde{G}_n(\epsilon). \tag{10.175}$$

The functions $\tilde{G}_n(\epsilon)$ behave as

$$\tilde{G}_n(\epsilon)=\frac{\beta_2^{n-1}(0)}{\epsilon^{n-1}}+\text{less singular terms}. \tag{10.176}$$

Alternatively we can expand G as

$$G(g_R)=g_R+\sum_{n=1}\frac{G_n(g_R)}{\epsilon^n}+\text{regular terms}\ ,\ G_n(g_R)=O(g_R^{n+1}). \tag{10.177}$$

This is equivalent to

$$\frac{Z_g}{Z^2} = 1 + \sum_{n=1} \frac{H_n(g_R)}{\epsilon^n} + \text{regular terms}\ ,\ H_n(g_R) = O(g_R^n). \tag{10.178}$$

We compute the beta function

$$\begin{aligned}
\beta(g_R) &= -\epsilon\bigg[g_R + \sum_{n=1} \frac{G_n(g_R)}{\epsilon^n}\bigg]\bigg[1 + \sum_{n=1} \frac{G_n^{'}(g_R)}{\epsilon^n}\bigg]^{-1} \\
&= -\epsilon\bigg[g_R + \frac{G_1}{\epsilon} + ...\bigg]\bigg[1 - \frac{G_1^{'}}{\epsilon} + \frac{(G_1^{'})^2}{\epsilon^2} - \frac{G_2^{'}}{\epsilon^2} + ...\bigg] \\
&= -\epsilon g_R - G_1(g_R) + g_R G_1^{'}(g_R) + \sum_{n=1} \frac{b_n(g_R)}{\epsilon^n}.
\end{aligned} \tag{10.179}$$

The beta function is finite in the limit $\epsilon \longrightarrow 0$ and as a consequence we must have $b_n(g_R) = 0$ for all n. The beta function must therefore be of the form

$$\beta(g_R) = -\epsilon g_R - G_1(g_R) + g_R G_1^{'}(g_R). \tag{10.180}$$

The beta function β is completely determined by the residue of the simple pole of G, i.e. by G_1. In fact all the functions G_n with $n \geq 2$ are determined uniquely by G_1 (from the condition $b_n = 0$).

Similarly from the finiteness of η in the limit $\epsilon \longrightarrow 0$ we conclude that the renormalization constant Z is of the form

$$Z(g_R) = 1 + \sum_{n=1} \frac{\alpha_n(g_R)}{\epsilon^n} + \text{regular terms}\ ,\ \alpha_n(g_R) = O(g_R^{n+1}). \tag{10.181}$$

We compute the anomalous dimension

$$\begin{aligned}
\eta &= \beta(g_R)\frac{d}{dg_R} \ln Z(g_R) \\
&= \bigg[-\epsilon g_R - G_1(g_R) + g_R G_1^{'}(g_R)\bigg]\bigg[\frac{1}{\epsilon}\alpha_1^{'} + ...\bigg].
\end{aligned} \tag{10.182}$$

Since η is finite in the limit $\epsilon \longrightarrow 0$ we must have

$$\eta = -g_R \alpha_1^{'}. \tag{10.183}$$

10.2.4 CS Renormalization Group Equation in ϕ^4 Theory

We will assume $d = 4$ in this section although much of what we will say is also valid in other dimensions. We will also use a cutoff regularization throughout.

Inhomogeneous CS RG Equation: Let us consider now ϕ^4 theory with ϕ^2 insertions. We add to the action (10.136) a source term of the form $\int d^dx K(x)\phi^2(x)/2$, i.e.

$$S[\phi, K] = \int d^dx \bigg[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}(\phi^2)^2 + \frac{1}{2}K\phi^2\bigg]. \tag{10.184}$$

Then we consider the path integral

$$Z[J,K] \quad = \quad \int \mathcal{D}\phi \exp(iS[\phi,K] + i\int d^dx J\phi). \tag{10.185}$$

It is clear that differentiation with respect to $K(x)$ generates insertions of the operator $-\phi^2/2$. The corresponding renormalized field theory will be given by the path integral

$$Z_R[J,K] \quad = \quad \int \mathcal{D}\phi_R \exp(iS[\phi_R,K] + i\int d^dx J\phi_R). \tag{10.186}$$

$$S[\phi_R,K] \quad = \quad \int d^dx \bigg[\frac{1}{2}\partial_\mu\phi_R\partial^\mu\phi_R - \frac{1}{2}m_R^2\phi_R^2 - \frac{\lambda_R}{4!}(\phi_R^2)^2 + \frac{Z_2}{2}K\phi_R^2\bigg] + \delta S. \tag{10.187}$$

Z_2 is a new renormalization constant associated with the operator $\int d^dx K(x)\phi^2(x)/2$. We have clearly the relations

$$W_R[J,K] = W[\frac{J}{\sqrt{Z}}, \frac{Z_2}{Z}K]. \tag{10.188}$$

$$\Gamma_R[\phi_c,K] = \Gamma[\sqrt{Z}\phi_c, \frac{Z_2}{Z}K]. \tag{10.189}$$

The renormalized $(l,n)-$point proper vertex $\Gamma_R^{(l,n)}$ is given in terms of the bare $(l,n)-$point proper vertex $\Gamma^{(l,n)}$ by

$$\Gamma_R^{(l,n)}(y_1,...,y_l;x_1,...,x_n) = Z^{\frac{n}{2}-l}Z_2^l\Gamma^{(l,n)}(y_1,...,y_l;x_1,...,x_n). \tag{10.190}$$

The proper vertex $\Gamma^{(1,2)}(y;x_1,x_2)$ is a new superficially divergent proper vertex which requires a new counterterm and a new renormalization condition. For consistency with the tree level action we choose the renormalization condition

$$\tilde{\Gamma}_R^{(1,2)}(q;p_1,p_2)|_{q=p_i=0} = 1. \tag{10.191}$$

Let us remark that correlation functions with one operator insertion $i\phi^2(y)/2$ are defined by

$$< \frac{i}{2}\phi^2(y)\phi(x_1)...\phi(x_n) >= \frac{1}{i^n}\frac{1}{Z[J,K]}\frac{\delta}{\delta K(y)}\frac{\delta}{\delta J(x_1)}...\frac{\delta}{\delta J(x_n)}Z[J,K]|_{J=K=0}. \tag{10.192}$$

This can be generalized easily to

$$< \frac{i^l}{2^l}\phi^2(y_1)...\phi^2(y_l)\phi(x_1)...\phi(x_n) >= \frac{1}{i^n}\frac{1}{Z[J,K]}\frac{\delta}{\delta K(y_1)}...\frac{\delta}{\delta K(y_l)}\frac{\delta}{\delta J(x_1)}...\frac{\delta}{\delta J(x_n)}Z[J,K]|_{J=K=0}. \tag{10.193}$$

From this formula we see that the generating functional of correlation functions with l operator insertions $i\phi^2(y)/2$ is defined by

$$Z[y_1,...,y_l;J] = \frac{\delta}{\delta K(y_1)}...\frac{\delta}{\delta K(y_l)}Z[J,K]|_{K=0}. \tag{10.194}$$

The generating functional of the connected correlation functions with l operator insertions $i\phi^2(y)/2$ is then defined by

$$W[y_1,...,y_l;J]=\frac{\delta}{\delta K(y_1)}...\frac{\delta}{\delta K(y_l)}W[J,K]|_{K=0}. \tag{10.195}$$

We write the effective action as

$$\Gamma[\phi_c,K] = \sum_{l,n=0}\frac{1}{l!n!}\int d^dy_1...\int d^dx_n\Gamma^{(l,n)}(y_1,...,y_l;x_1,...,x_n)K(y_1)...K(y_l)\phi_c(x_1)...\phi_c(x_n). \tag{10.196}$$

The generating functional of 1PI correlation functions with l operator insertions $i\phi_c^2(y)/2$ is defined by

$$\frac{\delta^l\Gamma[\phi_c,K]}{\delta K(y_1)...\delta K(y_l)} = \sum_{n=0}\frac{1}{n!}\int d^dx_1...\int d^dx_n\Gamma^{(l,n)}(y_1,...,y_l;x_1,...,x_n)\phi_c(x_1)...\phi_c(x_n). \tag{10.197}$$

Clearly

$$\frac{\delta^{l+n}\Gamma[\phi_c,K]}{\delta K(y_1)...\delta K(y_l)\delta\phi_c(x_1)...\delta\phi_c(x_n)} = \Gamma^{(l,n)}(y_1,...,y_l;x_1,...,x_n). \tag{10.198}$$

We also write

$$\Gamma[\phi_c,K] = \sum_{l,n=0}\frac{1}{l!n!}\int\frac{d^dq_1}{(2\pi)^d}...\int\frac{d^dp_n}{(2\pi)^d}\tilde{\Gamma}^{(l,n)}(q_1,...,q_l;p_1,...,p_n)\tilde{K}(q_1)...\tilde{K}(q_l)\tilde{\phi}_c(p_1)...\tilde{\phi}_c(p_n). \tag{10.199}$$

We have defined

$$\int d^dy_1...\int d^dx_n\Gamma^{(l,n)}(y_1,...,y_l;x_1,...,x_n)\ e^{iq_1y_1}...e^{iq_ly_l}e^{ip_1x_1}...e^{ip_nx_n}=\tilde{\Gamma}^{(l,n)}(q_1,...,q_l;p_1,...,p_n). \tag{10.200}$$

The definition of the proper vertex $\tilde{\Gamma}^{(l,n)}(q_1,...,q_l;p_1,...,p_n)$ in this equation includes a delta function. We recall that

$$\Gamma[\phi_c,K]=W[J,K]-\int d^dxJ(x)\phi_c(x)\ ,\ \phi_c(x)=\frac{\delta W[J,K]}{\delta J(x)}. \tag{10.201}$$

We calculate immediately

$$\frac{\partial W}{\partial m^2}|_{\lambda,\Lambda}=-\int d^dz\frac{\delta W}{\delta K(z)}\Rightarrow\frac{\partial\Gamma}{\partial m^2}|_{\lambda,\Lambda}=-\int d^dz\frac{\delta\Gamma}{\delta K(z)}. \tag{10.202}$$

As a consequence

$$\begin{aligned}\frac{\partial\Gamma^{(l,n)}(y_1,...,y_l;x_1,...,x_n)}{\partial m^2}|_{\lambda,\Lambda} &= -\int d^dz\frac{\delta\Gamma^{(l,n)}(y_1,...,y_l;x_1,...,x_n)}{\delta K(z)}\\ &= -\int d^dz\Gamma^{(l+1,n)}(z,y_1,...,y_l;x_1,...,x_n)\end{aligned} \tag{10.203}$$

Fourier transform then gives

$$\frac{\partial\tilde{\Gamma}^{(l,n)}(q_1,...,q_l;p_1,...,p_n)}{\partial m^2}|_{\lambda,\Lambda} = -\tilde{\Gamma}^{(l+1,n)}(0,q_1,...,q_l;p_1,...,p_n). \quad (10.204)$$

By using equation (10.190) to convert bare proper vertices into renormalized proper vertices we obtain

$$\bigg(\frac{\partial}{\partial m^2}-\frac{n}{2}\frac{\partial\ln Z}{\partial m^2}-l\frac{\partial\ln Z_2/Z}{\partial m^2}\bigg)_{\lambda,\Lambda}\tilde{\Gamma}_R^{(l,n)} = -ZZ_2^{-1}\tilde{\Gamma}_R^{(l+1,n)}. \quad (10.205)$$

The factor of -1 multiplying the right hand side of this equation will be absent in the Euclidean rotation of the theory [9]. The renormalized proper vertices $\tilde{\Gamma}_R^{(l,n)}$ depend on the momenta q_1,...,q_l,p_1,...,p_n but also on the renormalized mass m_R^2, the renormalized coupling constant λ_R and the cutoff Λ. They also depend on the subtraction mass scale μ^2. We will either assume that $\mu^2=0$ or $\mu^2=m_R^2$. We have then

$$\bigg(\frac{\partial m_R}{\partial m^2}\frac{\partial}{\partial m_R}+\frac{\partial\lambda_R}{\partial m^2}\frac{\partial}{\partial\lambda_R}-\frac{n}{2}\frac{\partial\ln Z}{\partial m^2}-l\frac{\partial\ln Z_2/Z}{\partial m^2}\bigg)_{\lambda,\Lambda}\tilde{\Gamma}_R^{(l,n)} = -ZZ_2^{-1}\tilde{\Gamma}_R^{(l+1,n)}. \quad (10.206)$$

We write this as

$$\frac{\partial m_R}{\partial m^2}\bigg(m_R\frac{\partial}{\partial m_R}+m_R\frac{\partial m^2}{\partial m_R}\frac{\partial\lambda_R}{\partial m^2}\frac{\partial}{\partial\lambda_R}-\frac{n}{2}m_R\frac{\partial m^2}{\partial m_R}\frac{\partial\ln Z}{\partial m^2}-lm_R\frac{\partial m^2}{\partial m_R}\frac{\partial\ln Z_2/Z}{\partial m^2}\bigg)_{\lambda,\Lambda}\tilde{\Gamma}_R^{(l,n)} = -m_RZZ_2^{-1}\tilde{\Gamma}_R^{(l+1,n)} \quad (10.207)$$

We define

$$\beta(\lambda_R,\frac{m_R}{\Lambda}) = \bigg(m_R\frac{\partial m^2}{\partial m_R}\frac{\partial\lambda_R}{\partial m^2}\bigg)_{\lambda,\Lambda} = \bigg(m_R\frac{\partial\lambda_R}{\partial m_R}\bigg)_{\lambda,\Lambda}. \quad (10.208)$$

$$\eta(\lambda_R,\frac{m_R}{\Lambda}) = \bigg(m_R\frac{\partial m^2}{\partial m_R}\frac{\partial\ln Z}{\partial m^2}\bigg)_{\lambda,\Lambda} = \bigg(m_R\frac{\partial}{\partial m_R}\ln Z+\beta\frac{\partial}{\partial\lambda_R}\ln Z\bigg)_{\lambda,\Lambda}. \quad (10.209)$$

$$\eta_2(\lambda_R,\frac{m_R}{\Lambda}) = \bigg(m_R\frac{\partial m^2}{\partial m_R}\frac{\partial\ln Z_2/Z}{\partial m^2}\bigg)_{\lambda,\Lambda} = \bigg(m_R\frac{\partial}{\partial m_R}\ln Z_2/Z+\beta\frac{\partial}{\partial\lambda_R}\ln Z_2/Z\bigg)_{\lambda,\Lambda}. \quad (10.210)$$

$$m_R^2\sigma(\lambda_R,\frac{m_R}{\Lambda}) = ZZ_2^{-1}\bigg(m_R\frac{\partial m^2}{\partial m_R}\bigg)_{\lambda,\Lambda}. \quad (10.211)$$

[9] Exercise: Check this.

The above differential equation becomes with these definitions

$$\Big(m_R\frac{\partial}{\partial m_R}+\beta\frac{\partial}{\partial \lambda_R}-\frac{n}{2}\eta-l\eta_2\Big)\tilde{\Gamma}_R^{(l,n)} \quad = \quad -m_R^2\sigma\tilde{\Gamma}_R^{(l+1,n)}. \tag{10.212}$$

This is the original Callan-Symanzik equation. This equation represents only the response of the proper vertices to rescaling ($\phi\longrightarrow\phi_R$) and to reparametrization ($m^2\longrightarrow m_R^2$, $\lambda\longrightarrow\lambda_R$). We still need to impose on the Callan-Symanzik equation the renormalization conditions in order to determine the renormalization constants and show that the renormalized proper vertices have a finite limit when $\Lambda\longrightarrow\infty$. The functions β, η, η_2 and σ can be expressed in terms of renormalized proper vertices and as such they have an infinite cutoff limit. The Callan-Symanzik equation (10.212) can be used to provide an inductive proof of renormalizability of ϕ^4 theory in 4 dimensions. We will not go through this involved exercise at this stage.

Homogeneous CS RG Equation-Massless Theory: The renormalization conditions for a massless ϕ^4 theory in $d=4$ are given by

$$\begin{aligned}
&\tilde{\Gamma}_R^{(2)}(p)|_{p^2=0}=0\\
&\frac{d}{dp^2}\tilde{\Gamma}^{(2)}(p)|_{p^2=\mu^2}=1\\
&\tilde{\Gamma}^{(4)}(p_1,...,p_4)|_{\rm SP}=\lambda_R.
\end{aligned} \tag{10.213}$$

The renormalized proper vertices $\tilde{\Gamma}_R^{(n)}$ depend on the momenta p_1,...,p_n, the mass scale μ^2, the renormalized coupling constant λ_R and the cutoff Λ. The bare proper vertices $\tilde{\Gamma}^{(n)}$ depend on the momenta p_1,...,p_n, the bare coupling constant λ and the cutoff Λ. The bare mass is fixed by the condition that the renormalized mass is 0. We have then

$$\tilde{\Gamma}_R^{(n)}(p_i,\mu^2;\lambda_R,\Lambda)=Z^{\frac{n}{2}}(\lambda,\frac{\Lambda^2}{\mu^2},\Lambda)\tilde{\Gamma}^{(n)}(p_i;\lambda,\Lambda). \tag{10.214}$$

The bare theory is obviously independent of the mass scale μ^2. This is expressed by the condition

$$\Big(\mu\frac{\partial}{\partial\mu}\tilde{\Gamma}^{(n)}(p_i;\lambda,\Lambda)\Big)_{\lambda,\Lambda}=0. \tag{10.215}$$

We differentiate equation (10.214) with respect to μ^2 keeping λ and Λ fixed. We get

$$\frac{\partial}{\partial\mu}\tilde{\Gamma}_R^{(n)}+\Big(\frac{\partial\lambda_R}{\partial\mu}\Big)_{\lambda,\Lambda}\frac{\partial}{\partial\lambda_R}\tilde{\Gamma}_R^{(n)}=\frac{n}{2}\Big(\frac{\partial\ln Z}{\partial\mu}\Big)_{\lambda,\Lambda}Z^{\frac{n}{2}}\tilde{\Gamma}^{(n)}. \tag{10.216}$$

We obtain immediately the differential equation

$$\Big(\mu\frac{\partial}{\partial\mu}+\beta(\lambda_R)\frac{\partial}{\partial\lambda_R}-\frac{n}{2}\eta(\lambda_R)\Big)\tilde{\Gamma}_R^{(n)}=0. \tag{10.217}$$

$$\beta(\lambda_R)=\Big(\mu\frac{\partial\lambda_R}{\partial\mu}\Big)_{\lambda,\Lambda}\;,\;\eta(\lambda_R)=\Big(\mu\frac{\partial\ln Z}{\partial\mu}\Big)_{\lambda,\Lambda}. \tag{10.218}$$

This is the Callan-Symanzik equation for the massless theory. The functions β and η do not depend on Λ/μ since they can be expressed in terms of renormalized proper vertices and as such they have an infinite cutoff limit.

For the massless theory with ϕ^2 insertions we need, as in the massive case, an extra renormalization constant Z_2 and an extra renormalization condition to fix it given by

$$\tilde{\Gamma}_R^{(1,2)}(q;p_1,p_2)|_{q^2=p_i^2=\mu^2} = 1. \tag{10.219}$$

We will also need an extra RG function given by

$$\eta_2(\lambda_R) = \bigg(\mu\frac{\partial}{\partial\mu}\ln Z_2/Z\bigg)_{\lambda,\Lambda}. \tag{10.220}$$

The Callan-Symanzik equation for the massless theory with ϕ^2 insertions is then given (by the almost obvious) equation

$$\bigg(\mu\frac{\partial}{\partial\mu}+\beta(\lambda_R)\frac{\partial}{\partial\lambda_R}-\frac{n}{2}\eta(\lambda_R)-l\eta_2(\lambda_R)\bigg)\tilde{\Gamma}_R^{(l,n)} = 0. \tag{10.221}$$

Homogeneous CS RG Equation-Massive Theory: We consider again a massless ϕ^4 theory in $d=4$ dimensions with ϕ^2 insertions. The action is given by the massless limit of the action (10.187), namely

$$S[\phi_R,K] = \int d^dx\bigg[\frac{1}{2}Z\partial_\mu\phi_R\partial^\mu\phi_R-\frac{1}{2}\delta_m\phi_R^2-\frac{\lambda_RZ_g}{4!}(\phi_R^2)^2+\frac{Z_2}{2}K\phi_R^2\bigg]. \tag{10.222}$$

The effective action is still given by

$$\Gamma[\phi_c,K] = \sum_{l,n=0}\frac{1}{l!n!}\int d^dy_1...\int d^dx_n\Gamma^{(l,n)}(y_1,...,y_l;x_1,...,x_n)K(y_1)...K(y_l)\phi_c(x_1)...\phi_c(x_n). \tag{10.223}$$

An arbitrary proper vertex $\tilde{\Gamma}_R^{(n)}(p_1,...,p_n;K)$ can be expanded in terms of the proper vertices $\Gamma_R^{(l,n)}(q_1,...,q_l;p_1,...,p_n)$ as follows

$$\tilde{\Gamma}_R^{(n)}(p_1,...,p_n;K) = \sum_{l=0}\frac{1}{l!}\int\frac{d^dq_1}{(2\pi)^d}...\int\frac{d^dq_l}{(2\pi)^d}\tilde{\Gamma}_R^{(l,n)}(q_1,...,q_l;p_1,...,p_n)\tilde{K}(q_1)...\tilde{K}(q_l). \tag{10.224}$$

We consider the differential operator

$$D = \mu\frac{\partial}{\partial\mu}+\beta(\lambda_R)\frac{\partial}{\partial\lambda_R}-\frac{n}{2}\eta(\lambda_R)-\eta_2(\lambda_R)\int d^dq\tilde{K}(q)\frac{\delta}{\delta\tilde{K}(q)}. \tag{10.225}$$

We compute

$$\begin{aligned}\int d^dq\tilde{K}(q)\frac{\delta}{\delta\tilde{K}(q)}\tilde{\Gamma}_R^{(n)}(p_1,...,p_n;K) &= \sum_{l=0}\frac{1}{l!}\int d^dq_1...\int d^dq_l\tilde{\Gamma}_R^{(l,n)}(q_1,...,q_l;p_1,...,p_n)\\ &\times \int d^dq\tilde{K}(q)\frac{\delta}{\delta\tilde{K}(q)}\tilde{K}(q_1)...\tilde{K}(q_l)\\ &= \sum_{l=0}\frac{1}{l!}\int d^dq_1...\int d^dq_l l\tilde{\Gamma}_R^{(l,n)}(q_1,...,q_l;p_1,...,p_n)\\ &\times \tilde{K}(q_1)...\tilde{K}(q_l).\end{aligned} \tag{10.226}$$

By using now the Callan-Symanzik equation (10.221) we get

$$\Big(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda_R}-\frac{n}{2}\eta-\eta_2\int d^dq\tilde{K}(q)\frac{\delta}{\delta\tilde{K}(q)}\Big)\tilde{\Gamma}_R^{(n)}(p_1,...,p_n;K)=0. \quad (10.227)$$

A massive theory can be obtained by setting the source $-K(x)$ equal to a constant which will play the role of the renormalized mass m_R^2. We will then set

$$K(x)=-m_R^2\Leftrightarrow\tilde{K}(q)=-m_R^2(2\pi)^d\delta^d(q). \quad (10.228)$$

We obtain therefore the Callan-Symanzik equation

$$\Big(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial\lambda_R}-\frac{n}{2}\eta-\eta_2m_R^2\frac{\partial}{\partial m_R^2}\Big)\tilde{\Gamma}_R^{(n)}(p_1,...,p_n;m_R^2)=0. \quad (10.229)$$

This needs to be compared with the renormalization group equation (10.170) and as a consequence the renormalization function $-\eta_2$ must be compared with the renormalization constant γ_m. The renormalized proper vertices $\tilde{\Gamma}_R^{(n)}$ will also depend on the coupling constant λ_R, the subtraction mass scale μ and the cutoff Λ.

10.2.5 Summary

We end this section by summarizing our main results so far. The bare action and with ϕ^2 insertion is

$$S[\phi,K] = \int d^dx\Big[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4!}(\phi^2)^2+\frac{1}{2}K\phi^2\Big]. \quad (10.230)$$

The renormalized action is

$$S_R[\phi_R,K] = \int d^dx\Big[\frac{1}{2}\partial_\mu\phi_R\partial^\mu\phi_R-\frac{1}{2}m_R^2\phi_R^2-\frac{\lambda_R}{4!}(\phi^2)^2+\frac{1}{2}Z_2K\phi_R^2\Big]. \quad (10.231)$$

The dimensionless coupling g_R and the renormalization constants Z, Z_g and Z_m are defined by the equations

$$g_R=\mu^{-\epsilon}\lambda_R\ ,\ \epsilon=4-d. \quad (10.232)$$

$$\begin{aligned}\phi&=\sqrt{Z}\phi_R\\ \lambda&=\lambda_R\frac{Z_g}{Z^2}\\ m^2&=m_R^2\frac{Z_m}{Z}.\end{aligned} \quad (10.233)$$

The arbitrary mass scale μ defines the renormalization scale. For example renormalization conditions must be imposed at the scale μ as follows

$$\begin{aligned}&\tilde{\Gamma}_R^{(2)}(p)|_{p^2=0}=m_R^2\\ &\frac{d}{dp^2}\tilde{\Gamma}^{(2)}(p)|_{p^2=\mu^2}=1\\ &\tilde{\Gamma}^{(4)}(p_1,...,p_4)|_{\rm SP}=\mu^\epsilon g_R\\ &\tilde{\Gamma}_R^{(1,2)}(q;p_1,p_2)|_{q=p_i=\mu^2}=1.\end{aligned} \quad (10.234)$$

However we will use in the following minimal subtraction to renormalize the theory instead of renormalization conditions. In minimal subtraction, which is due to 't Hooft, the renormalization functions β, γ_m, η and η_2 depend only on the coupling constant g_R and they are defined by

$$\beta(g_R) = \bigg(\mu \frac{\partial g_R}{\partial \mu}\bigg)_{\lambda, m^2} = -\epsilon \bigg[\frac{d}{dg_R} \ln G(g_R)\bigg]^{-1} \ , \ G = \frac{Z_g}{Z^2} g_R. \tag{10.235}$$

$$\gamma_m(g_R) = \bigg(\mu \frac{\partial}{\partial \mu} \ln m_R^2\bigg)_{\lambda, m^2} = -\beta(g_R) \frac{d}{dg_R} \ln \frac{Z_m}{Z}. \tag{10.236}$$

$$\eta(g_R) = \bigg(\mu \frac{\partial}{\partial \mu} \ln Z\bigg)_{\lambda, m^2} = \beta(g_R) \frac{d}{dg_R} \ln Z. \tag{10.237}$$

$$\eta_2(g_R) = \bigg(\mu \frac{\partial}{\partial \mu} \ln \frac{Z_2}{Z}\bigg)_{\lambda, m^2} = \beta(g_R) \frac{d}{dg_R} \ln \frac{Z_2}{Z}. \tag{10.238}$$

We may also use the renormalization function γ defined simply by

$$\gamma(g_R) = \frac{\eta(g_R)}{2}. \tag{10.239}$$

The renormalized proper vertices are given by

$$\tilde{\Gamma}_R^{(n)}(p_i, \mu^2; m_R^2, g_R) = Z^{n/2}(g_R, \epsilon) \tilde{\Gamma}^{(n)}(p_i; m^2, \lambda, \epsilon). \tag{10.240}$$

$$\tilde{\Gamma}_R^{(l,n)}(q_i; p_i; \mu^2; m_R^2, g_R) = Z^{\frac{n}{2}-l} Z_2^l \tilde{\Gamma}^{(l,n)}(q_i; p_i; m^2, \lambda, \epsilon). \tag{10.241}$$

They satisfy the renormalization group equations

$$\bigg(\mu \frac{\partial}{\partial \mu} + \gamma_m m_R^2 \frac{\partial}{\partial m_R^2} + \beta \frac{\partial}{\partial g_R} - \frac{n}{2} \eta\bigg) \tilde{\Gamma}_R^{(n)} = 0. \tag{10.242}$$

$$\bigg(m_R \frac{\partial}{\partial m_R} + \beta \frac{\partial}{\partial \lambda_R} - \frac{n}{2} \eta - l \eta_2\bigg) \tilde{\Gamma}_R^{(l,n)} = -m_R^2 \sigma \tilde{\Gamma}_R^{(l+1,n)}. \tag{10.243}$$

In the first equation we have set $K = 0$ and in the second equation the renormalization scale is $\mu = m_R$. The renormalization function σ is given by

$$\begin{aligned} \sigma(g_R) &= \frac{Z}{Z_2} \frac{1}{m_R^2} \bigg(m_R \frac{\partial m^2}{\partial m_R}\bigg)_\lambda \\ &= \frac{Z_m}{Z_2} \bigg[2 + \beta(g_R) \frac{d}{dg_R} \ln \frac{Z_m}{Z}\bigg]. \end{aligned} \tag{10.244}$$

An alternative renormalization group equation satisfied by the proper vertices $\tilde{\Gamma}_R^{(n)}$ can be obtained by starting from a massless theory, i.e. $m = m_R = 0$ with $K \neq 0$ and then setting $K = -m_R^2$ at the end. We obtain

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\lambda_R} - \frac{n}{2}\eta - \eta_2 m_R^2\frac{\partial}{\partial m_R^2}\right)\tilde{\Gamma}_R^{(n)} = 0. \tag{10.245}$$

In this form the massless limit is accessible. As we can see from (10.242) and (10.245) the renormalization functions $\gamma_m(g_R)$ and $-\eta_2(g_R)$ are essentially the same object. Indeed since the two equations describe the same theory one must have

$$\eta_2(g_R) = -\gamma_m(g_R). \tag{10.246}$$

Alternatively we see from equation (10.244) that the renormalization constant Z_2 is not an independent renormalization constant since σ is finite. In accordance with (10.246) we choose

$$Z_2 = Z_m. \tag{10.247}$$

Because $Z_2 = Z_m$ equation (10.244) becomes

$$\begin{aligned}\sigma(g_R) &= \left(m_R\frac{\partial}{\partial m_R}\ln m^2\right)_\lambda \\ &= 2 - \gamma_m.\end{aligned} \tag{10.248}$$

10.3 Renormalization Constants and Renormalization Functions at Two-Loop

10.3.1 The Divergent Part of the Effective Action

The 2 and 4−Point Proper Vertices: Now we will renormalize the $O(N)$ sigma model at the two-loop order using dimensional regularization and (modified) minimal subtraction. The main divergences in this theory occur in the 2−point proper vertex (quadratic) and the 4−point proper vertex (logarithmic). Indeed all other divergences in this theory stem from these two functions. Furthermore only the divergence in the 2−point proper vertex is momentum dependent.

The 2−point and 4−point (at zero momentum) proper vertices of the $O(N)$ sigma model at the two-loop order in Euclidean signature are given by equations (8.201) and (8.227), viz

$$\Gamma_{ij}^{(2)}(p) = \delta_{ij}\left[p^2 + m^2 + \frac{1}{2}\lambda\frac{N+2}{3}(a) - \frac{\lambda^2}{4}\left(\frac{N+2}{3}\right)^2(b) - \frac{\lambda^2}{6}\frac{N+2}{3}(c)\right]. \tag{10.249}$$

$$\begin{aligned}\Gamma_{i_1\ldots i_4}^{(4)}(0,0,0,0) &= \frac{\delta_{i_1i_2i_3i_4}}{3}\left[\lambda - \frac{3}{2}\frac{N+8}{9}\lambda^2(d) + \frac{3}{2}\lambda^3\frac{(N+2)(N+8)}{27}(g)\right. \\ &+ \left.\frac{3}{4}\lambda^3\frac{(N+2)(N+4)+12}{27}(e) + 3\lambda^3\frac{5N+22}{27}(f)\right].\end{aligned} \tag{10.250}$$

The Feynman diagrams corresponding to (a), (b), (c), (d), (g), (e) and (f) are shown on figure 16. Explicitly we have

$$(a) = I(m^2) = \int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+m^2}. \tag{10.251}$$

$$(d) = J(0,m^2) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2+m^2)^2}. \quad (10.252)$$

$$(b) = I(m^2)J(0,m^2) = (a)(d). \quad (10.253)$$

$$(c) = K(p^2,m) = \int \frac{d^dk}{(2\pi)^d} \frac{d^dl}{(2\pi)^d} \frac{1}{(l^2+m^2)(k^2+m^2)((l+k+p)^2+m^2)}. \quad (10.254)$$

$$(g) = I(m^2)L(0,m^2) = \int \frac{d^dk}{(2\pi)^d} \frac{1}{k^2+m^2} \int \frac{d^dl}{(2\pi)^d} \frac{1}{(l^2+m^2)^3}. \quad (10.255)$$

$$(e) = J(0,m^2)^2 = (d)^2. \quad (10.256)$$

$$(f) = M(0,0,m^2) = \int \frac{d^dl}{(2\pi)^d} \frac{d^dk}{(2\pi)^d} \frac{1}{(l^2+m^2)(k^2+m^2)((l+k)^2+m^2)}. \quad (10.257)$$

We remark that the two-loop graph (g) is a superposition of the one-loop graphs (a) and (d) and thus it will be made finite once (a) and (d) are renormalized. At the two-loop order only the diagram (c) is momentum dependent. We introduce the notation

$$\begin{aligned} (c) = \Sigma^{(2)}(p) &= \Sigma^{(2)}(0) + p^2 \frac{\partial}{\partial p^2} \Sigma^{(2)}(0) + ... \\ &= m^{2d-6} I_2 + p^2 m^{2d-8} I_3 + ... \end{aligned} \quad (10.258)$$

We will also introduce the notation

$$(a) = m^{d-2} I_1. \quad (10.259)$$

All other integrals can be expressed in terms of I_1 and I_2. Indeed we can show [10]

$$(d) = -\frac{\partial}{\partial m^2}(a) = (1-\frac{d}{2}) m^{d-4} I_1. \quad (10.260)$$

$$(b) = (1-\frac{d}{2}) m^{2d-6} I_1^2. \quad (10.261)$$

$$(e) = (1-\frac{d}{2})^2 m^{2d-8} I_1^2. \quad (10.262)$$

$$(f) = -\frac{1}{3} \frac{\partial}{\partial m^2} \Sigma^{(2)}(0) = -\frac{1}{3}(d-3) m^{2d-8} I_2. \quad (10.263)$$

$$(g) = -\frac{1}{2}(a) \frac{\partial}{\partial m^2}(d) = \frac{1}{2}(1-\frac{d}{2})(2-\frac{d}{2}) m^{2d-8} I_1^2. \quad (10.264)$$

[10] Exercise: Derive these results.

Calculation of The Poles: We have already met the integral I_1 before. We compute

$$\begin{aligned} I_1 &= \int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2+1} \\ &= \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}\int\frac{k^{d-1}dk}{k^2+1} \\ &= \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}\frac{1}{2}\int\frac{x^{d/2-1}dx}{x+1}. \end{aligned} \tag{10.265}$$

We use the formula

$$\int\frac{u^\alpha du}{(u+a)^\beta} = a^{\alpha+1-\beta}\frac{\Gamma(\alpha+1)\Gamma(\beta-\alpha-1)}{\Gamma(\beta)}. \tag{10.266}$$

Thus (with $d=4-\epsilon$)

$$(a) = \frac{m^2}{16\pi^2}\frac{(m^2)^{-\epsilon/2}}{(4\pi)^{-\epsilon/2}}\Gamma(-1+\epsilon/2) \tag{10.267}$$

We use the result

$$\Gamma(-1+\epsilon/2) = -\frac{2}{\epsilon}-1+\gamma+O(\epsilon). \tag{10.268}$$

Hence we obtain

$$(a) = \frac{m^2}{16\pi^2}\bigg[-\frac{2}{\epsilon}-1+\gamma+\ln\frac{m^2}{4\pi}+O(\epsilon)\bigg]. \tag{10.269}$$

The first Feynman graph is then given by

$$\lambda(a) = g\frac{m^2}{16\pi^2}\bigg[-\frac{2}{\epsilon}-1+\gamma-\ln 4\pi+\ln\frac{m^2}{\mu^2}+O(\epsilon)\bigg]. \tag{10.270}$$

In minimal subtraction (MS) we subtract only the pole term $-2/\epsilon$ whereas in modified minimal subtraction (MMS) we subtract also any other extra constant such as the term $-1+\gamma-\ln 4\pi$.

We introduce

$$N_d = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)}. \tag{10.271}$$

We compute

$$\begin{aligned} I_1 &= \frac{N_d}{2}\Gamma(d/2)\Gamma(1-d/2) \\ &= \frac{N_d}{2}(-\frac{2}{\epsilon}+O(\epsilon)). \end{aligned} \tag{10.272}$$

Then

$$\begin{aligned} \lambda(a) &= g\mu^\epsilon(m^2)^{d/2-1}I_1 \\ &= gm^2(\frac{\mu}{m})^\epsilon I_1 \\ &= \frac{N_d}{2}[-\frac{2}{\epsilon}+\ln\frac{m^2}{\mu^2}+O(\epsilon)]. \end{aligned} \tag{10.273}$$

From this formula it is now obvious that subtracting $-N_d/\epsilon$ is precisely the above modified minimal subtraction.

We have also met the integral $\Sigma^{(2)}(p)$ before (see (8.202)). By following the same steps that led to equation (8.205) we obtain

$$K(p^2,m^2) = \frac{1}{(4\pi)^d}\int dx_1dx_2dx_3\ \frac{e^{-m^2(x_1+x_2+x_3)-\frac{x_1x_2x_3}{\Delta}p^2}}{\Delta^{d/2}}\ ,\ \Delta=x_1x_2+x_1x_3+x_2x_3. \tag{10.274}$$

Thus

$$\begin{aligned} I_2 &= K(0,1) \\ &= \frac{1}{(4\pi)^d}\int dx_1dx_2dx_3\ \frac{e^{-(x_1+x_2+x_3)}}{\Delta^{d/2}}. \end{aligned} \tag{10.275}$$

$$\begin{aligned} I_3 &= \frac{\partial}{\partial p^2}K(p^2,1)|_{p^2=0} \\ &= -\frac{1}{(4\pi)^d}\int dx_1dx_2dx_3\ \frac{x_1x_2x_3e^{-(x_1+x_2+x_3)}}{\Delta^{1+d/2}}. \end{aligned} \tag{10.276}$$

We perform the change of variables $x_1=stu$, $x_2=st(1-u)$ and $x_3=s(1-t)$. Thus $x_1+x_2+x_3=s$, $dx_1dx_2dx_3=s^2tdsdtdu$ and $\Delta=s^2t(1-t+ut(1-u))$. The above integrals become

$$\begin{aligned} I_2 &= \frac{1}{(4\pi)^d}\int_0^\infty dse^{-s}s^{2-d}\int_0^1 du\int_0^1 dt\frac{t^{1-d/2}}{(1-t+ut(1-u))^{d/2}} \\ &= \frac{\Gamma(3-d)}{(4\pi)^d}\int_0^1 du\int_0^1 dt\frac{t^{1-d/2}}{(1-t+ut(1-u))^{d/2}}. \end{aligned} \tag{10.277}$$

$$\begin{aligned} I_3 &= -\frac{1}{(4\pi)^d}\int_0^\infty dse^{-s}s^{3-d}\int_0^1 duu(1-u)\int_0^1 dt\frac{t^{2-d/2}(1-t)}{(1-t+ut(1-u))^{1+d/2}} \\ &= -\frac{\Gamma(4-d)}{(4\pi)^d}\int_0^1 duu(1-u)\int_0^1 dt\frac{t^{2-d/2}(1-t)}{(1-t+ut(1-u))^{1+d/2}}. \end{aligned} \tag{10.278}$$

We want to evaluate the integral

$$\begin{aligned} J &= \int_0^1 du\int_0^1 dt\ t^{1-d/2}(1-t+ut(1-u))^{-d/2} \\ &= \int_0^1 du\int_0^1 dt\Big[t^{1-d/2}+\Big((1-t+ut(1-u))^{-d/2}-1\Big) \\ &+ (t^{1-d/2}-1)\Big((1-t+ut(1-u))^{-d/2}-1\Big)\Big]. \end{aligned} \tag{10.279}$$

The first term gives the first contribution to the pole term. The last term is finite at $d=4$.

Indeed with the change of variable $x = t\big(u(1-u)-1\big)+1$ we calculate

$$\begin{aligned}
\int_0^1 du \int_0^1 dt(t^{-1}-1)\bigg((1-t+ut(1-u))^{-2}-1\bigg) &= -\int_0^1 du \ln u(1-u) \\
&= -2\int_0^1 du \ln u \\
&= 2. \qquad (10.280)
\end{aligned}$$

We have then

$$\begin{aligned}
J &= \frac{2}{\epsilon} - 1 + \int_0^1 du \int_0^1 dt(1-t+ut(1-u))^{-d/2} + (2+O_1(\epsilon)) \\
&= \frac{2}{\epsilon} - 1 + \frac{2}{\epsilon-2}\int_0^1 du \frac{1}{u(1-u)-1}\bigg((u(1-u))^{1-d/2}-1\bigg) + (2+O_1(\epsilon)) \\
&= \frac{2}{\epsilon} - 1 - \frac{2}{\epsilon-2}\int_0^1 du\bigg((u(1-u))^{1-d/2}-1\bigg) + \frac{2}{\epsilon-2}\int_0^1 du \frac{u(1-u)}{u(1-u)-1} \\
&\times \bigg((u(1-u))^{1-d/2}-1\bigg) + (2+O_1(\epsilon)) \\
&= \frac{2}{\epsilon} - 1 + \frac{2}{\epsilon-2} - \frac{2}{\epsilon-2}\int_0^1 du (u(1-u))^{1-d/2} + \frac{2}{\epsilon-2}(-1+O_2(\epsilon)) + (2+O_1(\epsilon)) \\
&= \frac{2}{\epsilon} - 1 + \frac{2}{\epsilon-2} - \frac{2}{\epsilon-2}\frac{\Gamma^2(2-d/2)}{\Gamma(4-d)} + \frac{2}{\epsilon-2}(-1+O_2(\epsilon)) + (2+O_1(\epsilon)) \qquad (10.281)
\end{aligned}$$

In the last line we have used the result (8.275) whereas in the third line we have used the fact that the second remaining integral is finite at $d=4$. From this result we deduce that

$$J = \frac{6}{\epsilon} + 3 + O(\epsilon). \qquad (10.282)$$

$$\begin{aligned}
I_2 &= \frac{\Gamma(3-d)}{(4\pi)^d} J \\
&= \frac{\Gamma(-1+\epsilon)}{(4\pi)^d} J \\
&= \frac{1}{(4\pi)^d}\bigg(-\frac{6}{\epsilon^2} - \frac{9}{\epsilon} + \frac{6\gamma}{\epsilon} + O(1)\bigg). \qquad (10.283)
\end{aligned}$$

We compute $\Gamma(d/2) = 1 + \gamma\epsilon/2 - \epsilon/2 + O(\epsilon^2)$ and hence $\Gamma^2(d/2) = 1 + \gamma\epsilon - \epsilon + O(\epsilon^2)$. Thus the above result can be rewritten as

$$I_2 = -N_d^2 \frac{3}{2\epsilon^2}(1+\frac{\epsilon}{2}) + O(1). \qquad (10.284)$$

Now we compute the integral I_3. The only divergence has already been exhibited by the term

$\Gamma(4-d)$. Thus we have

$$\begin{aligned} I_3 &= -\frac{\Gamma(\epsilon)}{(4\pi)^d}\int_0^1 duu(1-u)\int_0^1 dt\frac{1-t}{(1-t+ut(1-u))^3}+O(1)\\ &= -\frac{\Gamma(\epsilon)}{(4\pi)^d}\int_0^1 du\frac{1}{2}+O(1)\\ &= -\frac{N_d^2\Gamma^2(d/2)}{8}\Gamma(\epsilon)+O(1)\\ &= -\frac{N_d^2}{8\epsilon}+O(1). \end{aligned} \tag{10.285}$$

10.3.2 Renormalization Constants

One-Loop Renormalization: We prefer to go back to the original expressions

$$\begin{aligned} \Gamma_{ij}^{(2)}(p) &= \delta_{ij}\Big[p^2+m^2+\frac{1}{2}\lambda\frac{N+2}{3}(a)\Big]\\ &= \delta_{ij}\Big[p^2+m^2+\frac{1}{2}\lambda\frac{N+2}{3}I(m^2)\Big]. \end{aligned} \tag{10.286}$$

$$\begin{aligned} \Gamma_{i_1\ldots i_4}^{(4)}(0,0,0,0) &= \frac{\delta_{i_1i_2i_3i_4}}{3}\Big[\lambda-\frac{3}{2}\frac{N+8}{9}\lambda^2(d)\Big]\\ &= \frac{\delta_{i_1i_2i_3i_4}}{3}\Big[\lambda-\frac{3}{2}\frac{N+8}{9}\lambda^2J(0,m^2)\Big]. \end{aligned} \tag{10.287}$$

The renormalized mass and the renormalized coupling constant are given by

$$m^2=m_R^2\frac{Z_m}{Z}\ ,\ \lambda=\lambda_R\frac{Z_g}{Z^2}. \tag{10.288}$$

We will expand the renormalization constants as

$$Z=1+\lambda_R^2Z^{(2)}. \tag{10.289}$$

$$Z_m=1+\lambda_RZ_m^{(1)}+\lambda_R^2Z_m^{(2)}. \tag{10.290}$$

$$Z_g=1+\lambda_RZ_g^{(1)}+\lambda_R^2Z_g^{(2)}. \tag{10.291}$$

At one-loop of course $Z^{(2)}=Z_m^{(2)}=Z_g^{(2)}=0$. We will also define the massless coupling constant by

$$g=\lambda m^{-\epsilon}. \tag{10.292}$$

The renormalized 2−point and 4−point proper vertices are given by

$$\begin{aligned} (\Gamma_R)_{ij}^{(2)}(p) &= Z\Gamma_{ij}^{(2)}(p)\\ &= \delta_{ij}\Big[p^2+m_R^2+\lambda_R\big(m_R^2Z_m^{(1)}+\frac{1}{2}\frac{N+2}{3}I(m_R^2)\big)\Big]. \end{aligned} \tag{10.293}$$

$$
\begin{aligned}
(\Gamma_R)^{(4)}_{i_1\ldots i_4}(0,0,0,0) &= Z^2\Gamma^{(4)}_{i_1\ldots i_4}(0,0,0,0)\\
&= \frac{\delta_{i_1i_2i_3i_4}}{3}\bigg[\lambda_R+\lambda_R^2\big(Z_g^{(1)}-\frac{3}{2}\frac{N+8}{9}J(0,m_R^2)\big)\bigg]. \qquad (10.294)
\end{aligned}
$$

Minimal subtraction gives immediately

$$
Z_m^{(1)}=-\frac{N+2}{6}\frac{I(m_R^2)}{m_R^2}=\frac{N+2}{6}m_R^{-\epsilon}\frac{N_d}{\epsilon}. \qquad (10.295)
$$

$$
Z_g^{(1)}=\frac{N+8}{6}J(0,m_R^2)=\frac{N+8}{6}m_R^{-\epsilon}\frac{N_d}{\epsilon}. \qquad (10.296)
$$

The renormalized mass and the renormalized coupling constant at one-loop order are given by

$$
m^2=m_R^2-\frac{N+2}{6}\lambda_R I(m_R^2)\ ,\ \lambda=\lambda_R+\frac{N+8}{6}\lambda_R^2 J(0,m_R^2). \qquad (10.297)
$$

Two-Loop Renormalization of The 2−Point Proper Vertex: The original expression of the 2−point vertex reads

$$
\Gamma^{(2)}_{ij}(p) = \delta_{ij}\bigg[p^2+m^2+\frac{1}{2}\lambda\frac{N+2}{3}(a)-\frac{\lambda^2}{4}(\frac{N+2}{3})^2(b)-\frac{\lambda^2}{6}\frac{N+2}{3}(c)\bigg]. \qquad (10.298)
$$

We use the result

$$
I(m^2)=I(m_R^2)+\frac{N+2}{6}\lambda_R I(m_R^2)J(0,m_R^2)+O(\lambda_R^2). \qquad (10.299)
$$

By using the one-loop results we find the renormalized 2−point proper vertex to be given by

$$
\begin{aligned}
(\Gamma_R)^{(2)}_{ij}(p) &= Z\Gamma^{(2)}_{ij}(p)\\
&= \delta_{ij}\bigg[p^2+m_R^2+Z^{(2)}\lambda_R^2p^2+Z_m^{(2)}\lambda_R^2m_R^2+\frac{N+2}{6}Z_g^{(1)}\lambda_R^2 I(m_R^2)+\frac{(N+2)^2}{36}\lambda_R^2 I(m_R^2)J(0,m_R^2)\\
&- \frac{\lambda_R^2}{4}(\frac{N+2}{3})^2(b)_R-\frac{\lambda_R^2}{6}\frac{N+2}{3}(c)_R\bigg]\\
&= \delta_{ij}\bigg[p^2+m_R^2+Z^{(2)}\lambda_R^2p^2+Z_m^{(2)}\lambda_R^2m_R^2+\frac{N+2}{6}Z_g^{(1)}\lambda_R^2 I(m_R^2)-\frac{\lambda_R^2}{6}\frac{N+2}{3}(m_R^{2d-6}I_2\\
&+ p^2m_R^{2d-8}I_3)\bigg]. \qquad (10.300)
\end{aligned}
$$

In the last equation we have used the results $(b)_R=I(m_R^2)J(0,m_R^2)$ and $(c)_R=m_R^{2d-6}I_2+p^2m_R^{2d-8}I_3$. By requiring finiteness of the kinetic term we obtain the result

$$
Z^{(2)}=\frac{N+2}{18}m_R^{2d-8}I_3=-\frac{N+2}{144}m_R^{-2\epsilon}\frac{N_d^2}{\epsilon}. \qquad (10.301)
$$

Cancellation of the remaining divergences gives

$$
\begin{aligned}
Z_m^{(2)} &= -\frac{N+2}{6}Z_g^{(1)}\frac{I(m_R^2)}{m_R^2}+\frac{N+2}{18}m_R^{2d-8}I_2\\
&= \frac{(N+2)(N+8)}{36}m_R^{-2\epsilon}\frac{N_d^2}{\epsilon^2}-\frac{N+2}{12}m_R^{-2\epsilon}\frac{N_d^2}{\epsilon^2}-\frac{N+2}{24}m_R^{-2\epsilon}\frac{N_d^2}{\epsilon}\\
&= \frac{(N+2)(N+5)}{36}m_R^{-2\epsilon}\frac{N_d^2}{\epsilon^2}-\frac{N+2}{24}m_R^{-2\epsilon}\frac{N_d^2}{\epsilon}. \qquad (10.302)
\end{aligned}
$$

Two-Loop Renormalization of The 4−**Point Proper Vertex:** The original expression of the 4−point vertex reads

$$\Gamma^{(4)}_{i_1...i_4}(0,0,0,0) = \frac{\delta_{i_1i_2i_3i_4}}{3}\bigg[\lambda - \frac{3}{2}\frac{N+8}{9}\lambda^2(d) + \frac{3}{2}\lambda^3\frac{(N+2)(N+8)}{27}(g) + \frac{3}{4}\lambda^3\frac{(N+2)(N+4)+12}{27}(e) + 3\lambda^3\frac{5N+22}{27}(f)\bigg]. \quad (10.303)$$

We use the result

$$J(0,m^2) = J(0,m_R^2) + \frac{N+2}{3}\lambda_R I(m_R^2)L(0,m_R^2) + O(\lambda_R^2). \quad (10.304)$$

By using the one-loop results we find the renormalized 4−point proper vertex to be given by

$$\begin{aligned}(\Gamma_R)^{(4)}_{i_1...i_4}(0,0,0,0) &= Z^2\Gamma^{(4)}_{i_1...i_4}(0,0,0,0)\\ &= \frac{\delta_{i_1i_2i_3i_4}}{3}\bigg[\lambda_R + \lambda_R^3 Z_g^{(2)} - \frac{N+8}{3}\lambda_R^3 Z_g^{(1)}(d)_R - \frac{(N+8)(N+2)}{18}\lambda_R^3 I(m_R^2)L(0,m_R^2)\\ &+ \frac{3}{2}\lambda_R^3\frac{(N+2)(N+8)}{27}(g)_R + \frac{3}{4}\lambda_R^3\frac{(N+2)(N+4)+12}{27}(e)_R + 3\lambda_R^3\frac{5N+22}{27}(f)_R\bigg]\\ &= \frac{\delta_{i_1i_2i_3i_4}}{3}\bigg[\lambda_R + \lambda_R^3 Z_g^{(2)} - \frac{N+8}{3}\lambda_R^3 Z_g^{(1)}(d)_R + \frac{3}{4}\lambda_R^3\frac{(N+2)(N+4)+12}{27}(e)_R\\ &+ 3\lambda_R^3\frac{5N+22}{27}(f)_R\bigg].\end{aligned} \quad (10.305)$$

Cancellation of the remaining divergences gives

$$\begin{aligned}Z_g^{(2)} &= \frac{N+8}{3}Z_g^{(1)}(d)_R - \frac{3}{4}\frac{(N+2)(N+4)+12}{27}(e)_R - 3\frac{5N+22}{27}(f)_R\\ &= -\frac{N+8}{3}Z_g^{(1)}(1-\frac{\epsilon}{2})m_R^{-\epsilon}I_1 - \frac{3}{4}\frac{(N+2)(N+4)+12}{27}(1-\frac{\epsilon}{2})^2 m_R^{-2\epsilon}I_1^2 + \frac{5N+22}{27}(1-\epsilon)m_R^{-2\epsilon}I_2\\ &= \frac{(N+8)^2}{18}m_R^{-2\epsilon}(\frac{N_d^2}{\epsilon^2} - \frac{N_d^2}{2\epsilon}) - \frac{N^2+6N+20}{36}m_R^{-2\epsilon}(\frac{N_d^2}{\epsilon^2} - \frac{N_d^2}{\epsilon}) - \frac{5N+22}{18}m_R^{-2\epsilon}(\frac{N_d^2}{\epsilon^2} - \frac{N_d^2}{2\epsilon})\\ &= \frac{(N+8)^2}{36}m_R^{-2\epsilon}\frac{N_d^2}{\epsilon^2} - \frac{5N+22}{36}m_R^{-2\epsilon}\frac{N_d^2}{\epsilon}.\end{aligned} \quad (10.306)$$

10.3.3 Renormalization Functions

The renormalization constants up to two-loop order are given by

$$Z = 1 - g_R^2\bigg(\frac{N+2}{144}\frac{N_d^2}{\epsilon}\bigg). \quad (10.307)$$

$$Z_g = 1 + \frac{N+8}{6}g_R\frac{N_d}{\epsilon} + g_R^2\bigg(\frac{(N+8)^2}{36}\frac{N_d^2}{\epsilon^2} - \frac{5N+22}{36}\frac{N_d^2}{\epsilon}\bigg). \quad (10.308)$$

$$Z_m = 1 + \frac{N+2}{6}g_R\frac{N_d}{\epsilon} + g_R^2\bigg(\frac{(N+2)(N+5)}{36}\frac{N_d^2}{\epsilon^2} - \frac{N+2}{24}\frac{N_d^2}{\epsilon}\bigg). \quad (10.309)$$

The beta function is given by

$$\beta(g_R) = -\frac{\epsilon g_R}{1 + g_R\frac{d}{dg_R}\ln Z_g - 2g_R\frac{d}{dg_R}\ln Z}. \tag{10.310}$$

We compute

$$g_R\frac{d}{dg_R}\ln Z_g = \frac{N+8}{6}g_R\frac{N_d}{\epsilon} + g_R^2\bigg(\frac{(N+8)^2}{36}\frac{N_d^2}{\epsilon^2} - \frac{5N+22}{18}\frac{N_d^2}{\epsilon}\bigg). \tag{10.311}$$

$$\frac{d}{dg_R}\ln Z = -\frac{N+2}{72}g_R\frac{N_d^2}{\epsilon} \Rightarrow -2g_R\frac{d}{dg_R}\ln Z = \frac{N+2}{36}g_R^2\frac{N_d^2}{\epsilon}. \tag{10.312}$$

We get then the fundamental result

$$\beta(g_R) = -\epsilon g_R + \frac{N+8}{6}g_R^2 N_d - \frac{3N+14}{12}g_R^3 N_d^2. \tag{10.313}$$

The second most important renormalization function is η. It is defined by

$$\begin{aligned}\eta(g_R) &= \beta(g_R)\frac{d}{dg_R}\ln Z\\ &= \bigg(-\epsilon g_R + \frac{N+8}{6}g_R^2 N_d - \frac{3N+14}{12}g_R^3 N_d^2\bigg)\bigg(-\frac{N+2}{72}g_R\frac{N_d^2}{\epsilon}\bigg)\\ &= \frac{N+2}{72}g_R^2 N_d^2.\end{aligned} \tag{10.314}$$

The renormalization constant Z_m is associated with the renormalization function γ_m defined by

$$\gamma_m(g_R) = -\beta(g_R)\frac{d}{dg_R}\ln\frac{Z_m}{Z}. \tag{10.315}$$

We compute

$$\frac{d}{dg_R}\ln Z_m = \frac{N+2}{6}\frac{N_d}{\epsilon} + g_R\bigg(\frac{(N+2)(N+8)}{36}\frac{N_d^2}{\epsilon^2} - \frac{N+2}{12}\frac{N_d^2}{\epsilon}\bigg). \tag{10.316}$$

The renormalization function γ_m at the two-loop order is then found to be given by

$$\gamma_m = \frac{N+2}{6}N_d g_R - \frac{5(N+2)}{72}g_R^2 N_d^2. \tag{10.317}$$

From this result we conclude immediately that

$$\eta_2 = -\gamma_m = -\frac{N+2}{6}N_d g_R + \frac{5(N+2)}{72}g_R^2 N_d^2. \tag{10.318}$$

10.4 Critical Exponents

10.4.1 Critical Theory and Fixed Points

We will postulate that quantum scalar field theory, in particular ϕ^4, describes the critical domain of second order phase transitions which includes the critical line $T = T_c$ where the correlation length ξ diverges and the scaling region with T near T_c where the correlation length ξ is large but finite. This is confirmed for example by mean field calculations. From now on we will work in Euclidean signature. We write the action in the form

$$S[\phi] = \beta\mathcal{H}(\phi) = \int d^dx\bigg[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 - \frac{\Lambda^\epsilon g}{4!}(\phi^2)^2\bigg]. \quad (10.319)$$

In the above action the cutoff Λ reflects the original lattice structure, i.e. $\Lambda = 1/a$. The cutoff procedure is irrelevant to the physics and as a consequence we will switch back and forth between cutoff regularization and dimensional regularization as needed. The critical domain is defined by the conditions

$$\begin{aligned} &|m^2 - m_c^2| << \Lambda^2 \\ &\text{momenta} << \Lambda \\ &< \phi(x) > << \Lambda^{\frac{d}{2}-1}. \end{aligned} \quad (10.320)$$

In above m_c^2 is the value of the mass parameter m^2 at the critical temperature T_c where $m_R^2 = 0$ or the correlation length ξ diverges. Clearly m_c^2 is essentially mass renormalization. We will set

$$m^2 = m_c^2 + t\ ,\ t \propto \frac{T - T_c}{T_c}. \quad (10.321)$$

The critical theory should be renormalized at a scale μ in such a way that the renormalized mass remains massless, viz

$$\begin{aligned} &\tilde{\Gamma}_R^{(2)}(p;\mu, g_R, \Lambda)|_{p^2=0} = 0 \\ &\frac{d}{dp^2}\tilde{\Gamma}^{(2)}(p;\mu, g_R, \Lambda)|_{p^2=\mu^2} = 1 \\ &\tilde{\Gamma}^{(4)}(p_1, ..., p_4;\mu, g_R, \Lambda)|_{\rm SP} = \mu^\epsilon g_R. \end{aligned} \quad (10.322)$$

The renormalized proper vertices are given by

$$\tilde{\Gamma}_R^{(n)}(p_i;\mu, g_R) = Z^{n/2}(g, \frac{\Lambda}{\mu})\tilde{\Gamma}^{(n)}(p_i; g, \Lambda). \quad (10.323)$$

The bare proper vertices $\tilde{\Gamma}^{(n)}$ are precisley the proper vertices of statistical mechanics. Now since the renormalized proper vertices $\tilde{\Gamma}_R^{(n)}$ are independent of Λ we should have

$$\bigg(\Lambda\frac{\partial}{\partial\Lambda}Z^{n/2}\tilde{\Gamma}^{(n)}\bigg)_{\mu, g_R} = 0. \quad (10.324)$$

We obtain the renormalization group equation [11]

$$\bigg(\Lambda\frac{\partial}{\partial\Lambda} + \beta\frac{\partial}{\partial g} - \frac{n}{2}\eta\bigg)\tilde{\Gamma}^{(n)} = 0. \quad (10.325)$$

The renormalization functions are now given by

$$\beta(g) = \bigg(\Lambda\frac{\partial g}{\partial\Lambda}\bigg)_{g_R, \mu}. \quad (10.326)$$

[11]Exercise: Show this result.

$$\eta(g) = -\left(\Lambda\frac{\partial}{\partial\Lambda}\ln Z\right)_{g_R,\mu}. \tag{10.327}$$

Clearly the functions β and η can not depend on the ratio Λ/μ since $\tilde{\Gamma}^{(n)}$ does not depend on μ. We state the (almost) obvious theorem

- The renormalization group equation (10.325) is a direct consequence of the existence of a renormalized field theory. Conversely the existence of a solution to this renormalization group equation implies the existence of a renormalized theory.

The fixed point $g = g_*$ and the critical exponent ω: The renormalization group equation (10.325) can be solved using the method of characteristics. We introduce a dilatation parameter λ, a running coupling constant $g(\lambda)$ and an auxiliary renormalization function $Z(\lambda)$ such that

$$\lambda\frac{d}{d\lambda}\left[Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}(p_i; g(\lambda), \lambda\Lambda)\right] = 0. \tag{10.328}$$

We can verify that proper vertices $\tilde{\Gamma}^{(n)}(p_i; g(\lambda), \lambda\Lambda)$ solves the renormalization group equation (10.325) provided that β and η solves the first order differential equations

$$\beta(g(\lambda)) = \lambda\frac{d}{d\lambda}g(\lambda)\ ,\ g(1) = g. \tag{10.329}$$

$$\eta(g(\lambda)) = \lambda\frac{d}{d\lambda}\ln Z(\lambda)\ ,\ Z(1) = Z. \tag{10.330}$$

We have the identification

$$\tilde{\Gamma}^{(n)}(p_i; g, \Lambda) = Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}(p_i; g(\lambda), \lambda\Lambda). \tag{10.331}$$

Equivalently

$$\tilde{\Gamma}^{(n)}(p_i; g, \frac{\Lambda}{\lambda}) = Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}(p_i; g(\lambda), \Lambda). \tag{10.332}$$

The limit $\Lambda \longrightarrow \infty$ is equivalent to the limit $\lambda \longrightarrow 0$. The functions β and η are assumed to be regular functions for $g \geq 0$.

The integration of (10.329) and (10.330) yields the integrated renormalization group equations

$$\ln\lambda = \int_g^{g(\lambda)}\frac{dx}{\beta(x)}. \tag{10.333}$$

$$\ln Z(\lambda) = \int_1^{\lambda}\frac{dx}{x}\eta(g(x)). \tag{10.334}$$

The zeros $g = g_*$ of the beta function β which satisfy $\beta(g_*) = 0$ are of central importance to quantum field theory and critical phenomena. Let us assume that the a zero $g = g_*$ of the beta function does indeed exist. We observe then that any value of the running coupling constant

$g(\lambda)$ near g_* will run into g_* in the limit $\lambda \longrightarrow 0$ regardless of the initial value $g = g(1)$ which can be either above or below g_*. This can be made precise as follows. We expand $\beta(g)$ about the zero as follows

$$\beta(g) = \beta(g_*) + (g - g_*)\omega + ... \tag{10.335}$$

$$\beta(g_*) = 0 \ , \ \omega = \beta^{'}(g_*). \tag{10.336}$$

We compute

$$\ln \lambda = \frac{1}{\omega}\frac{g(\lambda) - g_*}{g - g_*} \Rightarrow g(\lambda) - g_* \sim \lambda^{\omega} \ , \ \lambda \longrightarrow 0. \tag{10.337}$$

If $\omega > 0$ then $g(\lambda) \longrightarrow g_*$ when $\lambda \longrightarrow 0$. The point $g = g_*$ is then called an attractive or stable infrared (since the limit $\lambda \longrightarrow 0$ is equivalent to the massless limit $\lambda\Lambda \longrightarrow 0$) fixed point (since $d^n g(\lambda)/d\lambda^n|_{g_*} = 0$). If $\omega < 0$ then the point $g = g_*$ is called a repulsive infrared fixed point or equivalently a stable ultraviolet fixed point since $g(\lambda) \longrightarrow g_*$ when $\lambda \longrightarrow \infty$.

The slope $\omega = \beta^{'}(g_*)$ is our first critical exponent which controls leading corrections to scaling laws.

As an example let us consider the beta function

$$\beta(g) = -\epsilon g + bg^2 \ , \ b = \frac{3}{16\pi^2}. \tag{10.338}$$

There are in this case two fixed points the origin and $g_* = \epsilon/b$ with critical exponents $\omega = -\epsilon < 0$ (infrared repulsive) and $\omega = +\epsilon$ (infrared attractive) respectively. We compute immediately

$$\ln \lambda = \frac{1}{\epsilon}\int_{1/g}^{1/g(\lambda)} \frac{dx}{x - 1/g_*} \Rightarrow g(\lambda) = \frac{g_*}{1 + \lambda^{\epsilon}(g_*/g - 1)}. \tag{10.339}$$

Since $\epsilon = 4 - d > 0$, $g(\lambda) \longrightarrow g_*$ when $\lambda \longrightarrow 0$ and as a consequence $g_* = \epsilon/b$ is a stable infrared fixed point known as the non trivial (interacting) Wilson-Fisher fixed point. In the limit $\lambda \longrightarrow \infty$ we see that $g(\lambda) \longrightarrow 0$, i.e. the origin is a stable ultraviolet fixed point which is the famous trivial (free) Gaussian fixed point. See figure 15.

The fact that the origin is a repulsive (unstable) infrared fixed point is the source of the strong infrared divergence found in dimensions < 4 since perturbation theory in this case is an expansion around the wrong fixed point. Remark that for $d > 4$ the origin becomes an attractive (stable) infrared fixed point while $g = g_*$ becomes repulsive.

The critical exponent η: Now we solve the second integrated renormalization group equation (10.330). We expand η as $\eta(g(\lambda)) = \eta(g_*) + (g(\lambda) - g_*)\eta^{'}(g_*) + ...$ In the limit $\lambda \longrightarrow 0$ we obtain

$$\ln Z(\lambda) = \eta \ln \lambda + ... \Rightarrow Z(\lambda) = \lambda^{\eta}. \tag{10.340}$$

The critical exponent η is defined by

$$\eta = \eta(g_*). \tag{10.341}$$

The proper vertex (10.332) becomes

$$\tilde{\Gamma}^{(n)}(p_i; g, \frac{\Lambda}{\lambda}) = \lambda^{-\frac{n}{2}\eta}\tilde{\Gamma}^{(n)}(p_i; g(\lambda), \Lambda). \tag{10.342}$$

However from dimensional considerations we know the mass dimension of $\tilde{\Gamma}^{(n)}(p_i; g, \frac{\Lambda}{\lambda})$ to be $M^{d-n(d/2-1)}$ and hence the mass dimension of $\tilde{\Gamma}^{(n)}(\lambda p_i; g, \Lambda)$ is $(\lambda M)^{d-n(d/2-1)}$. We get therefore

$$\tilde{\Gamma}^{(n)}(p_i; g, \frac{\Lambda}{\lambda}) = \lambda^{-d+n(\frac{d}{2}-1)}\tilde{\Gamma}^{(n)}(\lambda p_i; g, \Lambda). \tag{10.343}$$

By combining these last two equations we obtain the crucial result

$$\tilde{\Gamma}^{(n)}(\lambda p_i; g, \Lambda) = \lambda^{-d+\frac{n}{2}(d-2+\eta)}\tilde{\Gamma}^{(n)}(p_i; g_*, \Lambda) \ , \ \lambda \longrightarrow 0. \tag{10.344}$$

The critical proper vertices have a power law behavior for small momenta which is independent of the original value g of the ϕ^4 coupling constant. This in turn is a manifestation of the universality of the critical behavior. The mass dimension of the field ϕ has also changed from the canonical (classical) value $(d-2)/2$ to the anomalous (quantum) value

$$d_\phi = \frac{1}{2}(d-2+\eta). \tag{10.345}$$

In the particular case $n=2$ we have the behavior

$$\tilde{\Gamma}^{(2)}(\lambda p; g, \Lambda) = \lambda^{\eta-2}\tilde{\Gamma}^{(2)}(p; g_*, \Lambda) \ , \ \lambda \longrightarrow 0. \tag{10.346}$$

Hence the $2-$point function must behave as

$$\tilde{G}^{(2)}(p) \sim \frac{1}{p^{2-\eta}} \ , \ p \longrightarrow 0. \tag{10.347}$$

The critical exponent ν: The full renormalized conditions of the massless (critical) theory when $K \neq 0$ are (10.322) plus the two extra conditions

$$\tilde{\Gamma}_R^{(1,2)}(q; p_1, p_2; \mu, g_R, \Lambda)|_{q^2=p_i^2=\mu^2, p_1p_2=-\frac{1}{3}\mu^2} = 1. \tag{10.348}$$

$$\tilde{\Gamma}_R^{(2,0)}(q; -q; \mu, g_R, \Lambda)|_{q^2=\frac{4}{3}\mu^2} = 0. \tag{10.349}$$

The first condition fixes the renormalization constant Z_2 while the second condition provides a renormalization of the $<\phi^2\phi^2>$ correlation function.

The renormalized proper vertices are defined by (with $l+n>2$)

$$\tilde{\Gamma}_R^{(l,n)}(q_i; p_i; \mu, g_R) = Z^{n/2-l}(g, \frac{\Lambda}{\mu})Z_2^l(g, \frac{\Lambda}{\mu})\tilde{\Gamma}^{(l,n)}(q_i; p_i; g, \Lambda). \tag{10.350}$$

We have clearly the condition

$$\left(\Lambda\frac{\partial}{\partial\Lambda}Z^{n/2-l}Z_2^l\tilde{\Gamma}^{(l,n)}\right)_{\mu, g_R} = 0. \tag{10.351}$$

We obtain the renormalization group equation

$$\left(\Lambda\frac{\partial}{\partial\Lambda} + \beta\frac{\partial}{\partial g} - \frac{n}{2}\eta - l\eta_2\right)\tilde{\Gamma}^{(l,n)} = 0. \tag{10.352}$$

The renormalization functions β and η are still given by equations (10.326) and (10.327) while the renormalization function η_2 is defined by

$$\eta_2(g) = -\left(\Lambda\frac{\partial}{\partial\Lambda}\ln\frac{Z_2}{Z}\right)_{g_R,\mu}. \quad (10.353)$$

As before we solve the above renormalization group equation (10.352) by the method of characteristics. We introduce a dilatation parameter λ, a running coupling constant $g(\lambda)$ and auxiliary renormalization functions $Z(\lambda)$ and $\zeta_2(\lambda)$ such that

$$\lambda\frac{d}{d\lambda}\left[Z^{-n/2}(\lambda)\zeta_2^{-l}(\lambda)\tilde{\Gamma}^{(l,n)}(q_i;p_i;g(\lambda),\lambda\Lambda)\right] = 0. \quad (10.354)$$

We can verify that proper vertices $\tilde{\Gamma}^{(l,n)}$ solves the above renormalization group equation (10.352) provided that β, η solve the first order differential equations (10.329) and (10.330) and η_2 solves the first order differential equation

$$\eta_2(g(\lambda)) = \lambda\frac{d}{d\lambda}\ln\zeta_2(\lambda)\ ,\ \zeta_2(1) = \zeta_2. \quad (10.355)$$

We have the identification

$$\tilde{\Gamma}^{(l,n)}(q_i;p_i;g,\Lambda) = Z^{-n/2}(\lambda)\zeta_2^{-l}(\lambda)\tilde{\Gamma}^{(l,n)}(q_i;p_i;g(\lambda),\lambda\Lambda). \quad (10.356)$$

Equivalently

$$\tilde{\Gamma}^{(l,n)}(q_i;p_i;g,\frac{\Lambda}{\lambda}) = Z^{-n/2}(\lambda)\zeta_2^{-l}(\lambda)\tilde{\Gamma}^{(l,n)}(q_i;p_i;g(\lambda),\Lambda). \quad (10.357)$$

The corresponding integrated renormalization group equation is

$$\ln\zeta_2 = \int_1^\lambda\frac{dx}{x}\eta_2(g(x)). \quad (10.358)$$

We obtain in the limit $\lambda\longrightarrow 0$ the behavior

$$\zeta_2(\lambda) = \lambda^{\eta_2}. \quad (10.359)$$

The new critical exponent η_2 is defined by

$$\eta_2 = \eta_2(g_*). \quad (10.360)$$

We introduce the mass critical exponent ν by the relation

$$\nu = \nu(g_*)\ ,\ \nu(g) = \frac{1}{2+\eta_2(g)}. \quad (10.361)$$

We have then the infrared behavior of the proper vertices given by

$$\tilde{\Gamma}^{(l,n)}(q_i;p_i;g,\frac{\Lambda}{\lambda}) = \lambda^{-\frac{n}{2}\eta-l\eta_2}\tilde{\Gamma}^{(l,n)}(q_i;p_i;g(\lambda),\Lambda). \quad (10.362)$$

From dimensional considerations the mass dimension of the proper vertex $\tilde{\Gamma}^{(l,n)}(q_i;p_i;g,\Lambda/\lambda)$ is $M^{d-n(d-2)/2-2l}$ and hence

$$\tilde{\Gamma}^{(l,n)}(q_i;p_i;g,\frac{\Lambda}{\lambda}) = \lambda^{-d+\frac{n}{2}(d-2)+2l}\tilde{\Gamma}^{(l,n)}(\lambda q_i;\lambda p_i;g(\lambda),\Lambda). \quad (10.363)$$

By combining the above two equations we obtain

$$\tilde{\Gamma}^{(l,n)}(\lambda q_i;\lambda p_i;g,\Lambda) = \lambda^{d-\frac{n}{2}(d-2+\eta)-\frac{l}{\nu}}\tilde{\Gamma}^{(l,n)}(q_i;p_i;g_*,\Lambda)\ ,\ \lambda\longrightarrow 0. \quad (10.364)$$

10.4.2 Scaling Domain ($T > T_c$)

In this section we will expand around the critical theory. The proper vertices for $T > T_c$ can be calculated in terms of the critical proper vertices with ϕ^2 insertions.

The correlation length: In order to allow a large but finite correlation length (non zero renormalized mass) in this massless theory without generating infrared divergences we consider the action

$$S[\phi] = \beta\mathcal{H}(\phi) = \int d^dx\bigg[\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}(m_c^2 + K(x))\phi^2 - \frac{\Lambda^\epsilon g}{4!}(\phi^2)^2\bigg]. \quad (10.365)$$

We want to set at the end

$$K(x) = t \propto \frac{T - T_c}{T_c}. \quad (10.366)$$

The $n-$point proper vertices are given by

$$\tilde{\Gamma}^{(n)}(p_i; K, g, \Lambda) = \sum_{l=0}\frac{1}{l!}\int\frac{d^dq_1}{(2\pi)^d}...\int\frac{d^dq_l}{(2\pi)^d}\tilde{\Gamma}^{(l,n)}(q_i; p_i; g, \Lambda)\tilde{K}(q_1)...\tilde{K}(q_l). \quad (10.367)$$

We consider the differential operator

$$D = \Lambda\frac{\partial}{\partial\Lambda} + \beta\frac{\partial}{\partial g} - \frac{n}{2}\eta - \eta_2\int d^dq\tilde{K}(q)\frac{\delta}{\delta\tilde{K}(q)}. \quad (10.368)$$

We compute

$$\begin{aligned}
\int d^dq\tilde{K}(q)\frac{\delta}{\delta\tilde{K}(q)}\tilde{\Gamma}^{(n)}(p_i; K, g, \Lambda) &= \sum_{l=0}\frac{1}{l!}\int d^dq_1...\int d^dq_l\tilde{\Gamma}^{(l,n)}(q_i; p_i; g, \Lambda)\\
&\times \int d^dq\tilde{K}(q)\frac{\delta}{\delta\tilde{K}(q)}\tilde{K}(q_1)...\tilde{K}(q_l)\\
&= \sum_{l=0}\frac{1}{l!}\int d^dq_1...\int d^dq_l(l\tilde{\Gamma}^{(l,n)}(q_i; p_i; g, \Lambda))\tilde{K}(q_1)...\tilde{K}(q_l).
\end{aligned} \quad (10.369)$$

By using now the Callan-Symanzik equation (10.352) we get

$$\bigg(\Lambda\frac{\partial}{\partial\Lambda} + \beta\frac{\partial}{\partial g} - \frac{n}{2}\eta - \eta_2\int d^dq\tilde{K}(q)\frac{\delta}{\delta\tilde{K}(q)}\bigg)\tilde{\Gamma}^{(n)}(p_i; K, g, \Lambda) = 0. \quad (10.370)$$

We now set $K(x) = t$ or equivalently $\tilde{K}(q) = t(2\pi)^d\delta^d(q)$ to obtain

$$\bigg(\Lambda\frac{\partial}{\partial\Lambda} + \beta\frac{\partial}{\partial g} - \frac{n}{2}\eta - \eta_2 t\frac{\partial}{\partial t}\bigg)\tilde{\Gamma}^{(n)}(p_i; t, g, \Lambda) = 0. \quad (10.371)$$

We employ again the method of characteristics in order to solve this renormalization group equation. We introduce a dilatation parameter λ, a running coupling constant $g(\lambda)$, a running mass $t(\lambda)$ and an auxiliary renormalization functions $Z(\lambda)$ such that

$$\lambda\frac{d}{d\lambda}\bigg[Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}(p_i; t(\lambda), g(\lambda), \lambda\Lambda)\bigg] = 0. \quad (10.372)$$

Then $\tilde{\Gamma}^{(n)}(p_i;t(\lambda),g(\lambda),\lambda\Lambda)$ will solve the renormalization group equation (10.371) provided the renormalization functions β, η and η_2 satisfy

$$\beta(g(\lambda))=\lambda\frac{d}{d\lambda}g(\lambda)\ ,\ g(1)=g. \tag{10.373}$$

$$\eta(g(\lambda))=\lambda\frac{d}{d\lambda}\ln Z(\lambda)\ ,\ Z(1)=Z. \tag{10.374}$$

$$\eta_2(g(\lambda))=-\lambda\frac{d}{d\lambda}\ln t(\lambda)\ ,\ t(1)=t. \tag{10.375}$$

The new definition of η_2 given in the last equation is very similar to the definition of γ_m given in equation (10.246). We make the identification

$$\tilde{\Gamma}^{(n)}(p_i;t,g,\Lambda)=Z^{-n/2}(\lambda)\tilde{\Gamma}^{(n)}(p_i;t(\lambda),g(\lambda),\lambda\Lambda). \tag{10.376}$$

From dimensional considerations we have

$$\tilde{\Gamma}^{(n)}(p_i;t(\lambda),g(\lambda),\lambda\Lambda)=(\lambda\Lambda)^{d-\frac{n}{2}(d-2)}\tilde{\Gamma}^{(n)}(\frac{p_i}{\lambda\Lambda};\frac{t(\lambda)}{\lambda^2\Lambda^2},g(\lambda),1). \tag{10.377}$$

Thus

$$\tilde{\Gamma}^{(n)}(p_i;t,g,\Lambda)=Z^{-n/2}(\lambda)m^{d-\frac{n}{2}(d-2)}\tilde{\Gamma}^{(n)}(\frac{p_i}{m};\frac{t(\lambda)}{m^2},g(\lambda),1). \tag{10.378}$$

We have used the notation $m=\lambda\Lambda$. We use the freedom of choice of λ to choose

$$t(\lambda)=m^2=\lambda^2\Lambda^2. \tag{10.379}$$

The theory at scale λ is therefore not critical since the critical regime is defined by the requirement $t<<\Lambda^2$.

The integrated form of the renormalization group equation (10.375) is given by

$$t(\lambda)=t\exp-\int_1^{\lambda}\frac{dx}{x}\eta_2(g(x)). \tag{10.380}$$

This can be rewritten as

$$\ln\frac{t\lambda^2}{t(\lambda)}=\int_1^{\lambda}\frac{dx}{x}\frac{1}{\nu(g(x))}. \tag{10.381}$$

Equivalently

$$\ln\frac{t}{\Lambda^2}=\int_1^{\lambda}\frac{dx}{x}\frac{1}{\nu(g(x))}. \tag{10.382}$$

This is an equation for λ. In the critical regime $\ln t/\Lambda^2\longrightarrow-\infty$. For $\nu(g)>0$ this means that $\lambda\longrightarrow 0$ and hence $g(\lambda)\longrightarrow g_*$. By expanding around the fixed point $g(\lambda)=g_*$ we obtain in the limit $\lambda\longrightarrow 0$ the result

$$\lambda=\big(\frac{t}{\Lambda^2}\big)^{\nu}. \tag{10.383}$$

By using this result in (10.378) we conclude that the proper vertices $\tilde{\Gamma}^{(n)}(p_i;t,g,\Lambda=1)$ must have the infrared ($\lambda\longrightarrow 0$) scaling

$$\tilde{\Gamma}^{(n)}(p_i;t,g,\Lambda=1)=m^{d-\frac{n}{2}(d-2+\eta)}\tilde{\Gamma}^{(n)}(\frac{p_i}{m};1,g_*,1)\ ,\ t<<1\ ,\ p<<1. \tag{10.384}$$

The mass m is thus the physical mass ξ^{-1} where ξ is the correlation length. We have then

$$m(\Lambda=1)=\xi^{-1}=t^{\nu}. \tag{10.385}$$

Clearly $\xi\longrightarrow\infty$ when $t\longrightarrow 0$ or equivalently $T\longrightarrow T_c$ since $\nu>0$.

The critical exponents α and γ: At zero momentum the above proper vertices are finite because of the non zero mass t. They have the infrared scaling

$$\begin{aligned}\tilde{\Gamma}^{(n)}(0;t,g,\Lambda=1) &= m^{d-\frac{n}{2}(d-2+\eta)}\\ &= t^{\nu(d-\frac{n}{2}(d-2+\eta))}\ ,\ t<<1\ ,\ p<<1.\end{aligned} \tag{10.386}$$

The case $n=2$ is of particular interest since it is related to the inverse susceptibility, viz

$$\begin{aligned}\chi^{-1} &= \tilde{\Gamma}^{(2)}(0;t,g,\Lambda=1)\\ &= t^{\gamma}.\end{aligned} \tag{10.387}$$

The critical exponent γ is given by

$$\gamma=\nu(2-\eta). \tag{10.388}$$

The obvious generalization of the renormalization group equation (10.371) is

$$\bigg(\Lambda\frac{\partial}{\partial\Lambda}+\beta\frac{\partial}{\partial g}-\frac{n}{2}\eta-\eta_2(l+t\frac{\partial}{\partial t})\bigg)\tilde{\Gamma}^{(l,n)}(q_i;p_i;t,g,\Lambda)=0. \tag{10.389}$$

This is valid for all $n+l>2$. The case $l=2$, $n=0$ is special because of the non multiplicative nature of the renormalization required in this case and as a consequence the corresponding renormalization group equation will be inhomogeneous. However we will not pay attention to this difference since the above renormalization group equation is sufficient to reproduce the leading infrared behavior, and as a consequence the relevant critical exponent, of the proper vertex with $l=2$ and $n=0$.

We find after some calculation, similar to the calculation used for the case $l=0$, the leading infrared ($\lambda\longrightarrow 0$) behavior

$$\tilde{\Gamma}^{(l,n)}(q_i;p_i;t,g,\Lambda=1)=m^{-\frac{l}{\nu}+d-\frac{n}{2}(d-2+\eta)}\tilde{\Gamma}^{(l,n)}(\frac{q_i}{m};\frac{p_i}{m};1,g_*,1)\ ,\ t<<1\ ,\ q,p<<1. \tag{10.390}$$

By applying this formula naively to the case $l=2$, $n=0$ we get the desired leading infrared behavior of $\tilde{\Gamma}^{(2,0)}$ which corresponds to the most infrared singular part of the energy-energy correlation function. We obtain

$$\tilde{\Gamma}^{(2,0)}(q;t,g,\Lambda=1)=m^{-\frac{2}{\nu}+d}\tilde{\Gamma}^{(2,0)}(\frac{q}{m};1,g_*,1)\ ,\ t<<1\ ,\ q<<1. \tag{10.391}$$

By substituting $\tilde{K}(q)=t(2\pi)^d\delta^d(q)$ in (10.367) we obtain

$$\tilde{\Gamma}^{(n)}(p_i;K,g,\Lambda)=\sum_{l=0}\frac{t^l}{l!}\tilde{\Gamma}^{(l,n)}(0;p_i;g,\Lambda). \tag{10.392}$$

Hence

$$\frac{\partial^2\Gamma(K,g,\Lambda)}{\partial t^2}|_{t=0} = \tilde{\Gamma}^{(2,0)}(0;g,\Lambda). \quad (10.393)$$

In other words $\tilde{\Gamma}^{(2,0)}$ at zero momentum is the specific heat since t is the temperature and Γ is the thermodynamic energy (effective action). The infrared behavior of the specific heat is therefore given by

$$\begin{aligned} C_v &= \tilde{\Gamma}^{(2,0)}(0;t,g,\Lambda=1) \\ &= t^{-\alpha}\tilde{\Gamma}^{(2,0)}(0;1,g_*,1)\ ,\ t<<1\ ,\ q<<1. \end{aligned} \quad (10.394)$$

The new critical exponent α is defined by

$$\alpha = 2-\nu d. \quad (10.395)$$

10.4.3 Scaling Below T_c

In order to describe in a continuous way the ordered phase corresponding to $T<T_c$ starting from the disordered phase ($T>T_c$) we introduce a magnetic field B, i.e. a source $J=B$. The corresponding magnetization M is precisely the classical field $\phi_c = <\phi(x)>$, viz

$$M(x) = <\phi(x)>. \quad (10.396)$$

The Helmholtz free energy (vacuum energy) will depend on the magnetic field B, viz $W = W(B) = -\ln Z(B)$. We know that the magnetization and the magnetic field are conjugate variables, i.e. $M(x) = \partial W(B)/\partial B(x)$. The Gibbs free energy or thermodynamic energy (effective action) is the Legendre transform of $W(B)$, viz $\Gamma(M) = \int d^dx M(x)B(x) - W(B)$. We compute then

$$B(x) = \frac{\partial\Gamma(M)}{\partial M(x)}. \quad (10.397)$$

The effective action can be expanded as

$$\Gamma[M,t,g,\Lambda] = \sum_{n=0}\frac{1}{n!}\int\frac{d^dp_1}{(2\pi)^d}...\int\frac{d^dp_n}{(2\pi)^d}\tilde{\Gamma}^{(n)}(p_i;t,g,\Lambda)M(p_1)...M(p_n). \quad (10.398)$$

Thus

$$B(p) = \sum_{n=0}\frac{1}{n!}\int\frac{d^dp_1}{(2\pi)^d}...\int\frac{d^dp_n}{(2\pi)^d}\tilde{\Gamma}^{(n+1)}(p_i,p;t,g,\Lambda)M(p_1)...M(p_n). \quad (10.399)$$

By assuming that the magnetization is uniform we obtain

$$\Gamma[M,t,g,\Lambda] = \sum_{n=0}\frac{M^n}{n!}\tilde{\Gamma}^{(n)}(p_i=0;t,g,\Lambda). \quad (10.400)$$

$$B[M,t,g,\Lambda] = \sum_{n=0}\frac{M^n}{n!}\tilde{\Gamma}^{(n+1)}(p_i=0;t,g,\Lambda). \quad (10.401)$$

By employing the renormalization group equation (10.371) we get

$$\Big(\Lambda\frac{\partial}{\partial\Lambda}+\beta\frac{\partial}{\partial g}-\frac{n+1}{2}\eta-\eta_2 t\frac{\partial}{\partial t}\Big)B = \sum_{n=0}\frac{M^n}{n!}\Big(\Lambda\frac{\partial}{\partial\Lambda}+\beta\frac{\partial}{\partial g}-\frac{n+1}{2}\eta-\eta_2 t\frac{\partial}{\partial t}\Big)\tilde{\Gamma}^{(n+1)} = 0. \quad (10.402)$$

Clearly

$$M\frac{\partial}{\partial M}B = \sum_{n=0} n\frac{M^n}{n!}\tilde{\Gamma}^{(n+1)}(p_i=0;t,g,\Lambda). \quad (10.403)$$

Hence the magnetic field obeys the renormalization group equation

$$\Big(\Lambda\frac{\partial}{\partial\Lambda}+\beta\frac{\partial}{\partial g}-\frac{1}{2}(1+M\frac{\partial}{\partial M})\eta-\eta_2 t\frac{\partial}{\partial t}\Big)B = 0. \quad (10.404)$$

By using the method of characteristics we introduce as before a running coupling constant $g(\lambda)$, a running mass $t(\lambda)$ and an auxiliary renormalization functions $Z(\lambda)$ such as equations (10.373), (10.374) and (10.375) are satisfied. However in this case we need also to introduce a running magnetization $M(\lambda)$ such that

$$\lambda\frac{d}{d\lambda}\ln M(\lambda) = -\frac{1}{2}\eta[g(\lambda)]. \quad (10.405)$$

By comparing (10.374) and (10.405) we obtain

$$M(\lambda) = MZ^{-\frac{1}{2}}(\lambda) \quad (10.406)$$

We must impose

$$\lambda\frac{d}{d\lambda}\Big[Z^{-1/2}(\lambda)B(M(\lambda),t(\lambda),g(\lambda),\lambda\Lambda)\Big] = 0. \quad (10.407)$$

In other words we make the identification

$$B(M,t,g,\Lambda) = Z^{-1/2}(\lambda)B(M(\lambda),t(\lambda),g(\lambda),\lambda\Lambda). \quad (10.408)$$

From dimensional analysis we know that $[\tilde{\Gamma}^{(n)}] = M^{d-n(d-2)/2}$ and $[M] = M^{(d-2)/2} = M^{1-\epsilon/2}$ and hence $[B] = M^{(d+2)/2} = M^{3-\epsilon/2}$. Hence

$$B(M,t,g,\Lambda) = \Lambda^{3-\epsilon/2}B(\frac{M}{\Lambda^{1-\epsilon/2}},\frac{t}{\Lambda^2},g,1). \quad (10.409)$$

By combining the above two equations we get

$$B(M,t,g,\Lambda) = Z^{-1/2}(\lambda)(\lambda\Lambda)^{3-\epsilon/2}B(\frac{M(\lambda)}{(\lambda\Lambda)^{1-\epsilon/2}},\frac{t(\lambda)}{\lambda^2\Lambda^2},g(\lambda),1). \quad (10.410)$$

Again we use the arbitrariness of λ to make the theory non critical and as a consequence avoid infrared divergences. We choose λ such that

$$\frac{M(\lambda)}{(\lambda\Lambda)^{1-\epsilon/2}} = 1. \quad (10.411)$$

The solution of equation (10.405) then reads

$$\ln\frac{M(\lambda)}{M}=-\frac{1}{2}\int_1^{\lambda}\frac{dx}{x}\eta(g(x))\Rightarrow\ln\frac{M}{\Lambda^{1-\epsilon/2}}=\frac{1}{2}\int_1^{\lambda}\frac{dx}{x}[d-2+\eta(g(x))].\tag{10.412}$$

The critical domain is defined obviously by $M<<\Lambda^{1-\epsilon/2}$. For $d-2+\eta$ positive we conclude that λ must be small and thus $g(\lambda)$ is close to the fixed point g_*. This equation then leads to the infrared behavior

$$\frac{M}{\Lambda^{1-\epsilon/2}}=\lambda^{\frac{d-2+\eta}{2}}.\tag{10.413}$$

From equation (10.381) we get the infrared behavior

$$\frac{t(\lambda)}{t\lambda^2}=\lambda^{-\frac{1}{\nu}}.\tag{10.414}$$

We know also the infrared behavior

$$Z(\lambda)=\lambda^{\eta}.\tag{10.415}$$

The infrared behavior of equation (10.410) is therefore given by

$$B(M,t,g,\Lambda)=\lambda^{\frac{2+d-\eta}{2}}\Lambda^{3-\epsilon/2}B(1,\frac{t}{\Lambda^2}\lambda^{-1/\nu},g_*,1).\tag{10.416}$$

This can also be rewritten as

$$B(M,t,g,1)=M^{\delta}f(tM^{-\frac{1}{\beta}}).\tag{10.417}$$

This is the equation of state. The two new critical exponents β and δ are defined by

$$\beta=\frac{\nu}{2}(d-2+\eta).\tag{10.418}$$

$$\delta=\frac{d+2-\eta}{d-2+\eta}.\tag{10.419}$$

From equations (10.413) and (10.414) we observe that

$$M=t^{\beta}(\frac{\lambda^2}{t(\lambda)})^{\beta}.\tag{10.420}$$

For negative t ($T<T_c$) the appearance of a spontaneous magnetization $M\neq 0$ at $B=0$ means that the function $f(x)$, where $x=tM^{-1/\beta}$, admits a negative zero x_0. Indeed the condition $B=0$, $M\neq 0$ around $x=x_0$ reads explicitly

$$0=f(x_0)+(x-x_0)f^{'}(x_0)+..\tag{10.421}$$

This is equivalent to

$$M=|x_0|^{-\beta}(-t)^{\beta}.\tag{10.422}$$

We state without proof that correlation functions below T_c have the same scaling behavior as above T_c. In particular the critical exponents ν, γ and α below T_c are the same as those defined earlier above T_c. We only remark that in the presence of a magnetic field B we have two mass scales t^{ν} (as before) and $m=M^{\nu/\beta}$ where M is the magnetization which is the correct choice in this phase. In the limit $B\longrightarrow 0$ (with $T<T_c$) the magnetization becomes spontaneous and m becomes the physical mass

10.4.4 Critical Exponents from 2−Loop and Comparison with Experiment

The most important critical exponents are the mass critical exponent ν and the anomalous dimension η. As we have shown these two critical exponents define the infrared behavior of proper vertices. At $T = T_c$ we find the scaling

$$\tilde{\Gamma}^{(l,n)}(\lambda q_i; \lambda p_i; g, \Lambda) = \lambda^{d-\frac{n}{2}(d-2+\eta)-\frac{l}{\nu}}\tilde{\Gamma}^{(l,n)}(q_i; p_i; g_*, \Lambda) \ , \ \lambda \longrightarrow 0. \tag{10.423}$$

The critical exponent η provides the quantum mass dimension of the field operator, viz

$$[\phi] = M^{d_\phi} \ , \ d_\phi = \frac{1}{2}(d-2+\eta). \tag{10.424}$$

The scaling of the wave function renormalization is also determined by the anomalous dimension, viz

$$Z(\lambda) \simeq \lambda^\eta. \tag{10.425}$$

The 2−point function at $T = T_c$ behaves therefore as

$$G^{(2)}(p) = \frac{1}{p^{2-\eta}} \Leftrightarrow G^{(2)}(r) = \frac{1}{r^{d-2+\eta}}. \tag{10.426}$$

The critical exponent ν determines the scaling behavior of the correlation length. For $T > T_c$ we find the scaling

$$\tilde{\Gamma}^{(n)}(p_i; t, g, \Lambda = 1) = m^{d-\frac{n}{2}(d-2+\eta)}F^{(n)}(\frac{p_i}{m}) \ , \ t = \frac{T-T_c}{T_c} << 1 \ , \ p_i << 1. \tag{10.427}$$

The mass m is proportional to the mass scale t^ν. From this equation we see that m is the physical mass ξ^{-1} where ξ is the correlation length ξ. We have then

$$m = \xi^{-1} \sim t^\nu. \tag{10.428}$$

The 2−point function for $T > T_c$ behaves therefore as [12]

$$G^{(2)}(r) = \frac{1}{r^{d-2+\eta}}\exp(-r/\xi). \tag{10.429}$$

The scaling behavior of correlation functions for $T < T_c$ is the same as for $T > T_c$ except that there exists a non zero spontaneous magnetization M in this regime which sets an extra mass scale given by $M^{1/\beta}$ besides t^ν. The exponent β is another critical exponent associated with the magnetization M given by the scaling law

$$\beta = \frac{\nu}{2}(d-2+\eta). \tag{10.430}$$

In other words for T close to T_c from below we must have

$$M \sim (-t)^\beta \tag{10.431}$$

For $T < T_c$ the physical mass m is given by

$$m = \xi^{-1} \sim M^{\nu/\beta} \sim (-t)^\nu. \tag{10.432}$$

[12]Exercise: Give an explicit proof.

There are three more critical exponents α (associated with the specific heat), γ (associated with the susceptibility) and δ (associated with the equation of state) which are not independent but given by the scaling laws

$$\alpha = 2 - \nu d. \tag{10.433}$$

$$\gamma = \nu(2-\eta). \tag{10.434}$$

$$\delta = \frac{d+2-\eta}{d-2+\eta}. \tag{10.435}$$

The last critical exponent of interest is ω which is given by the slope of the beta function at the fixed point and measures the approach to scaling.

The beta function at two-loop order of the $O(N)$ sigma model is given by

$$\beta(g_R) = -\epsilon g_R + \frac{N+8}{6}g_R^2 N_d - \frac{3N+14}{12}g_R^3 N_d^2. \tag{10.436}$$

The fixed point g_* is defined by

$$\beta(g_{R*}) = 0 \Rightarrow \frac{3N+14}{12}g_{R*}^2 N_d^2 - \frac{N+8}{6}g_{R*}N_d + \epsilon = 0. \tag{10.437}$$

The solution must be of the form

$$g_{R*}N_d = a\epsilon + b\epsilon^2 + ... \tag{10.438}$$

We find the solution

$$a = \frac{6}{N+8}\ ,\ b = \frac{18(3N+14)}{(N+8)^3}. \tag{10.439}$$

Thus

$$g_{R*}N_d = \frac{6}{N+8}\epsilon + \frac{18(3N+14)}{(N+8)^3}\epsilon^2 + ... \tag{10.440}$$

The critical exponent ω is given by

$$\begin{aligned} \omega &= \beta^{'}(g_{R*}) \\ &= \epsilon - \frac{3(3N+14)}{(N+8)^2}\epsilon^2 + ... \end{aligned} \tag{10.441}$$

The critical exponent η is given by

$$\eta = \eta(g_{R*}). \tag{10.442}$$

The renormalization function $\eta(g)$ is given by

$$\eta(g_R) = \frac{N+2}{72}g_R^2 N_d^2. \tag{10.443}$$

We substitute now the value of the fixed point. We obtain immediately

$$\eta = \frac{N+2}{2(N+8)^2}\epsilon^2. \tag{10.444}$$

The critical exponent ν is given by

$$\nu = \frac{1}{2+\eta_2}. \tag{10.445}$$

$$\nu = \nu(g_{R*}) \ , \ \eta_2 = \eta_2(g_{R*}). \tag{10.446}$$

The renormalization function $\eta_2(g)$ is given by

$$\eta_2 = -\frac{N+2}{6}N_d g_R + \frac{5(N+2)}{72}g_R^2 N_d^2. \tag{10.447}$$

By substituting the value of the fixed point we compute immediately

$$\eta_2 = -\frac{N+2}{N+8}\epsilon - \frac{(N+2)(13N+44)}{2(N+8)^3}\epsilon^2 + ... \tag{10.448}$$

$$\nu = \frac{1}{2} + \frac{N+2}{4(N+8)}\epsilon + \frac{(N+2)(N^2+23N+60)}{8(N+8)^3}\epsilon^2 + ... \tag{10.449}$$

All critical exponents can be determined in terms of ν and η. They only depend on the dimension of space d and on the dimension of the symmetry space N which is precisely the statement of universality. The epsilon expansion is divergent for all ϵ and as a consequence a resummation is required before we can coherently compare with experiments. This is a technical exercise which we will not delve into here and content ourselves by using what we have already established and also by quoting some results.

The most important predictions (in our view) correspond to $d = 3$ ($\epsilon = 1$) and $N = 1, 2, 3$.

- The case $N = 1$ describes Ising-like systems such as the liquid-vapor transitions in classical fluids. Experimentally we observe

$$\begin{aligned}\nu &= 0.625 \pm 0.006\\ \gamma &= 1.23 - 1.25.\end{aligned} \tag{10.450}$$

  The theoretical calculation gives

$$\begin{aligned}\nu &= \frac{1}{2} + \frac{1}{12} + \frac{7}{162} + ... = \frac{203}{324} + ... = 0.6265 \pm 0.0432\\ \eta &= \frac{1}{54} + ... = 0.019 \Leftrightarrow \gamma = \nu(2-\eta) = 1.241.\end{aligned} \tag{10.451}$$

  The agreement for ν and η up to order ϵ^2 is very reasonable and is a consequence of the asymptotic convergence of the ϵ series. The error is estimated by the last term available in the epsilon expansion.

- The case $N=2$ corresponds to the Helium superfluid transition. This system allows precise measurement near T_c of ν and α given by

$$\nu = 0.672 \pm 0.001$$
$$\alpha = -0.013 \pm 0.003. \quad (10.452)$$

The theoretical calculation gives

$$\nu = \frac{1}{2} + \frac{1}{10} + \frac{11}{200} + ... = \frac{131}{200} + ... = 0.6550 \pm 0.0550$$
$$\alpha = 2 - \nu d = -0.035. \quad (10.453)$$

Here the agreement up to order ϵ^2 is not very good. After proper resummation of the ϵ expansion we find excellent agreement with the experimental values. We quote the improved theoretical predictions

$$\nu = 0.664 - 0.671$$
$$\alpha = -(0.008 - 0.013). \quad (10.454)$$

- The case $N=3$ corresponds to magnetic systems. The experimental values are

$$\nu = 0.7 - 0.725$$
$$\gamma = 1.36 - 1.42. \quad (10.455)$$

The theoretical calculation gives

$$\nu = \frac{1}{2} + \frac{5}{44} + \frac{345}{5324} + ... = \frac{903}{1331} + ... = 0.6874 \pm 0.0648$$
$$\eta = \frac{5}{242} = 0.021 \Leftrightarrow \gamma = 1.36. \quad (10.456)$$

There is a very good agreement.

10.5 The Wilson Approximate Recursion Formulas

10.5.1 Kadanoff-Wilson Phase Space Analysis

We start by describing a particular phase space cell decomposition due to Wilson which is largely motivated by Kadanoff block spins.

We assume a hard cutoff 2Λ. Thus if $\phi(x)$ is the field (spin) variable and $\tilde{\phi}(k)$ is its Fourier transform we will assume that $\tilde{\phi}(k)$ is zero for $k > 2\Lambda$.

We expand the field as

$$\phi(x) = \sum_{\vec{m}} \sum_{l=0}^{\infty} \psi_{\vec{m}l}(x)\phi_{\vec{m}l}. \quad (10.457)$$

The wave functions $\psi_{\vec{m}l}(x)$ satisfy the orthonormality condition

$$\int d^d x \psi^*_{\vec{m}_1 l_1}(x)\psi_{\vec{m}_2 l_2}(x) = \delta_{\vec{m}_1\vec{m}_2}\delta_{l_1 l_2}. \quad (10.458)$$

The Fourier transform $\tilde{\psi}_{\vec{m}l}(k)$ is defined by

$$\tilde{\psi}_{\vec{m}l}(k) = \int d^dx \psi_{\vec{m}l}(x)\ e^{ikx}. \tag{10.459}$$

The interpretation of l and $\vec{m}$ is as follows. We decompose momentum space into thin spherical shells, i.e. logarithmically as

$$\frac{1}{2^l} \leq \frac{|k|}{\Lambda} \leq \frac{1}{2^{l-1}}. \tag{10.460}$$

The functions $\tilde{\psi}_{\vec{m}l}(k)$ for a fixed l are non zero only inside the shell l, i.e. for $1/2^l \leq |k|/\Lambda \leq 1/2^{l-1}$. We will assume furthermore that the functions $\tilde{\psi}_{\vec{m}l}(k)$ are constant within its shell and satisfy the normalization condition

$$\int \frac{d^dk}{(2\pi)^d} |\tilde{\psi}_{\vec{m}l}(k)|^2 = 1. \tag{10.461}$$

The functions $\tilde{\psi}_{\vec{m}l}(k)$ and $\psi_{\vec{m}l}(x)$ for a fixed l and a fixed $\vec{m}$ should be thought of as minimal wave packets, i.e. if Δk is the width of $\tilde{\psi}_{\vec{m}l}(k)$ and Δx is the width of $\psi_{\vec{m}l}(x)$ then one must have by the uncertainty principle the requirement $\Delta x \Delta k = (2\pi)^d$. Thus for each shell we divide position space into blocks of equal size each with volume inversely proportional to the volume of the corresponding shell. The volume of the lth momentum shell is proportional to R^d where $R = 1/2^l$, viz $\Delta k = (2\pi)^d 2^{-ld} w$ where w is a constant. Hence the volume of the corresponding position space box is $\Delta x = 2^{ld} w^{-1}$. The functions $\psi_{\vec{m}l}(x)$ are non zero (constant) only inside this box by construction. This position space box is characterized by the index $\vec{m}$ as is obvious from the normalization condition

$$\int d^dx |\psi_{\vec{m}l}(x)|^2 = \int_{\vec{x}\in \text{box } \vec{m}} d^dx |\psi_{\vec{m}l}(x)|^2 = 1. \tag{10.462}$$

In other words

$$\int d^dx = \sum_{\vec{m}} \int_{\vec{x}\in \text{box } \vec{m}} d^dx. \tag{10.463}$$

The normalization conditions in momentum and position spaces lead to the relations

$$|\tilde{\psi}_{\vec{m}l}(k)| = 2^{ld/2} w^{-1/2}. \tag{10.464}$$

$$|\psi_{\vec{m}l}(x)| = 2^{-ld/2} w^{1/2}. \tag{10.465}$$

Obviously $|\tilde{\psi}_{\vec{m}l}(0)| = 0$. Thus $\int d^dx \psi_{\vec{m}l}(x) = \tilde{\psi}_{\vec{m}l}(0) = 0$ and as a consequence we will assume that $\psi_{\vec{m}l}(x)$ is equal to $+2^{-ld/2} w^{1/2}$ in one half of the box and $-2^{-ld/2} w^{1/2}$ in the other half.

The meaning of the index $\vec{m}$ which labels the position space boxes can be clarified further by the following argument. By an appropriate scale transformation in momentum space we can scale the momenta such that the lth shell becomes the largest shell $l = 0$. Clearly the correct scale transformation is $k \longrightarrow 2^l k$ since $1 \leq |2^l k|/\Lambda \leq 2$. This corresponds to a scale transformation in position space of the form $x \longrightarrow x/2^l$. We obtain therefore the relation $\psi_{\vec{m}l}(x) = \psi_{\vec{m}0}(x/2^l)$. Next we perform an appropriate translation in position space to bring the box $\vec{m}$ to the box $\vec{0}$. This is clearly given by the translation $\vec{x} \longrightarrow \vec{x} - a_0 \vec{m}$. We obtain therefore the relation

$$\psi_{\vec{m}l}(x) = \psi_{\vec{0}0}(x/2^l - a_0 \vec{m}). \tag{10.466}$$

The functions $\tilde{\psi}_{\vec{m}l}(k)$ and $\psi_{\vec{m}l}(x)$ correspond to a single degree of freedom in phase space occupying a volume $(2\pi)^d$, i.e. a single cell in phase space is characterized by l and $\vec{m}$. Each momentum shell l corresponds to a lattice in the position space with a lattice spacing given by

$$a_l=(\Delta x)^{1/d}=2^l a_0\ ,\ a_0=w^{-1/d}. \tag{10.467}$$

The largest shell $l=0$ correspond to a lattice spacing a_0 and each time l is increased by 1 the lattice spacing gets doubled which is the original spin blocking idea of Kadanoff.

We are interested in integrating out only the $l=0$ modes. We write then

$$\phi(x)=\sum_{\vec{m}}\psi_{\vec{m}0}(x)\phi_{\vec{m}0}+\phi_1(x). \tag{10.468}$$

$$\phi_1(x)=\sum_{\vec{m}}\sum_{l=1}^{\infty}\psi_{\vec{m}l}(x)\phi_{\vec{m}l}. \tag{10.469}$$

From the normalization (10.465) and the scaling law (10.466) we have

$$\psi_{\vec{m}l}(x)=2^{-d/2}\psi_{\vec{m}l-1}(x/2). \tag{10.470}$$

We define $\phi^{'}(x/2)$ by

$$\phi_1(x)=2^{-d/2}\alpha_0\phi^{'}(x/2). \tag{10.471}$$

In other words

$$\phi^{'}(x/2)=\sum_{\vec{m}}\sum_{l=1}^{\infty}\psi_{\vec{m}l-1}(x/2)\phi^{'}_{\vec{m}l-1}\ ,\ \phi^{'}_{\vec{m}l-1}=\alpha_0^{-1}\phi_{\vec{m}l}. \tag{10.472}$$

We have then

$$\phi(x)=\sum_{\vec{m}}\psi_{\vec{m}0}(x)\phi_{\vec{m}0}+2^{-d/2}\alpha_0\phi^{'}(x/2). \tag{10.473}$$

10.5.2 Recursion Formulas

We will be interested in actions of the form

$$S_0(\phi(x))=\frac{K}{2}\int d^dxR_0(\phi(x))\partial_\mu\phi(x)\partial^\mu\phi(x)+\int d^dxP_0(\phi(x)). \tag{10.474}$$

We will assume that P_0 and R_0 are even polynomials of the field and that $dR_0/d\phi$ is much smaller than P_0 for all relevant configurations. The partition function is

$$\begin{aligned}Z_0&=\int\mathcal{D}\phi(x)\ e^{-S_0(\phi(x))}\\&=\int\mathcal{D}\phi^{'}(x/2)\int\prod_{\vec{m}}d\phi_{\vec{m}0}\ e^{-S_0(\phi(x))}.\end{aligned} \tag{10.475}$$

The degrees of freedom contained in the fluctuation $\phi^{'}(x/2)$ correspond to momentum shells $l\geq 1$ and thus correspond to position space wave packets larger than the box $\vec{m}$ by at least a factor of 2. We can thus assume that $\phi^{'}(x/2)$ is almost constant over the box $\vec{m}$. If x_0 is the center of the box $\vec{m}$ we can expand $\phi^{'}(x/2)$ as

$$\phi^{'}(x/2) = \phi^{'}(x_0/2) + (x-x_0)^{\mu}\partial_{\mu}\phi^{'}(x_0/2) + \frac{1}{2}(x-x_0)^{\mu}(x-x_0)^{\nu}\partial_{\mu}\partial_{\nu}\phi^{'}(x_0/2) + ... \tag{10.476}$$

The partition function becomes

$$Z_0 = \int \mathcal{D}\phi^{'}(x_0/2)\int\prod_{\vec{m}} d\phi_{\vec{m}0}\ e^{-S_0(\phi(x))}. \tag{10.477}$$

The Kinetic Term: We compute

$$\begin{aligned}\int d^dxR_0\partial_{\mu}\phi(x)\partial^{\mu}\phi(x) &= \int d^dxR_0\Big[\sum_{\vec{m},\vec{m}^{'}}\partial_{\mu}\psi_{\vec{m}0}(x)\partial^{\mu}\psi_{\vec{m}^{'}0}(x).\phi_{\vec{m}0}\phi_{\vec{m}^{'}0}\\ &+ 2^{1-d/2}\alpha_0\sum_{\vec{m}}\phi_{\vec{m}0}\partial_{\mu}\psi_{\vec{m}0}(x)\partial_{\mu}\phi^{'}(x/2) + 2^{-d}\alpha_0^2\partial_{\mu}\phi^{'}(x/2)\partial^{\mu}\phi^{'}(x/2)\Big].\end{aligned} \tag{10.478}$$

The integral $\int d^dx$ is $\sum_{\vec{m}}\int_{\vec{x}\in \text{box }\vec{m}}$. In the box $\vec{m}$ the function $R_0(\phi(x))$ can be replaced by the function $R_0(\psi_{\vec{m}0}(x)\phi_{\vec{m}0} + 2^{-d/2}\alpha_0\phi^{'}(x/2))$. In this box $\psi_{\vec{m}0}(x)$ is approximated by $+w^{1/2}$ in one half of the box and by $-w^{1/2}$ in the other half, i.e. by a step function. Thus the third term in the above equation can be approximated by (dropping also higher derivative corrections and defining $u_{\vec{m}} = 2^{-d/2}\alpha_0\phi^{'}(x_0/2)$)

$$2^{-d-1}\alpha_0^2w^{-1}\sum_{\vec{m}}\Big[R_0(w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}) + R_0(-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}})\Big]\partial_{\mu}\phi^{'}(x_0/2)\partial^{\mu}\phi^{'}(x_0/2). \tag{10.479}$$

The contribution of $\partial_{\mu}\psi_{\vec{m}0}(x)$ is however only appreciable when $\psi_{\vec{m}0}(x) = 0$, i.e. at the center of the box. In the first and second terms of the above equation we can then replace $R_0(\phi(x))$ by $R_0(u_{\vec{m}})$. The second term in the above equation (10.478) vanishes by conservation of momentum. In the first term we can neglect all the coupling terms $\phi_{\vec{m}0}\phi_{\vec{m}^{'}0}$ with $\vec{m}\neq\vec{m}^{'}$ since $\psi_{\vec{m}^{'}0}(x)$ is zero inside the box $\vec{m}$. We define the integral

$$\rho = \int d^dx\partial_{\mu}\psi_{\vec{m}0}(x)\partial^{\mu}\psi_{\vec{m}0}(x). \tag{10.480}$$

This is independent of $\vec{m}$ because of the relation (10.466). The first term becomes therefore $\rho\sum_{\vec{m}}R_0(u_{\vec{m}})\phi_{\vec{m}0}^2$. Equation (10.478) becomes

$$\begin{aligned}\int d^dxR_0\partial_{\mu}\phi(x)\partial^{\mu}\phi(x) &= \rho\sum_{\vec{m}}R_0(u_{\vec{m}})\phi_{\vec{m}0}^2\\ &+ 2^{-d-1}\alpha_0^2w^{-1}\sum_{\vec{m}}\Big[R_0(w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}) + R_0(-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}})\Big]\\ &\times \partial_{\mu}\phi^{'}(x_0/2)\partial^{\mu}\phi^{'}(x_0/2).\end{aligned} \tag{10.481}$$

The Interaction Term: Next we want to compute

$$\int_{\vec{x}\in \text{box}\ \vec{m}} d^dx P_0(\phi^{(m)}(x)) = \int_{\vec{x}\in \text{box}\ \vec{m}} d^dx P_0(\psi_{\vec{m}0}(x)\phi_{\vec{m}0} + 2^{-d/2}\alpha_0\phi^{'}(x/2)). \tag{10.482}$$

We note again that within the box $\vec{m}$ the field is given by

$$\phi^{(m)}(x) = \psi_{\vec{m}0}(x)\phi_{\vec{m}0} + 2^{-d/2}\alpha_0\phi^{'}(x/2). \tag{10.483}$$

Introduce $z_0 = \psi_{\vec{m}0}(x)\phi_{\vec{m}0} + u_{\vec{m}}$. We have then

$$\begin{aligned}\int_{\vec{x}\in \text{box}\ \vec{m}} d^dx P_0(\phi^{(m)}(x)) &= \int_{\vec{x}\in \text{box}\ \vec{m}} d^dx\Big[P_0(z_0) + \Big((x-x_0)^\mu \partial_\mu u_{\vec{m}} + \frac{1}{2}(x-x_0)^\mu (x-x_0)^\nu\Big) \\ &\times \partial_\mu\partial_\nu u_{\vec{m}}\frac{dP_0}{dz}|_{z_0} + \frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu\partial_\mu u_{\vec{m}}\partial_\nu u_{\vec{m}}\frac{d^2P_0}{dz^2}|z_0 + ...\Big].\end{aligned} \tag{10.484}$$

As stated earlier $\psi_{\vec{m}0}(x)$ is approximated by $+w^{1/2}$ in one half of the box and by $-w^{1/2}$ in the other half. Also recall that the volume of the box is w^{-1}. The first term can be approximated by

$$\int_{\vec{x}\in \text{box}\ \vec{m}} d^dx P_0(z_0) = \frac{w^{-1}}{2}\Big[P_0(w^{1/2}\phi_{\vec{m}0} + u_{\vec{m}}) + P_0(-w^{1/2}\phi_{\vec{m}0} + u_{\vec{m}})\Big]. \tag{10.485}$$

We compute now the third term in (10.484). We start from the obvious identity

$$\int_{\text{box}} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu = \int_{\text{box}_+} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu + \int_{\text{box}_-} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu. \tag{10.486}$$

We will think of the box $\vec{m}$ as a sphere of volume w^{-1}. Thus

$$\int_{\text{box}} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu = \frac{1}{2}V\eta^{\mu\nu}. \tag{10.487}$$

We have the definitions

$$V = \frac{1}{d}\int_{\text{box}} r^2 d^dx\ ,\ w^{-1} = \int_{\text{box}} d^dx. \tag{10.488}$$

Explicitly we have

$$V = \frac{1}{d+2}\Big(\frac{dw^{-1}}{\Omega_{d-1}}\Big)^{(d+2)/d}\frac{\Omega_{d-1}}{d}. \tag{10.489}$$

The above identity becomes then

$$\frac{1}{2}V\eta^{\mu\nu} = \int_{\text{box}_+} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu + \int_{\text{box}_-} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu. \tag{10.490}$$

The sum of these two integrals is rotational invariant. As stated before we think of the box as a sphere divided into two regions of equal volume. The first region (the first half of the box) is a concentric smaller sphere whereas the second region (the second half of the box) is a thin spherical shell. Both regions are spherically symmetric and thus we can assume that

$$\int_{\text{box}_\pm} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu=\frac{1}{2}V_\pm\eta^{\mu\nu}. \tag{10.491}$$

Clearly $V=V_++V_-$. The integral of interest is

$$\begin{aligned}\int_{\vec{x}\in\text{box}} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu\partial_\mu\partial_\nu u_{\vec{m}}\frac{dP_0}{dz}|_{z_0} &= \frac{dP_0}{dz}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\partial_\nu u_{\vec{m}}\int_{\text{box}_+} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu\\ &+ \frac{dP_0}{dz}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\partial_\nu u_{\vec{m}}\int_{\text{box}_-} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu\\ &= \frac{V_+}{2}\frac{dP_0}{dz}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\partial^\mu u_{\vec{m}}+\frac{V_-}{2}\frac{dP_0}{dz}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\partial^\mu u_{\vec{m}}\\ &= 2^{-1-d/2}\alpha_0V_+\frac{dP_0}{dz}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\partial^\mu\phi^{'}(x_0/2)\\ &+ 2^{-1-d/2}\alpha_0V_-\frac{dP_0}{dz}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\partial^\mu\phi^{'}(x_0/2).\end{aligned} \tag{10.492}$$

Similarly the fourth term in (10.484) is computed as follows. We have

$$\begin{aligned}\int_{\vec{x}\in\text{box}\ \vec{m}} d^dx\frac{1}{2}(x-x_0)^\mu(x-x_0)^\nu\partial_\mu u_{\vec{m}}\partial_\nu u_{\vec{m}}\frac{d^2P_0}{dz^2}|z_0 &= \frac{d^2P_0}{dz^2}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu u_{\vec{m}}\partial_\nu u_{\vec{m}}\int_{\text{box}_+} d^dx\frac{1}{2}(x-x_0)^\mu\\ &\times (x-x_0)^\nu\\ &+ \frac{d^2P_0}{dz^2}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu u_{\vec{m}}\partial_\nu u_{\vec{m}}\int_{\text{box}_-} d^dx\frac{1}{2}(x-x_0)^\mu\\ &\times (x-x_0)^\nu\\ &= \frac{V_+}{2}\frac{d^2P_0}{dz^2}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu u_{\vec{m}}\partial^\mu u_{\vec{m}}\\ &+ \frac{V_-}{2}\frac{d^2P_0}{dz^2}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu u_{\vec{m}}\partial^\mu u_{\vec{m}}\\ &= 2^{-1-d}\alpha_0^2V_+\frac{d^2P_0}{dz^2}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\phi^{'}(x_0/2)\partial^\mu\phi^{'}(x_0/2)\\ &+ 2^{-1-d}\alpha_0^2V_-\frac{d^2P_0}{dz^2}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu\phi^{'}(x_0/2)\partial^\mu\phi^{'}(x_0/2).\end{aligned} \tag{10.493}$$

Finally we compute the first term in (10.484). we have

$$\begin{aligned}\int_{\vec{x}\in\text{box}\ \vec{m}} d^dx(x-x_0)^\mu\partial_\mu u_{\vec{m}}\frac{dP_0}{dz}|_{z_0} &= \frac{dP_0}{dz}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu u_{\vec{m}}\int_{\text{box}_+} d^dx(x-x_0)^\mu\\ &+ \frac{dP_0}{dz}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\partial_\mu u_{\vec{m}}\int_{\text{box}_-} d^dx(x-x_0)^\mu.\end{aligned} \tag{10.494}$$

Clearly we must have

$$\int_{\text{box}_+} d^dx(x-x_0)^\mu+\int_{\text{box}_-} d^dx(x-x_0)^\mu=0. \tag{10.495}$$

We will assume that both integrals vanish again by the same previous argument. The linear term is therefore 0, viz

$$\int_{\vec{x}\in \text{box } \vec{m}} d^dx(x-x_0)^{\mu}\partial_{\mu}u_{\vec{m}}\frac{dP_0}{dz}|_{z_0} = 0. \quad (10.496)$$

The final result is

$$\begin{aligned}\int_{\vec{x}\in \text{box } \vec{m}} d^dxP_0(\phi^{(m)}(x)) &= \frac{w^{-1}}{2}\Big[P_0(w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}})+P_0(-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}})\Big]\\ &+ 2^{-1-d/2}\alpha_0\Big[V_+\frac{dP_0}{dz}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}+V_-\frac{dP_0}{dz}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\Big]\partial_{\mu}\partial^{\mu}\phi^{'}(x_0/2)\\ &+ 2^{-1-d}\alpha_0^2\Big[V_+\frac{d^2P_0}{dz^2}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}+V_-\frac{d^2P_0}{dz^2}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\Big]\partial_{\mu}\phi^{'}(x_0/2)\partial^{\mu}\phi^{'}(x_0/2).\end{aligned} \quad (10.497)$$

The Action: By putting all the previous results together we obtain the expansion of the action. We get

$$\begin{aligned}S_0(\phi(x)) &= \frac{K}{2}\int d^dxR_0(\phi(x))\partial_{\mu}\phi(x)\partial^{\mu}\phi(x)+\int d^dxP_0(\phi(x))\\ &= \frac{K\rho}{2}\sum_{\vec{m}}R_0(u_{\vec{m}})\phi_{\vec{m}0}^2\\ &+ 2^{-d-2}K\alpha_0^2w^{-1}\sum_{\vec{m}}\Big[R_0(w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}})+R_0(-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}})\Big]\partial_{\mu}\phi^{'}(x_0/2)\partial^{\mu}\phi^{'}(x_0/2)\\ &+ \frac{w^{-1}}{2}\sum_{\vec{m}}\Big(P_0(w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}})+P_0(-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}})\Big)\\ &+ 2^{-1-d/2}\alpha_0\sum_{\vec{m}}\Big(V_+\frac{dP_0}{dz}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}+V_-\frac{dP_0}{dz}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\Big)\partial_{\mu}\partial^{\mu}\phi^{'}(x_0/2)\\ &+ 2^{-1-d}\alpha_0^2\sum_{\vec{m}}\Big(V_+\frac{d^2P_0}{dz^2}|_{w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}+V_-\frac{d^2P_0}{dz^2}|_{-w^{1/2}\phi_{\vec{m}0}+u_{\vec{m}}}\Big)\partial_{\mu}\phi^{'}(x_0/2)\partial^{\mu}\phi^{'}(x_0/2).\end{aligned} \quad (10.498)$$

The Path Integral: We need now to evaluate the path integral

$$\int\prod_{\vec{m}}d\phi_{\vec{m}0}\ e^{-S_0(\phi(x))}. \quad (10.499)$$

We introduce the variables

$$u_{\vec{m}}\longrightarrow z_{\vec{m}}=(\frac{K\rho}{2w})^{1/2}u_{\vec{m}}. \quad (10.500)$$

$$\phi_{\vec{m}0}\longrightarrow y_{\vec{m}}=(\frac{K\rho}{2})^{1/2}\phi_{\vec{m}0}. \quad (10.501)$$

$$R_0\longrightarrow W_0(x)=R_0((\frac{2w}{K\rho})^{1/2}x). \quad (10.502)$$

$$P_0 \longrightarrow Q_0(x) = w^{-1}P_0\big(\big(\frac{2w}{K\rho}\big)^{1/2}x\big). \tag{10.503}$$

We compute then

$$\begin{aligned}
\int \prod_{\vec{m}} d\phi_{\vec{m}0}\; e^{-S_0(\phi(x))} &= \prod_{\vec{m}} \big(\frac{2}{K\rho}\big)^{1/2} \int dy_{\vec{m}}\;\exp\bigg(- y_{\vec{m}}^2 W_0(z_{\vec{m}}) - \frac{1}{2}Q_0(y_{\vec{m}}+z_{\vec{m}}) - \frac{1}{2}Q_0(-y_{\vec{m}}+z_{\vec{m}}) \\
&- 2^{-(3+d)/2}\alpha_0(K\rho w)^{1/2}\bigg[V_+\frac{dQ_0}{dy_{\vec{m}}}(y_{\vec{m}}+z_{\vec{m}}) + V_-\frac{dQ_0}{dy_{\vec{m}}}(-y_{\vec{m}}+z_{\vec{m}})\bigg]\partial_\mu\partial^\mu\phi^{'}(x_0/2) \\
&- 2^{-d-2}\alpha_0^2 K\rho\bigg[V_+\frac{d^2Q_0}{dy_{\vec{m}}^2}(y_{\vec{m}}+z_{\vec{m}}) + V_-\frac{d^2Q_0}{dy_{\vec{m}}^2}(-y_{\vec{m}}+z_{\vec{m}})\bigg]\partial_\mu\phi^{'}(x_0/2)\partial^\mu\phi^{'}(x_0/2) \\
&- 2^{-d-2}\alpha_0^2 K w^{-1}\bigg[W_0(y_{\vec{m}}+z_{\vec{m}}) + W_0(-y_{\vec{m}}+z_{\vec{m}})\bigg]\partial_\mu\phi^{'}(x_0/2)\partial^\mu\phi^{'}(x_0/2)\bigg).
\end{aligned} \tag{10.504}$$

Define

$$\begin{aligned}
M_0(z_{\vec{m}}) &= \int dy_{\vec{m}}\;\exp\bigg(- y_{\vec{m}}^2 W_0(z_{\vec{m}}) - \frac{1}{2}Q_0(y_{\vec{m}}+z_{\vec{m}}) - \frac{1}{2}Q_0(-y_{\vec{m}}+z_{\vec{m}}) \\
&- 2^{-(3+d)/2}\alpha_0(K\rho w)^{1/2}\bigg[V_+\frac{dQ_0}{dy_{\vec{m}}}(y_{\vec{m}}+z_{\vec{m}}) + V_-\frac{dQ_0}{dy_{\vec{m}}}(-y_{\vec{m}}+z_{\vec{m}})\bigg]\partial_\mu\partial^\mu\phi^{'}(x_0/2) \\
&- 2^{-d-2}\alpha_0^2 K\rho\bigg[V_+\frac{d^2Q_0}{dy_{\vec{m}}^2}(y_{\vec{m}}+z_{\vec{m}}) + V_-\frac{d^2Q_0}{dy_{\vec{m}}^2}(-y_{\vec{m}}+z_{\vec{m}})\bigg]\partial_\mu\phi^{'}(x_0/2)\partial^\mu\phi^{'}(x_0/2) \\
&- 2^{-d-2}\alpha_0^2 K w^{-1}\bigg[W_0(y_{\vec{m}}+z_{\vec{m}}) + W_0(-y_{\vec{m}}+z_{\vec{m}})\bigg]\partial_\mu\phi^{'}(x_0/2)\partial^\mu\phi^{'}(x_0/2)\bigg).
\end{aligned} \tag{10.505}$$

In this equation $\phi^{'}(x_0/2)$ is given in terms of $z_{\vec{m}}$ by

$$\phi^{'}(x_0/2) = \frac{2^{d/2}}{\alpha_0}\big(\frac{2w}{K\rho}\big)^{1/2} z_{\vec{m}}. \tag{10.506}$$

The remaining dependence on the box $\vec{m}$ is only through the center of the box x_0. Then

$$\begin{aligned}
\int \prod_{\vec{m}} d\phi_{\vec{m}0}\; e^{-S_0(\phi(x))} &= \prod_{\vec{m}} \big(\frac{2}{K\rho}\big)^{1/2} M_0(z_{\vec{m}}) \\
&= \prod_{\vec{m}} \big(\frac{2}{K\rho}\big)^{1/2} \exp(\ln M_0(z_{\vec{m}})) \\
&= \exp\big(\sum_{\vec{m}} \ln\frac{M_0(z_{\vec{m}})}{I_0(0)}\big) \prod_{\vec{m}} \big(\frac{2}{K\rho}\big)^{1/2} I_0(0).
\end{aligned} \tag{10.507}$$

The function $I_0(z)$ is defined by

$$I_0(z) = \int dy\;\exp\bigg(- y^2 W_0(z) - \frac{1}{2}Q_0(y+z) - \frac{1}{2}Q_0(-y+z)\bigg). \tag{10.508}$$

In order to compute M_0 we will assume that the derivative terms are small and expand the exponential around the ultra local approximation. We compute

$$\begin{aligned} M_0(z_{\vec{m}}) &= I_0(z_{\vec{m}})\Big[1-2^{-(3+d)/2}\alpha_0(K\rho w)^{1/2}V < \frac{dQ_0}{dy_{\vec{m}}}(y_{\vec{m}}+z_{\vec{m}}) > \partial_\mu\partial^\mu\phi^{'}(x_0/2)\\ &- 2^{-d-1}\alpha_0^2 K\Big[\frac{\rho V}{2} < \frac{d^2Q_0}{dy_{\vec{m}}^2}(y_{\vec{m}}+z_{\vec{m}}) > +w^{-1} < W_0(y_{\vec{m}}+z_{\vec{m}}) > \Big]\partial_\mu\phi^{'}(x_0/2)\partial^\mu\phi^{'}(x_0/2)+...\Big]. \end{aligned} \quad (10.509)$$

The path integral (10.499) becomes

$$\begin{aligned} \int\prod_{\vec{m}} d\phi_{\vec{m}0}\, e^{-S_0(\phi(x))} &= \prod_{\vec{m}}(\frac{2}{K\rho})^{1/2} I_0(0)\times\exp\bigg(\sum_{\vec{m}}\ln\frac{I_0(z_{\vec{m}})}{I_0(0)} - 2^{-(3+d)/2}\alpha_0(K\rho w)^{1/2}V\sum_{\vec{m}}\\ &\times < \frac{dQ_0}{dy_{\vec{m}}}(y_{\vec{m}}+z_{\vec{m}}) > \partial_\mu\partial^\mu\phi^{'}(x_0/2) - 2^{-d-1}\alpha_0^2 K\sum_{\vec{m}}\Big[\frac{\rho V}{2} < \frac{d^2Q_0}{dy_{\vec{m}}^2}(y_{\vec{m}}+z_{\vec{m}}) >\\ &+ w^{-1} < W_0(y_{\vec{m}}+z_{\vec{m}}) > \Big]\partial_\mu\phi^{'}(x_0/2)\partial^\mu\phi^{'}(x_0/2)\bigg). \end{aligned} \quad (10.510)$$

We make the change of variable $x_0/2 \longrightarrow x$ (this means that the position space wave packet with $l=1$ corresponding to the highest not integrated momentum will now fit into the box) to obtain

$$\begin{aligned} \int\prod_{\vec{m}} d\phi_{\vec{m}0}\, e^{-S_0(\phi(x))} &= \prod_{\vec{m}}(\frac{2}{K\rho})^{1/2} I_0(0)\times\exp\bigg(\sum_{\vec{m}}\ln\frac{I_0(z_{\vec{m}})}{I_0(0)} - 2^{-(3+d)/2}\frac{\alpha_0(K\rho w)^{1/2}V}{4}\sum_{\vec{m}}\\ &\times < \frac{dQ_0}{dy_{\vec{m}}}(y_{\vec{m}}+z_{\vec{m}}) > \partial_\mu\partial^\mu\phi^{'}(x) - 2^{-d-1}\frac{\alpha_0^2 K}{4}\sum_{\vec{m}}\Big[\frac{\rho V}{2} < \frac{d^2Q_0}{dy_{\vec{m}}^2}(y_{\vec{m}}+z_{\vec{m}}) >\\ &+ w^{-1} < W_0(y_{\vec{m}}+z_{\vec{m}}) > \Big]\partial_\mu\phi^{'}(x)\partial^\mu\phi^{'}(x)\bigg). \end{aligned} \quad (10.511)$$

Now $z_{\vec{m}}$ is given by $z_{\vec{m}} = (K\rho/2w)^{1/2}2^{-d/2}\alpha_0\phi^{'}(x)$. It is clear that $dQ_0(y_{\vec{m}}+z_{\vec{m}})/dy_{\vec{m}} = dQ_0(y_{\vec{m}}+z_{\vec{m}})/dz_{\vec{m}}$, etc. Recall that the volume of the box $\vec{m}$ is w^{-1} and thus we can make the identification $w\int d^dx = \sum_{\vec{m}}$. However we have also made the rescaling $x_0 \longrightarrow 2x$ and hence we must make instead the identification $2^d w\int d^dx = \sum_{\vec{m}}$. We obtain

$$\begin{aligned} \int\prod_{\vec{m}} d\phi_{\vec{m}0}\, e^{-S_0(\phi(x))} &= \prod_{\vec{m}}(\frac{2}{K\rho})^{1/2} I_0(0)\times\exp\bigg(2^d w\int d^dx\ln\frac{I_0(z)}{I_0(0)} - 2^{-(3+d)/2}\frac{\alpha_0(K\rho w)^{1/2}V}{4}2^d w\\ &\times \int d^dx < \frac{dQ_0}{dz}(z) > \partial_\mu\partial^\mu\phi^{'}(x) - 2^{-d-1}\frac{\alpha_0^2 K}{4}2^d w\int d^dx\Big[\frac{\rho V}{2} < \frac{d^2Q_0}{dz^2}(z) >\\ &+ w^{-1} < W_0(z) > \Big]\partial_\mu\phi^{'}(x)\partial^\mu\phi^{'}(x)\bigg). \end{aligned} \quad (10.512)$$

Now z is given by $z = (K\rho/2w)^{1/2}2^{-d/2}\alpha_0\phi^{'}(x)$. The expectation values $< O^n(z) >$ are defined by

$$< O^n(z) > = \frac{1}{I(z)}\int dy\,\bigg(\frac{O(y+z)+O(-y+z)}{2}\bigg)^n\exp\bigg(-y^2W_0(z)-\frac{1}{2}Q_0(y+z)-\frac{1}{2}Q_0(-y+z)\bigg). \quad (10.513)$$

Next we derive in a straightforward way the formula

$$\frac{d}{dz} < \frac{dQ_0}{dz}(z) > \;=\; < \frac{d^2Q_0}{dz^2}(z) > - < (\frac{dQ_0}{dz})^2(z) > + < \frac{dQ_0}{dz}(z) >^2 + \frac{dW_0}{dz}\Big(< \frac{dQ_0}{dz}(z) > \frac{1}{I_0(z)} \int dy y^2 e^{\cdots} - \frac{1}{I_0(z)} \int dy \frac{dQ_0}{dz}(y+z) y^2 e^{\cdots}\Big). \quad (10.514)$$

By integrating by part the second term in (10.512) we can see that the first term in (10.514) cancels the 3rd term in (10.512). The last term can be neglected if we assume that dW_0/dz is much smaller than Q_0. We obtain then

$$\int \prod_{\vec{m}} d\phi_{\vec{m}0}\; e^{-S_0(\phi(x))} \;=\; \prod_{\vec{m}} (\frac{2}{K\rho})^{1/2} I_0(0) \times \exp\Big(2^d w \int d^dx \ln\frac{I_0(z)}{I_0(0)} - \frac{\alpha_0^2 K w}{8} \int d^dx \Big[\frac{\rho V}{2}\big(< (\frac{dQ_0}{dz})^2(z) > - < \frac{dQ_0}{dz}(z) >^2 \big) + w^{-1} < W_0(z) > \Big] \times \partial_\mu \phi^{'}(x) \partial^\mu \phi^{'}(x)\Big). \quad (10.515)$$

The Recursion Formulas: The full path integral is therefore given by

$$Z_0 = \int \mathcal{D}\phi^{'}(x_0/2) \int \prod_{\vec{m}} d\phi_{\vec{m}0}\; e^{-S_0(\phi(x))} \propto \int \mathcal{D}\phi^{'}(x) \int \exp\Big(2^d w \int d^dx \ln\frac{I_0(z)}{I_0(0)} - \frac{\alpha_0^2 K w}{8} \int d^dx \Big[\frac{\rho V}{2}\big(< (\frac{dQ_0}{dz})^2(z) > - < \frac{dQ_0}{dz}(z) >^2 \big) + w^{-1} < W_0(z) > \Big] \partial_\mu \phi^{'}(x) \partial^\mu \phi^{'}(x)\Big). \quad (10.516)$$

We write this as

$$Z_1 = \int \mathcal{D}\phi^{'}(x)\; e^{-S_1(\phi^{'}(x))}. \quad (10.517)$$

The new action S_1 has the same form as the action S_0, viz

$$S_1(\phi^{'}(x)) = \frac{K}{2} \int d^dx R_1(\phi^{'}(x)) \partial_\mu \phi^{'}(x) \partial^\mu \phi^{'}(x) + \int d^dx P_1(\phi^{'}(x)). \quad (10.518)$$

The new polynomials P_1 and R_1 (or equivalently Q_1 and W_1) are given in terms of the old ones P_0 and R_0 (or equivalently Q_0 and W_0) by the relations

$$W_1(2^{d/2}\alpha_0^{-1} z) = R_1(\phi^{'}(x)) = \frac{\alpha_0^2 C_d}{8}\big(< (\frac{dQ_0}{dz})^2(z) > - < \frac{dQ_0}{dz}(z) >^2 \big) + \frac{\alpha_0^2}{4} < W_0(z) > . \quad (10.519)$$

$$Q_1(2^{d/2}\alpha_0^{-1} z) = w^{-1} P_1(\phi^{'}(x)) = -2^d \ln\frac{I_0(z)}{I_0(0)}. \quad (10.520)$$

The constant C_d is given by

$$C_d = w\rho V. \tag{10.521}$$

Before we write down the recursion formulas we also introduce the notation

$$\phi^{(0)} = \phi \ , \ \phi^{(1)} = \phi^{'}. \tag{10.522}$$

The above procedure can be repeated to integrate out the momentum shell $l = 1$ and get from S_1 to S_2. The modes with $l \geq 2$ will involve a new constant α_1 and instead of P_0, R_0, S_0 and I_0 we will have P_1, R_1, S_1 and I_1. Aside from this trivial relabeling everything else will be the same including the constants w, V and C_d since $l = 1$ can be mapped to $l = 0$ due to the scaling $x_0 \longrightarrow 2x$ (see equations (10.470) and (10.472)). This whole process can be repeated an arbitrary number of times to get a renormalization group flow of the action given explicitly by the sequences $P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow ... \longrightarrow P_i$ and $R_0 \longrightarrow R_1 \longrightarrow R_2 \longrightarrow ... \longrightarrow R_i$. By assuming that $dR_i/d\phi^{(i)}$ is sufficiently small compared to P_i for all i the recursion formulas which relates the different operators at the renormalization group steps i and $i+1$ are obviously given by

$$\begin{aligned} W_{i+1}(2^{d/2}\alpha_i^{-1}z) &= R_{i+1}(\phi^{(i+1)}(x)) \\ &= \frac{\alpha_i^2 C_d}{8}\big(< (\frac{dQ_i}{dz})^2(z) > - < \frac{dQ_i}{dz}(z) >^2 \big) + \frac{\alpha_i^2}{4} < W_i(z) > . \end{aligned} \tag{10.523}$$

$$\begin{aligned} Q_{i+1}(2^{d/2}\alpha_i^{-1}z) &= w^{-1}P_{i+1}(\phi^{(i+1)}(x)) \\ &= -2^d \ln\frac{I_i(z)}{I_i(0)}. \end{aligned} \tag{10.524}$$

The function $I_i(z)$ is given by the same formula (10.508) with the substitutions $I_0 \longrightarrow I_i$, $W_0 \longrightarrow W_i$ and $Q_0 \longrightarrow Q_i$.

The field $\phi^{(i+1)}$ and the variable z are related by $z = (K\rho/2w)^{1/2}2^{-d/2}\alpha_i\phi^{(i+1)}$. The full action at the renormalization group step i is

$$S_i(\phi^{(i)}(x)) = \frac{K}{2}\int d^dx R_i(\phi^{(i)}(x))\partial_\mu\phi^{(i)}(x)\partial^\mu\phi^{(i)}(x) + \int d^dx P_i(\phi^{(i)}(x)). \tag{10.525}$$

The constants α_i will be determined from the normalization condition

$$W_{i+1}(0) = 1. \tag{10.526}$$

Since Q is even this normalization condition is equivalent to

$$\frac{\alpha_i^2}{4} < W_i(z) > |_{z=0} = 1. \tag{10.527}$$

The Ultra Local Recursion Formula: This corresponds to keeping in the expansion (10.476) only the first term. The resulting recursion formula is obtained from the above recursion formulas by dropping from equation (10.523) the fluctuation term

$$< (\frac{dQ_i}{dz})^2(z) > - < \frac{dQ_i}{dz}(z) >^2 . \tag{10.528}$$

The recursion formula (10.523) becomes

$$W_{i+1}(2^{d/2}\alpha_i^{-1}z) \quad = \quad \frac{\alpha_i^2}{4} < W_i(z) > . \tag{10.529}$$

The solution of this equation together with the normalization condition (10.527) is given by

$$W_i = 1 \ , \ \alpha_i = 2. \tag{10.530}$$

The remaining recursion formula is given by

$$Q_{i+1}(2^{d/2}2^{-1}z) \quad = \quad -2^d \ln \frac{I_i(z)}{I_i(0)}. \tag{10.531}$$

We state without proof that the use of this recursion formula is completely equivalent to the use in perturbation theory of the Polyakov-Wilson rules given by the approximations:

- We replace every internal propagator $1/(k^2+r_0^2)$ by $1/(\Lambda^2+r_0^2)$.
- We replace every momentum integral $\int_{\Lambda/2}^{\Lambda} d^dp/(2\pi)^d$ by the volume $c/4$ where $c = 4\Omega_{d-1}\Lambda^d(1-2^{-d})/(d(2\pi)^d)$.

10.5.3 The Wilson-Fisher Fixed Point

Let us start with a ϕ^4 action given by

$$S_0[\phi_0] = \int d^dx\bigg(\frac{1}{2}(\partial_\mu\phi_0)^2 + \frac{1}{2}r_0\phi_0^2 + u_0\phi_0^4\bigg). \tag{10.532}$$

The Fourier transform of the field is given by

$$\phi_0(x) = \int_0^\Lambda \frac{d^dp}{(2\pi)^d}\tilde{\phi}_0(p)\ e^{ipx}. \tag{10.533}$$

We will decompose the field as

$$\phi_0(x) = \phi_1^{'}(x) + \Phi(x). \tag{10.534}$$

The background field $\phi_1^{'}(x)$ corresponds to the low frequency modes $\tilde{\phi}_0(p)$ where $0 \leq p \leq \Lambda/2$ whereas the fluctuation field $\Phi(x)$ corresponds to high frequency modes $\tilde{\phi}_0(p)$ where $\Lambda/2 < p \leq \Lambda$, viz

$$\phi_1^{'}(x) = \int_0^{\Lambda/2} \frac{d^dp}{(2\pi)^d}\tilde{\phi}_0(p)\ e^{ipx} \ , \ \Phi(x) = \int_{\Lambda/2}^{\Lambda} \frac{d^dp}{(2\pi)^d}\tilde{\phi}_0(p)\ e^{ipx}. \tag{10.535}$$

The goal is integrate out the high frequency modes from the partition function. The partition function is given by

$$\begin{aligned} Z_0 &= \int \mathcal{D}\phi_0\ e^{-S_0[\phi_0]} \\ &= \int \mathcal{D}\phi_1^{'} \int \mathcal{D}\Phi\ e^{-S_0[\phi_1^{'}+\Phi]} \\ &= \int \mathcal{D}\phi_1^{'}\ e^{-S_1[\phi_1^{'}]}. \end{aligned} \tag{10.536}$$

The first goal is to determine the action $S_1[\phi_1^{'}]$. We have

$$
\begin{aligned}
e^{-S_1[\phi_1^{'}]} &= \int d\Phi \; e^{-S_0[\phi_1^{'}+\Phi]} \\
&= e^{-S_0[\phi_1^{'}]} \int d\Phi \; e^{-S_0[\Phi]} e^{-u_0 \int d^dx [4\Phi^3\phi_1^{'}+4\Phi\phi_1^{'3}+6\Phi^2\phi_1^{'2}]}.
\end{aligned} \tag{10.537}
$$

We will expand in the field $\phi_1^{'}$ up to the fourth power. We define expectation values with respect to the partition function

$$
Z = \int d\Phi \; e^{-S_0[\Phi]}. \tag{10.538}
$$

Let us also introduce

$$
\begin{aligned}
V_1 &= -4u_0 \int d^dx \Phi^3 \phi_1^{'} \\
V_2 &= -6u_0 \int d^dx \Phi^2 \phi_1^{'2} \\
V_3 &= -4u_0 \int d^dx \Phi \phi_1^{'3}.
\end{aligned} \tag{10.539}
$$

Then we compute

$$
e^{-S_1[\phi_1^{'}]} = Ze^{-S_0[\phi_1^{'}]} < \left[1+V_1+V_2+V_3+\frac{1}{2}(V_1^2+2V_1V_2+V_2^2+2V_1V_3)+\frac{1}{6}(V_1^3+3V_1^2V_2)+\frac{1}{24}V_1^4\right] > . \tag{10.540}
$$

By using the symmetry $\Phi \longrightarrow -\Phi$ we obtain

$$
e^{-S_1[\phi_1^{'}]} = Ze^{-S_0[\phi_1^{'}]} < \left[1+V_2+\frac{1}{2}(V_1^2+V_2^2+2V_1V_3)+\frac{1}{6}(3V_1^2V_2)+\frac{1}{24}V_1^4\right] > . \tag{10.541}
$$

The term $< V_1V_3 >$ vanishes by momentum conservation. We rewrite the different expectation values in terms of connected functions. We have

$$
\begin{aligned}
&< V_2 >=< V_2 >_{\rm co} \\
&< V_1^2 >=< V_1^2 >_{\rm co} \\
&< V_2^2 >=< V_2^2 >_{\rm co} + < V_2 >_{\rm co}^2 \\
&< V_1^2V_2 >=< V_1^2V_2 >_{\rm co} + < V_1^2 >_{\rm co}< V_2 >_{\rm co} \\
&< V_1^4 >=< V_1^4 >_{\rm co} +3 < V_1^2 >_{\rm co}^2 .
\end{aligned} \tag{10.542}
$$

By using these results the partition function becomes

$$
e^{-S_1[\phi_1^{'}]} = Ze^{-S_0[\phi_1^{'}]} \; e^{<V_2>_{\rm co}+\frac{1}{2}(<V_1^2>_{\rm co}+<V_2^2>_{\rm co})+\frac{1}{2}<V_1^2V_2>_{\rm co}+\frac{1}{24}<V_1^4>_{\rm co}}. \tag{10.543}
$$

In other words the partition function is expressible only in terms of irreducible connected functions. This is sometimes known as the cumulant expansion. The action $S_1[\phi_1^{'}]$ is given by

$$
S_1[\phi_1^{'}] = S_0[\phi_1^{'}] - < V_2 >_{\rm co} -\frac{1}{2}(< V_1^2 >_{\rm co} + < V_2^2 >_{\rm co}) - \frac{1}{2} < V_1^2V_2 >_{\rm co} -\frac{1}{24} < V_1^4 >_{\rm co} . \tag{10.544}
$$

We need the propagator

$$< \Phi(x)\Phi(y) >_{\rm co} \;=\; < \Phi(x)\Phi(y) >_0 -12u_0\int d^dz < \Phi(x)\Phi(z) >_0 < \Phi(y)\Phi(z) >_0 < \Phi(z)\Phi(z) >_0 +O(u_0^2). \tag{10.545}$$

The free propagator is obviously given by

$$< \Phi(x)\Phi(y) >_0 \;=\; \int_{\Lambda/2}^{\Lambda}\frac{d^dp}{(2\pi)^d}\frac{1}{p^2+r_0}e^{ip(x-y)}. \tag{10.546}$$

Thus

$$< \Phi(x)\Phi(y) >_{\rm co} \;=\; \bigg[\int\frac{d^dp_1}{(2\pi)^d}\frac{1}{p_1^2+r_0}-12u_0\int\frac{d^dp_1}{(2\pi)^d}\frac{d^dp_2}{(2\pi)^d}\frac{1}{(p_1^2+r_0)^2(p_2^2+r_0)}+O(u_0^2)\bigg]e^{ip_1(x-y)}. \tag{10.547}$$

We can now compute

$$\begin{aligned}
- < V_2 >_{\rm co} \;&=\; 6u_0\int d^dx\phi_1^{'2}(x) < \Phi^2(x) >_{\rm co}\\
&=\; 6u_0\int\frac{d^dp}{(2\pi)^d}|\tilde{\phi}_1^{'}(p)|^2\bigg[\int\frac{d^dp_1}{(2\pi)^d}\frac{1}{p_1^2+r_0}-12u_0\int\frac{d^dp_1}{(2\pi)^d}\frac{d^dp_2}{(2\pi)^d}\frac{1}{(p_1^2+r_0)^2(p_2^2+r_0)}+O(u_0^2)\bigg].
\end{aligned} \tag{10.548}$$

$$\begin{aligned}
-\frac{1}{2} < V_1^2 >_{\rm co} \;&=\; -8u_0^2\int d^dx_1d^dx_2\phi_1^{'}(x_1)\phi_1^{'}(x_2) < \Phi^3(x_1)\Phi^3(x_2) >_{\rm co}\\
&=\; -8u_0^2\int d^dx_1d^dx_2\phi_1^{'}(x_1)\phi_1^{'}(x_2)\bigg[6 < \Phi(x_1)\Phi(x_2) >_0^3 +3 < \Phi(x_1)\Phi(x_2) >_0 < \Phi(x_1)\Phi(x_1) >_0\\
&\times\; < \Phi(x_2)\Phi(x_2) >_0\bigg]\\
&=\; -8u_0^2\int\frac{d^dp}{(2\pi)^d}|\tilde{\phi}_1^{'}(p)|^2\bigg[6\int\frac{d^dp_1}{(2\pi)^d}\frac{d^dp_2}{(2\pi)^d}\frac{1}{(p_1^2+r_0)(p_2^2+r_0)((p+p_1+p_2)^2+r_0)}+O(u_0)\bigg].
\end{aligned} \tag{10.549}$$

The second term of the second line of the above equation did not contribute because of momentum conservation. Next we compute

$$\begin{aligned}
-\frac{1}{2} < V_2^2 >_{\rm co} \;&=\; -18u_0^2\int d^dx_1d^dx_2\phi_1^{'2}(x_1)\phi_1^{'2}(x_2) < \Phi^2(x_1)\Phi^2(x_2) >_{\rm co}\\
&=\; -36u_0^2\int d^dx_1d^dx_2\phi_1^{'2}(x_1)\phi_1^{'2}(x_2) < \Phi(x_1)\Phi(x_2) >_0^2\\
&=\; -12u_0^2\int\frac{d^dp_1}{(2\pi)^d}...\frac{d^dp_3}{(2\pi)^d}\tilde{\phi}_1^{'}(p_1)...\tilde{\phi}_1^{'}(p_3)\tilde{\phi}_1^{'}(-p_1-p_2-p_3)\bigg[\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+r_0}\\
&\times\; \frac{1}{(k+p_1+p_2)^2+r_0}+2\text{ permutations}+O(u_0)\bigg].
\end{aligned} \tag{10.550}$$

The last two terms of the cumulant expansion are of order u_0^3 and u_0^4 respectively which we are not computing. The action $S_1[\phi_1^{'}]$ reads then explicitly

$$\begin{aligned} S_1[\phi_1^{'}] &= \frac{1}{2}\int_0^{\Lambda/2}\frac{d^dp}{(2\pi)^d}|\tilde{\phi}_1^{'}(p)|^2\bigg\{p^2+r_0+12u_0\int\frac{d^dk_1}{(2\pi)^d}\frac{1}{k_1^2+r_0}-144u_0^2\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{1}{(k_1^2+r_0)^2(k_2^2+r_0)}\\ &- 96u_0^2\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{1}{(k_1^2+r_0)(k_2^2+r_0)((p+k_1+k_2)^2+r_0)}+O(u_0^3)\bigg\}\\ &+ \int_0^{\Lambda/2}\frac{d^dp_1}{(2\pi)^d}...\frac{d^dp_3}{(2\pi)^d}\tilde{\phi}_1^{'}(p_1)...\tilde{\phi}_1^{'}(p_3)\tilde{\phi}_1^{'}(-p_1-p_2-p_3)\bigg[u_0-12u_0^2\bigg[\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+r_0}\\ &\times \frac{1}{(k+p_1+p_2)^2+r_0}+2\text{ permutations}\bigg]+O(u_0^3)\bigg]. \end{aligned} \quad (10.551)$$

The Fourier mode $\tilde{\phi}_1^{'}(p)$ is of course equal $\tilde{\phi}_0(p)$ for $0\leq p\leq\Lambda/2$ and 0 otherwise. We scale now the field as

$$\tilde{\phi}_1^{'}(p)=\alpha_0\tilde{\phi}_1(2p). \quad (10.552)$$

The action becomes

$$\begin{aligned} S_1[\phi_1] &= \frac{1}{2}\alpha_0^2 2^{-d}\int_0^{\Lambda}\frac{d^dp}{(2\pi)^d}|\tilde{\phi}_1(p)|^2\bigg\{\frac{p^2}{4}+r_0+12u_0\int\frac{d^dk_1}{(2\pi)^d}\frac{1}{k_1^2+r_0}-144u_0^2\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\\ &\times \frac{1}{(k_1^2+r_0)^2(k_2^2+r_0)}-96u_0^2\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{1}{(k_1^2+r_0)(k_2^2+r_0)((\frac{1}{2}p+k_1+k_2)^2+r_0)}+O(u_0^3)\bigg\}\\ &+ \alpha_0^4 2^{-3d}\int_0^{\Lambda}\frac{d^dp_1}{(2\pi)^d}...\frac{d^dp_3}{(2\pi)^d}\tilde{\phi}_1(p_1)...\tilde{\phi}_1(p_3)\tilde{\phi}_1(-p_1-p_2-p_3)\bigg[u_0-12u_0^2\bigg[\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+r_0}\\ &\times \frac{1}{(k+\frac{1}{2}p_1+\frac{1}{2}p_2)^2+r_0}+2\text{ permutations}\bigg]+O(u_0^3)\bigg]. \end{aligned} \quad (10.553)$$

In the above equation the internal momenta k_i are still unscaled in the interval $[\Lambda/2,\Lambda]$. The one-loop truncation of this result is given by

$$\begin{aligned} S_1[\phi_1] &= \frac{1}{2}\alpha_0^2 2^{-d}\int_0^{\Lambda}\frac{d^dp}{(2\pi)^d}|\tilde{\phi}_1(p)|^2\bigg\{\frac{p^2}{4}+r_0+12u_0\int\frac{d^dk_1}{(2\pi)^d}\frac{1}{k_1^2+r_0}+O(u_0^2)\bigg\}\\ &+ \alpha_0^4 2^{-3d}\int_0^{\Lambda}\frac{d^dp_1}{(2\pi)^d}...\frac{d^dp_3}{(2\pi)^d}\tilde{\phi}_1(p_1)...\tilde{\phi}_1(p_3)\tilde{\phi}_1(-p_1-p_2-p_3)\bigg[u_0-12u_0^2\bigg[\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2+r_0}\\ &\times \frac{1}{(k+\frac{1}{2}p_1+\frac{1}{2}p_2)^2+r_0}+2\text{ permutations}\bigg]+O(u_0^3)\bigg]. \end{aligned} \quad (10.554)$$

We bring the kinetic term to the canonical form by choose α_0 as

$$\alpha_0=2^{1+d/2}. \quad (10.555)$$

Furthermore we truncate the interaction term in the action by setting the external momenta to zero since we are only interested in the renormalization group flow of the operators present in the original action. We get then

$$S_1[\phi_1] = \frac{1}{2}\int_0^{\Lambda}\frac{d^dp}{(2\pi)^d}|\tilde{\phi}_1(p)|^2(p^2+r_1)+u_1\int_0^{\Lambda}\frac{d^dp_1}{(2\pi)^d}...\frac{d^dp_3}{(2\pi)^d}\tilde{\phi}_1(p_1)...\tilde{\phi}_1(p_3)\tilde{\phi}_1(-p_1-p_2-p_3). \quad (10.556)$$

The new mass parameter r_1 and the new coupling constant u_1 are given by

$$r_1 = 4r_0 + 48u_0 \int \frac{d^dk_1}{(2\pi)^d} \frac{1}{k_1^2 + r_0} + O(u_0^2). \tag{10.557}$$

$$u_1 = 2^{4-d}\bigg[u_0 - 36u_0^2 \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2 + r_0)^2} + O(u_0^3)\bigg]. \tag{10.558}$$

Now we employ the Wilson-Polyakov rules corresponding to the ultra local Wilson recursion formula (10.531) consisting of making the following approximations:

- We replace every internal propagator $1/(k^2 + r_0^2)$ by $1/(\Lambda^2 + r_0^2)$.
- We replace every momentum integral $\int_{\Lambda/2}^{\Lambda} d^dp/(2\pi)^d$ by the volume $c/4$ where $c = 4\Omega_{d-1}\Lambda^d(1-2^{-d})/(d(2\pi)^d)$.

The mass parameter r_1 and the coupling constant u_1 become

$$r_1 = 4\bigg[r_0 + 3c\frac{u_0}{\Lambda^2 + r_0} + O(u_0^2)\bigg]. \tag{10.559}$$

$$u_1 = 2^{4-d}\bigg[u_0 - 9c\frac{u_0^2}{(\Lambda^2 + r_0)^2} + O(u_0^3)\bigg]. \tag{10.560}$$

This is the result of our first renormalization group step. Since the action $S_1[\phi_1]$ is of the same form as the action $S_0[\phi_0]$ the renormalization group calculation can be repeated without any change to go from r_1 and u_1 to a new mass parameter r_2 and a new coupling constant u_2. This whole process can evidently be iterated an arbitrary number of times to define a renormalization group flow $(r_0, u_0) \longrightarrow (r_1, u_1) \longrightarrow ....(r_l, u_l) \longrightarrow (r_{l+1}, u_{l+1})$..... The renormalization group recursion equations relating (r_{l+1}, u_{l+1}) to (r_l, u_l) are given precisely by the above equations, viz

$$r_{l+1} = 4\bigg[r_l + 3c\frac{u_l}{\Lambda^2 + r_l}\bigg]. \tag{10.561}$$

$$u_{l+1} = 2^{4-d}\bigg[u_l - 9c\frac{u_l^2}{(\Lambda^2 + r_l)^2}\bigg]. \tag{10.562}$$

The fixed points of the renormalization group equations is define obviously by

$$r_* = 4\bigg[r_* + 3c\frac{u_*}{\Lambda^2 + r_*}\bigg]. \tag{10.563}$$

$$u_* = 2^{4-d}\bigg[u_* - 9c\frac{u_*^2}{(\Lambda^2 + r_*)^2}\bigg]. \tag{10.564}$$

We find the solutions

$$\text{Gaussian fixed point} \ : \ r_* = 0 \ , \ u_* = 0, \tag{10.565}$$

and (by assuming that u_* is sufficiently small)

$$\text{Wilson} - \text{Fisher fixed point} \ : \ r_* = -\frac{4cu_*}{\Lambda^2} \ , \ u_* = \frac{\Lambda^4}{9c}(1 - 2^{d-4}). \tag{10.566}$$

For $\epsilon = 4 - d$ small the non trivial (interacting) Wilson-Fisher fixed point approaches the trivial (free) Gaussian fixed point as

$$r_* = -\frac{4}{9}\Lambda^2 \epsilon \ln 2 \ , \ u_* = \frac{\Lambda^4}{9c}\epsilon \ln 2. \tag{10.567}$$

The value u_* controls the strength of the interaction of the low energy (infrared) physics of the system.

10.5.4 The Critical Exponents ν

In the Gaussian model the recursion formula reads simply $r_{l+1} = 4r_l$ and hence we have two possible solutions. At $T = T_c$ the mass parameter r_0 must be zero and hence $r_l = 0$ for all l, i.e. $r_0 = 0$ is a fixed point. For $T \neq T_c$ the mass parameter r_0 is non zero and hence $r_l = 4^l r_0 \longrightarrow \infty$ for $l \longrightarrow \infty$ ($r_0 = \infty$ is the second fixed point). For T near T_c the mass parameter r_0 is linear in $T - T_c$.

In the ϕ^4 model the situation is naturally more complicated. We can be at the critical temperature $T = T_c$ without having the parameters r_0 and u_0 at their fixed point values. Indeed, as we have already seen, for any value u_0 there will be a critical value $r_{0c} = r_{0c}(u_0)$ of r_0 corresponding to $T = T_c$. At $T = T_c$ we have $r_l \longrightarrow r_*$ and $u_l \longrightarrow u_*$ for $l \longrightarrow \infty$. For $T \neq T_c$ we will have in general a different limit for large l.

The critical exponent ν can be calculated by studying the behavior of the theory only for T near T_c. As stated above $r_l(T_c) \longrightarrow r_*$ and $u_l(T_c) \longrightarrow u_*$ for $l \longrightarrow \infty$. From the analytic property of the recursion formulas we conclude that $r_l(T)$ and $u_l(T)$ are analytic functions of the temperature and hence near T_c we should have $r_l(T) = r_l(T_c) + (T - T_c)r_l^{'}(T_c) + ...$ and $u_l(T) = u_l(T_c) + (T - T_c)u_l^{'}(T_c) + ...$ and as a consequence $r_l(T)$ and $u_l(T)$ are close to the fixed point values for sufficiently large l and sufficiently small $T - T_c$. We are thus led in a natural way to studying the recursion formulas only around the fixed point, i.e. to studying the linearized recursion formulas.

The Linearized Recursion Formulas: Now we linearize the recursion formulas around the fixed point. We find without any approximation

$$r_{l+1} - r_* = \left[4 - \frac{12cu_*}{(\Lambda^2 + r_l)(\Lambda^2 + r_*)}\right](r_l - r_*) + \frac{12c}{\Lambda^2 + r_*}(u_l - u_*) + 12cu_*\left[\frac{1}{\Lambda^2 + r_l} - \frac{1}{\Lambda^2 + r_*}\right]. \tag{10.568}$$

$$u_{l+1} - u_* = 2^{4-d}\frac{9cu_*^2(2\Lambda^2 + r_* + r_l)}{(\Lambda^2 + r_l)^2(\Lambda^2 + r_*)^2}(r_l - r_*) + 2^{4-d}\left[1 - \frac{9c}{(\Lambda^2 + r_l)^2}(u_l + u_*)\right](u_l - u_*). \tag{10.569}$$

Keeping only linear terms we find

$$r_{l+1} - r_* = \left[4 - \frac{12cu_*}{(\Lambda^2 + r_*)^2}\right](r_l - r_*) + \frac{12c}{\Lambda^2 + r_*}(u_l - u_*). \tag{10.570}$$

$$u_{l+1} - u_* = 2^{4-d}\frac{18cu_*^2}{(\Lambda^2+r_*)^3}(r_l - r_*) + 2^{4-d}\bigg[1 - \frac{18cu_*}{(\Lambda^2+r_*)^2}\bigg](u_l - u_*). \tag{10.571}$$

This can be put into the matrix form

$$\begin{pmatrix} r_{l+1} - r_* \\ u_{l+1} - u_* \end{pmatrix} = M \begin{pmatrix} r_l - r_* \\ u_l - u_* \end{pmatrix}. \tag{10.572}$$

The matrix M is given by

$$M = \begin{pmatrix} 4 - \frac{12cu_*}{(\Lambda^2+r_*)^2} & \frac{12c}{\Lambda^2+r_*} \\ 2^{4-d}\frac{18cu_*^2}{(\Lambda^2+r_*)^3} & 2^{4-d}(1 - \frac{18cu_*}{(\Lambda^2+r_*)^2}) \end{pmatrix} = \begin{pmatrix} 4 - \frac{4}{3}\epsilon\ln 2 & \frac{12c}{\Lambda^2}(1 + \frac{4}{9}\epsilon\ln 2) \\ 0 & 1 - \epsilon\ln 2 \end{pmatrix}. \tag{10.573}$$

After n steps of the renormalization group we will have

$$\begin{pmatrix} r_{l+n} - r_* \\ u_{l+n} - u_* \end{pmatrix} = M^n \begin{pmatrix} r_l - r_* \\ u_l - u_* \end{pmatrix}. \tag{10.574}$$

In other words for large n the matrix M^n is completely dominated by the largest eigenvalue of M.

Let λ_1 and λ_2 be the eigenvalues of M with eigenvectors w_1 and w_2 respectively such that $\lambda_1 > \lambda_2$. Clearly for $u_* = 0$ we have

$$\lambda_1 = 4 \ , \ w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\lambda_2 = 1 \ , \ w_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{10.575}$$

The matrix M is not symmetric and thus diagonalization is achieved by an invertible (and not an orthogonal) matrix U. We write

$$M = UDU^{-1}. \tag{10.576}$$

The eigenvalues λ_1 and λ_2 can be determined from the trace and determinant which are given by

$$\lambda_1 + \lambda_2 = M_{11} + M_{22} \ , \ \lambda_1\lambda_2 = M_{11}M_{22} - M_{12}M_{21}. \tag{10.577}$$

We obtain immediately

$$\lambda_1 = 4 - \frac{4}{3}\epsilon\ln 2 \ , \ \lambda_2 = 1 - \epsilon\ln 2. \tag{10.578}$$

The corresponding eigenvectors are

$$w_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \ w_2 = \begin{pmatrix} -\frac{4c}{\Lambda^2}(1 + \frac{5}{9}\epsilon\ln 2) \\ 1 \end{pmatrix}. \tag{10.579}$$

We write the equation $Mw_k = \lambda_k w_k$ as (with $(w_k)_j = w_{jk}$)

$$M_{ij}w_{jk} = \lambda_k w_{ik}. \tag{10.580}$$

The identity $M=\sum_k \lambda_k|\lambda_k><\lambda_k|$ can be rewritten as

$$M_{ij}=\sum_k \lambda_k w_{ik}v_{kj}=\lambda_1 w_{i1}v_{1j}+\lambda_2 w_{i2}v_{2j}. \quad (10.581)$$

The vectors v_k are the eigenvectors of M^T with eigenvalues λ_k respectively, viz (with $(v_k)_j=v_{kj}$)

$$M^T_{ij}v_{kj}=v_{kj}M_{ji}=\lambda_k v_{ki}. \quad (10.582)$$

We find explicitly

$$v_1=\begin{pmatrix} 1 \\ \frac{4c}{\Lambda^2}(1+\frac{5}{9}\epsilon\ln 2)\end{pmatrix}\ ,\ v_2=\begin{pmatrix} 0 \\ 1\end{pmatrix}. \quad (10.583)$$

The orthonormality condition is then

$$\sum_j v_{kj}w_{jl}=\delta_{kl}. \quad (10.584)$$

From the result (10.581) we deduce immediately that

$$\begin{aligned} M^n_{ij} &= \lambda_1^n w_{i1}v_{1j}+\lambda_2^n w_{i2}v_{2j} \\ &\simeq \lambda_1^n w_{i1}v_{1j}. \end{aligned} \quad (10.585)$$

The linearized recursion formulas take then the form

$$r_{l+n}-r_*\simeq \lambda_1^n w_{11}\big(v_{11}(r_l-r_*)+v_{12}(u_l-u_*)\big). \quad (10.586)$$

$$u_{l+n}-u_*\simeq \lambda_1^n w_{21}\big(v_{11}(r_l-r_*)+v_{12}(u_l-u_*)\big). \quad (10.587)$$

Since $r_l=r_l(T)$ and $u_l=u_l(T)$ are close to the fixed point values for sufficiently large l and sufficiently small $T-T_c$ we conclude that r_l-r_* and u_l-u_* are both linear in $T-T_c$ and as a consequence

$$v_{11}(r_l-r_*)+v_{12}(u_l-u_*)=c_l(T-T_c). \quad (10.588)$$

The linearized recursion formulas become

$$r_{l+n}-r_*\simeq c_l\lambda_1^n w_{11}(T-T_c). \quad (10.589)$$

$$u_{l+n}-u_*\simeq c_l\lambda_1^n w_{21}(T-T_c). \quad (10.590)$$

The Critical Exponent ν: The correlation length corresponding to the initial action is given by

$$\xi_0(T)=X(r_0(T),u_0(T)). \quad (10.591)$$

After $l+n$ renormalization group steps the correlation length becomes

$$\xi_{l+n}(T)=X(r_{l+n}(T),u_{l+n}(T)). \quad (10.592)$$

At each renormalization group step we scale the momenta as $p \longrightarrow 2p$ which corresponds to scaling the distances as $x \longrightarrow x/2$. The correlation length is a measure of distance and thus one must have

$$X(r_{l+n}, u_{l+n}) = 2^{-l-n} X(r_0, u_0). \tag{10.593}$$

From equations (10.589) and (10.590) we have

$$(r_{l+n+1} - r_*)|_{T-T_c=\tau/\lambda_1} = (r_{l+n} - r_*)|_{T-T_c=\tau} \ , \ (u_{l+n+1} - u_*)|_{T-T_c=\tau/\lambda_1} = (u_{l+n} - u_*)|_{T-T_c=\tau}. \tag{10.594}$$

Hence

$$X(r_{l+n+1}, u_{l+n+1})|_{T=T_c+\tau/\lambda_1} = X(r_{l+n}, u_{l+n})|_{T=T_c+\tau}. \tag{10.595}$$

By using the two results (10.593) and (10.595) we obtain

$$2^{-l-n-1}\xi_0(T_c + \tau/\lambda_1) = 2^{-l-n}\xi_0(T_c + \tau). \tag{10.596}$$

We expect

$$\xi_0(T_c + \tau) \propto \tau^{-\nu}. \tag{10.597}$$

In other words

$$\frac{1}{2}(\frac{\tau}{\lambda_1})^{-\nu} = \tau^{-\nu} \Leftrightarrow \lambda_1^{\nu} = 2 \Leftrightarrow \nu = \frac{\ln 2}{\ln \lambda_1}. \tag{10.598}$$

10.5.5 The Critical Exponent η

The ultra local recursion formula (10.531) used so far do not lead to a wave function renormalization since all momentum dependence of Feynman diagrams has been dropped and as a consequence the value of the anomalous dimension η within this approximation is 0. This can also be seen from the field scaling (10.552) with the choice (10.555) which are made at every renormalization group step and hence the wave function renormalization is independent of the momentum.

In any case we can see from equation (10.554) that the wave function renormalization at the first renormalization group step is given by

$$Z = \frac{\alpha_0^2}{2^{2+d}}. \tag{10.599}$$

From the other hand we have already established that the scaling behavior of $Z(\lambda)$ for small λ (the limit in which we approach the infrared stable fixed point) is λ^{η}. In our case $\lambda = 1/2$ and hence we must have

$$Z = 2^{-\eta}. \tag{10.600}$$

Let α_* be the fixed value of the sequence α_i. Then from the above two equations we obtain the formula

$$\eta = -\frac{\ln Z}{\ln 2} = d + 2 - \frac{2\ln \alpha_*}{\ln 2}. \tag{10.601}$$

As discussed above since $\alpha_* = 2^{1+d/2}$ for the ultra local recursion formula (10.531) we get immediately $\eta = 0$.

To incorporate a non zero value of the critical exponent η we must go to the more accurate yet more complicated recursion formulas (10.523) and (10.524). The field scaling at each renormalization group step is a different number α_i. These numbers are determined from the normalization condition (10.527).

Recall that integrating out the momenta $1 \leq |k|/\Lambda \leq 2$ resulted in the field $\phi_1(x) = \sum_{\vec{m}} \sum_{l=1}^{\infty} \psi_{\vec{m}l}(x)\phi_{\vec{m}l}$ which was expressed in terms of the field $\phi^{'}(x) = \phi^{(1)}$ which appears in the final action as $\phi_1(x) = 2^{-d/2}\alpha_0\phi^{'}(x/2)$. After n renormalization group steps we integrate out the momenta $2^{1-n} \leq |k|/\Lambda \leq 2$ which results in the field $\phi_n(x) = \sum_{\vec{m}} \sum_{l=n}^{\infty} \psi_{\vec{m}l}(x)\phi_{\vec{m}l}$. However the action will be expressed in terms of the field $\phi^{'} = \phi^{(n)}$ defined by

$$\phi_n(x) = 2^{-nd/2}\alpha_0\alpha_1....\alpha_{n-1}\phi^{(n)}(x/2^n). \tag{10.602}$$

We are interested in the 2−point function

$$< \phi_{\vec{m}l}\phi_{\vec{m}'l'} > \ = \ \frac{1}{Z}\int \prod_{l_1=0}^{\infty}\prod_{\vec{m}_1} d\phi_{\vec{m}_1 l_1}\phi_{\vec{m}l}\phi_{\vec{m}'l'}\, e^{-S_0[\phi]}. \tag{10.603}$$

Let us concentrate on the integral with $l_1 = l$ and $\vec{m}_1 = \vec{m}$ and assume that $l^{'} > l$. We have then the integral

$$...\int d\phi_{\vec{m}l}\phi_{\vec{m}l}\int \prod_{l_1=0}^{l-1}\prod_{\vec{m}_1} d\phi_{\vec{m}_1 l_1} e^{-S_0[\phi]} \ = \ ...\int d\phi_{\vec{m}l}\phi_{\vec{m}l}\ e^{-S_l[\phi^{'}]}. \tag{10.604}$$

We have $\phi^{'} = \phi^{(l)}$ where $\phi^{(l)}$ contains the momenta $|k|/\Lambda \leq 1/2^{l-1}$. Since $\phi_{\vec{m}l}$ is not integrated we have $\phi_{\vec{m}l} = \alpha_0...\alpha_{l-1}(K\rho/2)^{-1/2}y_{\vec{m}}$ which is the generalization of $\phi_{\vec{m}l} = \alpha_0\phi^{'}_{\vec{m}l-1}$. We want now to further integrate $\phi_{\vec{m}l}$. The final result is similar to (10.504) except that we have an extra factor of $y_{\vec{m}}$ and $z_{\vec{m}}$ contains all the modes with $l_1 > l$. The integral thus clearly vanishes because it is odd under $y_{\vec{m}} \longrightarrow -\vec{y}_{\vec{m}}$.

We conclude that we must have $l^{'} = l$ and $\vec{m}^{'} = \vec{m}$ otherwise the above 2−point function vanishes. After few more calculations we obtain

$$< \phi_{\vec{m}l}\phi_{\vec{m}'l'} > \ = \ \delta_{ll'}\delta_{\vec{m}\vec{m}'}\alpha_0^2...\alpha_{l-1}^2(\frac{K\rho}{2})^{-1}\frac{\int\prod_{l_1=l+1}\prod_{\vec{m}_1} d\phi_{\vec{m}_1 l_1}\prod_{\vec{m}_1} M_l(z_{\vec{m}_1}).R_l(z_{\vec{m}})}{\int\prod_{l_1=l+1}\prod_{\vec{m}_1} d\phi_{\vec{m}_1 l_1}\prod_{\vec{m}_1} M_l(z_{\vec{m}_1})}. \tag{10.605}$$

The function M_l is given by the same formula (10.505) with the substitutions $M_0 \longrightarrow M_l$, $W_0 \longrightarrow W_l$ and $Q_0 \longrightarrow Q_l$. The variable $z_{\vec{m}}$ is given explicitly by

$$z_{\vec{m}} = \big(\frac{K\rho}{2w}\big)^{1/2}2^{-d/2}\alpha_l\phi^{(l+1)}(x_0/2). \tag{10.606}$$

The function $R_l(z)$ is defined by

$$R_l(z) = M_l^{-1}(z)\int dyy^2\exp\big(...\big). \tag{10.607}$$

The exponent is given by the same exponent of equation (10.505) with the substitutions $M_0 \longrightarrow M_l$, $W_0 \longrightarrow W_l$ and $Q_0 \longrightarrow Q_l$.

An order of magnitude formula for the 2−point function can be obtained by replacing the function $R_l(z)$ by $R_l(0)$. We obtain then

$$< \phi_{\vec{m}l}\phi_{\vec{m}'l'} > \ = \ \delta_{ll'}\delta_{\vec{m}\vec{m}'}\alpha_0^2...\alpha_{l-1}^2(\frac{K\rho}{2})^{-1}R_l(0). \quad (10.608)$$

At a fixed point of the recursion formulas we must have

$$W_l \longrightarrow W_* \ , \ Q_l \longrightarrow Q_* \Leftrightarrow R_l \longrightarrow R_*, \quad (10.609)$$

and

$$\alpha_l \longrightarrow \alpha_*. \quad (10.610)$$

The 2−point function is therefore given by

$$< \phi_{\vec{m}l}\phi_{\vec{m}'l'} > \ \propto \ \delta_{ll'}\delta_{\vec{m}\vec{m}'}\alpha_*^{2l}(\frac{K\rho}{2})^{-1}R_*(0). \quad (10.611)$$

The modes $< \phi_{\vec{m}l} >$ correspond to the momentum shell $2^{-l} \leq |k|/\Lambda \leq 2^{1-l}$, i.e. $k \sim \Lambda 2^{-l}$. From the other hand the 2−point function is expected to behave as

$$< \phi_{\vec{m}l}\phi_{\vec{m}'l'} > \ \propto \ \delta_{ll'}\delta_{\vec{m}\vec{m}'}\frac{1}{k^{2-\eta}}, \quad (10.612)$$

where η is precisely the anomalous dimension. By substituting $k \sim \Lambda 2^{-l}$ in this last formula we obtain

$$< \phi_{\vec{m}l}\phi_{\vec{m}'l'} > \ \propto \ \delta_{ll'}\delta_{\vec{m}\vec{m}'}\Lambda^{\eta-2}2^{l(2-\eta)}. \quad (10.613)$$

By comparing the l−dependent bits in (10.611) and (10.613) we find that the anomalous dimension is given by

$$2^{2-\eta} = \alpha_*^2 \Rightarrow \eta = 2 - \frac{2\ln\alpha_*}{\ln 2}. \quad (10.614)$$

10.6 Exercises and Problems

Power Counting Theorems for Dirac and Vector Fields

- Derive power counting theorems for theories involving scalar as well as Dirac and vector fields by analogy with what we have done for pure scalar field theories.
- What are renormalizable field theories in $d = 4$ dimensions involving spin 0, 1/2 and 1 particles.
- Discuss the case of QED.

Renormalization Group Analysis for The Effective Action

- In order to study the system in the borken phase we must perform a renormalization group analysis of the effective action and study its behavior as a function of the mass parameter. Carry out explicitly this program.

A

Exams

Midterm Examination QFT
Master 2
2012-2013
2 h

Solve 3 exercises out of 6 as follows:

- Choose between 1 and 2.
- Choose between 3 and 4.
- Choose between 5 and 6.

Exercise 1: We consider the two Euclidean integrals

$$I(m^2)=\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}.$$

$$J(p^2,m^2)=\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}\frac{1}{(p-k)^2+m^2}.$$

- Determine in each case the divergent behavior of the integral.
- Use dimensional regularization to compute the above integrals. Determine in each case the divergent part of the integral. In the case of $J(p^2,m^2)$ assume for simplicity zero external momentum $p=0$.

Exercise 2: The two integrals in exercise 1 can also be regularized using a cutoff Λ. First we perform Laplace transform as follows

$$\frac{1}{k^2+m^2}=\int_0^\infty d\alpha e^{-\alpha(k^2+m^2)}.$$

- Do the integral over k in $I(m^2)$ and $J(p^2,m^2)$. In the case of $J(p^2,m^2)$ assume for simplicity zero external momentum $p=0$.
- The remaining integral over α is regularized by replacing the lower bound $\alpha=0$ by $\alpha=1/\Lambda^2$. Perform the integral over α explicitly. Determine the divergent part in each case.

  Hint: Use the exponential-integral function

$$Ei(-x)=\int_{-\infty}^{-x}\frac{e^t}{t}dt=\mathbf{C}+\ln x+\int_0^x dt\frac{e^{-t}-1}{t}.$$

Exercise 3: Let z_i be a set of complex numbers, θ_i be a set of anticommuting Grassmann numbers and let M be a hermitian matrix. Perform the following integrals

$$\int\prod_i dz_i^+dz_ie^{-M_{ij}z_i^+z_j-z_i^+j_i-j_i^+z_i}.$$

$$\int\prod_i d\theta_i^+d\theta_ie^{-M_{ij}\theta_i^+\theta_i-\theta_i^+\eta_i-\eta_i^+\theta_i}.$$

Exercise 4**:** Let $S(r,\theta)$ be an action dependent on two degrees of freedom r and θ which is invariant under 2−dimensional rotations, i.e. $\vec{r}=(r,\theta)$. We propose to gauge fix the following 2−dimensional path integral

$$W=\int e^{iS(\vec{r})}d^2\vec{r}.$$

We will impose the gauge condition

$$g(r,\theta)=0.$$

- Show that

$$|\frac{\partial g(r,\theta)}{\partial\theta}|_{g=0}\int d\phi\delta\Big(g(r,\theta+\phi)\Big)=1.$$

- Use the above identity to gauge fix the path integral W.

Exercise 5**:** The gauge fixed path integral of quantum electrodynamics is given by

$$Z[J] = \int\prod_\mu\mathcal{D}A_\mu\exp\Big(-i\int d^4x\frac{(\partial_\mu A^\mu)^2}{2\xi}-\frac{i}{4}\int d^4xF_{\mu\nu}F^{\mu\nu}-i\int d^4xJ_\mu A^\mu\Big).$$

- Derive the equations of motion.
- Compute $Z[J]$ in a closed form.
- Derive the photon propagator.

Exercise 6**:** We consider phi-four interaction in 4 dimensions. The action is given by

$$S[\phi]=\int d^4x\Big[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi-\frac{1}{2}m^2\phi^2-\frac{\lambda}{4!}(\phi^2)^2\Big].$$

- Write down Feynman rules in momentum space.
- Use Feynman rules to derive the 2−point proper vertex $\Gamma^2(p)$ up to the one-loop order. Draw the corresponding Feynman diagrams.
- Use Feynman rules to derive the 4−point proper vertex $\Gamma^4(p_1,p_2,p_3,p_4)$ up to the one-loop order. Draw the corresponding Feynman diagrams.
- By assuming that the momentum loop integrals are regularized perform one-loop renormalization of the theory. Impose the two conditions

$$\Gamma^2(0)=m_R^2\ ,\ \Gamma^4(0,0,0,0)=\lambda_R.$$

  Determine the bare coupling constants m^2 and λ in terms of the renormalized coupling constants m_R^2 and λ_R.
- Determine $\Gamma^2(p)$ and $\Gamma^4(p_1,p_2,p_3,p_4)$ in terms of the renormalized coupling constants.

Final Examination QFT
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2012-2013
2 h

Exercise 1: We consider the two Euclidean integrals

$$I(m^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2}.$$

$$J(p^2, m^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2} \frac{1}{(p-k)^2 + m^2}.$$

- Determine in each case the divergent behavior of the integral.
- Use dimensional regularization to compute the above integrals. Determine in each case the divergent part of the integral. In the case of $J(p^2, m^2)$ assume for simplicity zero external momentum $p = 0$.

Exercise 2: The gauge fixed path integral of quantum electrodynamics is given by

$$Z[J] \quad = \quad \int \prod_{\mu} \mathcal{D}A_\mu \exp\left(-i\int d^4x \frac{(\partial_\mu A^\mu)^2}{2\xi} - \frac{i}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} - i\int d^4x J_\mu A^\mu\right).$$

- Derive the equations of motion.
- Compute $Z[J]$ in a closed form.
- Derive the photon propagator.

Midterm 1 Examination QFT
Master 1
2011-2012
2 h

Exercise 1:

- Write down an expression of the free scalar field in terms of creation and annihilation.
- Compute the 2−point function
$$D_F(x_1-x_2) =< 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)|0> .$$
- Compute in terms of D_F the 4−point function
$$D(x_1,x_2,x_3,x_4) =< 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)\hat{\phi}(x_4)|0> .$$
- Without calculation what is the value of the 3−point function $< 0|T\hat{\phi}(x_1)\hat{\phi}(x_2)\hat{\phi}(x_3)|0 >$. Explain.

Exercise 2: The electromagnetic field is a vector in four dimensional Minkowski spacetime denoted by
$$A^\mu = (A^0, \vec{A}).$$
A^0 is the electric potential and $\vec{A}$ is the magnetic vector potential. The Dirac Lagrangian density with non zero external electromagnetic field is given
$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma_\mu\psi A^\mu.$$

- Derive the Euler-Lagrange equation of motion. This will be precisely the Dirac equation in an external electromagnetic field.

Exercise 3:

- Compute the integral over p^0:
$$\int d^3\vec{p}\int dp^0\ \delta(p^2-m^2).$$
What do you conclude for the action of Lorentz transformations on:
$$\frac{d^3\vec{p}}{2E_p}.$$

Midterm 2 Examination QFT
Master 1
2011-2012
2 h

Exercise 1:

- Write down an expression of the free scalar field in terms of creation and annihilation.
- Compute for time like intervals $(x-y)^2 > 0$ the commutator

$$[\hat{\phi}(x), \hat{\phi}(y)].$$

  Can we measure simultaneously the field at the two points x and y.

- What happens for space like intervals.

Exercise 2: The Yukawa Lagrangian density describes the interaction between spinorial and scalar fields. It is given by

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - \phi^2) - g\phi\bar{\psi}\psi.$$

- Derive the Euler-Lagrange equation of motion.

Exercise 3:

- Show that Feynman propagator in one dimension is given by

$$G_{\vec{p}}(t-t') = \int \frac{dE}{2\pi}\frac{i}{E^2 - E_{\vec{p}}^2 + i\epsilon}e^{-iE(t-t')} = \frac{e^{-iE_{\vec{p}}|t-t'|}}{2E_{\vec{p}}}.$$

Exercise 4:

- What is the condition satisfied by the Dirac matrices in order for the Dirac equation to be covariant.
- Write down the spin representation of the infinitesimal Lorentz transformations

$$\Lambda = 1 - \frac{i}{2}\epsilon_{\mu\nu}\mathcal{J}^{\mu\nu}.$$

Exercise 5:

- Show that gamma matrices in two dimensions are given by

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \ \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

- Write down the general solution of Dirac equation in two dimensions in the massless limit.

Final Examination QFT
Master 1
2011-2012
2 h

Exercise 1:

- Write down the vacuum stability condition.
- Write down Gell-Mann-Low formulas.
- Write down the scattering $S-$matrix.
- Write down the Lehmannn-Symanzik-Zimmermann (LSZ) reduction formula which expresses the transition probability amplitude between $1-$particle states in terms of the $2-$point function.
- Write down the Lehmannn-Symanzik-Zimmermann (LSZ) reduction formula which expresses the transition probability amplitude between $2-$particle states in terms of the $4-$point function.
- Write down Wick's theorem. Apply for 2, 4 and 6 fields.

Exercise 2: We consider phi-cube theory in four dimensions where the interaction is given by the Lagrangian density

$$\mathcal{L}_{\rm int} = -\frac{\lambda}{3!}\phi^3.$$

- Compute the $0-$point function up to the second order of perturbation theory and express the result in terms of Feynman diagrams.
- Compute the $1-$point function up to the second order of perturbation theory and express the result in terms of Feynman diagrams.
- Compute the $2-$point function up to the second order of perturbation theory and express the result in terms of Feynman diagrams.
- Compute the connected $2-$point function up to the second order of perturbation theory and express the result in terms of Feynman diagrams.

Exercise 3: We consider phi-four theory in four dimensions where the interaction is given by the Lagrangian density

$$\mathcal{L}_{\rm int} = -\frac{\lambda}{4!}\phi^4.$$

- Compute the $4-$point function up to the first order of perturbation theory and express the result in terms of Feynman diagrams.

First Examination QFT
Master 1
2010-2011
2.5h

Problem 1 For a real scalar field theory the one-particle states are defined by $|\vec{p}>=\sqrt{2E(\vec{p})}\hat{a}(\vec{p})^+|0>$.

- Compute the energy of this state. We give

$$[\hat{a}(\vec{p}),\hat{a}(\vec{q})^+]=(2\pi)^3\delta^3(\vec{p}-\vec{q}).$$

- Show that the scalar product $<\vec{p}|\vec{q}>$ is Lorentz invariant. We give

$$x^{0'}=\gamma(x^0-\beta x^1)\ ,\ x^{1'}=\gamma(x^1-\beta x^0)\ ,\ x^{2'}=x^2\ ,\ x^{3'}=x^3.$$

Problem 2 Show the Lorentz invariance of the D'Alembertian $\partial_\mu\partial^\mu=\partial_t^2-\vec{\nabla}^2$.

Problem 3 Determine the transformation rule under Lorentz transformations of $\bar{\psi}$, $\bar{\psi}\psi$, $\bar{\psi}\gamma^\mu\psi$. We give

$$\psi(x)\longrightarrow\psi^{'}(x^{'})=S(\Lambda)\psi(x).$$

Problem 4 **(optional)** Show that

$$<0|T\hat{\phi}(x)\hat{\phi}(y)|0>=\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-y)}.$$

We give

$$\hat{\phi}(x)\quad=\quad\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{p})}}\bigg(\hat{a}(\vec{p})e^{-ipx}+\hat{a}(\vec{p})^+e^{ipx}\bigg).$$

Problem 5**(optional)** Compute the total momentum operator of a quantum real scalar field in terms of the creation and annihilation operators $\hat{a}(\vec{p})^+$ and $\hat{a}(\vec{p})$. We give

$$\hat{P}_i=\int d^3x\hat{\pi}\partial_i\hat{\phi}.$$

Second Examination QFT
Master 1
2010-2011
2.5h

Problem 1 Show that the scalar field operator $\hat{\phi}_I(x)$ and the conjugate momentum field operator $\hat{\pi}_I(x)$ (operators in the interaction picture) are free field operators.

Problem 2 Calculate the 2−point function $< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2))|0 >$ in $\phi-$four theory up to the second order in preturbation theory using the Gell-Mann Low formula and Wick's theorem. Express each order in perturbation theory in terms of Feynman diagrams.

Final Examination QFT
Master 1
2010-2011
2.5h

Problem 1 We consider a single forced harmonic oscillator given by the equation of motion

$$(\partial_t^2 + E^2)Q(t) = J(t). \tag{A.1}$$

1) Show that the $S-$matrix defined by the matrix elements $S_{mn} =< m \text{ out}|n \text{ in} >$ is unitary.

2) Determine S from solving the equation

$$S^{-1}\hat{a}_{\rm in}S = \hat{a}_{\rm out} = \hat{a}_{\rm in} + \frac{i}{\sqrt{2E}}j(E). \tag{A.2}$$

3) Compute the probability $| < n\ out|0\ in > |^2$.

4) Determine the evolution operator in the interaction picture $\Omega(t)$ from solving the Schrodinger equation

$$i\partial_t\Omega(t) = \hat{V}_I(t)\Omega(t)\ ,\ \hat{V}_I(t) = -J(t)\hat{Q}_I(t). \tag{A.3}$$

5) Deduce from 4) the $S-$matrix and compare with the result of 2).

Problem 2 The probability amplitudes for a Dirac particle (antiparticle) to propagate from the spacetime point y (x) to the spacetime x (y) are

$$S_{ab}(x-y) =< 0|\hat{\psi}_a(x)\bar{\hat{\psi}}_b(y)|0 > . \tag{A.4}$$

$$\bar{S}_{ba}(y-x) =< 0|\bar{\hat{\psi}}_b(y)\hat{\psi}_a(x)|0 > . \tag{A.5}$$

1) Compute S and $\bar{S}$ in terms of the Klein-Gordon propagator $D(x-y)$ given by

$$D(x-y) \quad = \quad \int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}p(x-y)}. \tag{A.6}$$

2) Show that the retarded Green's function of the Dirac equation is given by

$$(S_R)_{ab}(x-y) \quad = \quad < 0|\{\hat{\psi}_a(x), \bar{\hat{\psi}}_b(y)\}|0 > . \tag{A.7}$$

3) Verify that S_R satisfies the Dirac equation

$$(i\hbar\gamma^\mu\partial_\mu^x - mc)_{ca}(S_R)_{ab}(x-y) \quad = \quad i\frac{\hbar}{c}\delta^4(x-y)\delta_{cb}. \tag{A.8}$$

4) Derive an expression of the Feynman propagator in terms of the Dirac fields $\hat{\psi}$ and $\bar{\hat{\psi}}$ and then write down its Fourier Expansion.

Recess Examination QFT
Master 1
2010-2011
2.0h

Problem 1 We consider a single forced harmonic oscillator given by the equation of motion

$$(\partial_t^2 + E^2)Q(t) = J(t). \tag{A.9}$$

1) Determine S from solving the equation

$$S^{-1}\hat{a}_{\rm in}S = \hat{a}_{\rm out} = \hat{a}_{\rm in} + \frac{i}{\sqrt{2E}}j(E). \tag{A.10}$$

2) Compute the probability $| < n\ out|0\ in > |^2$.

3) Determine the evolution operator in the interaction picture $\Omega(t)$ from solving the Schrodinger equation

$$i\partial_t\Omega(t) = \hat{V}_I(t)\Omega(t)\ ,\ \hat{V}_I(t) = -J(t)\hat{Q}_I(t). \tag{A.11}$$

4) Deduce from 3) the $S-$matrix and compare with the result of 1).

Problem 2 Calculate the $2-$point function $< 0|T(\hat{\phi}(x_1)\hat{\phi}(x_2))|0 >$ in $\phi-$four theory up to the 1st order in preturbation theory using the Gell-Mann Low formula and Wick's theorem. Express each order in perturbation theory in terms of Feynman diagrams.

Problem 3 Show that

$$< 0|T\hat{\phi}(x)\hat{\phi}(y)|0 >= \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-y)}.$$

We give

$$\hat{\phi}(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E(\vec{p})}}\bigg(\hat{a}(\vec{p})e^{-ipx} + \hat{a}(\vec{p})^+e^{ipx}\bigg).$$

Examination QFT
Master 2
2011-2012
Take Home

Problem 1:

1) Compute the electron $2-$point function in configuration space up to one-loop using the Gell-Mann Low formula and Wick's theorem. Write down the corresponding Feynman diagrams.

2) Compute the electron $2-$point function in momentum space up to one-loop using Feynman rules.

3) Use dimensional regularization to evaluate the electron self-energy. Add a small photon mass to regularize the IR behvavior. What is the UV behavior of the electron self-energy.

4) Determine the physical mass of the electron at one-loop.

5) Determine the wave-function renormalization Z_2 and the counter term $\delta_2 = 1 - Z_2$ up to one-loop.

Problem 2

1) Write down all Feynman diagrams up to one-loop which contribute to the probability amplitude of the process $e^-(p) + \mu^-(k) \longrightarrow e^-(p^{'}) + \mu^-(k^{'})$.

2) Write down using Feynman rules the tree level probability amplitude of the process $e^-(p) + \mu^-(k) \longrightarrow e^-(p^{'}) + \mu^-(k^{'})$. Write down the probability amplitude of this process at one-loop due to the electron vertex correction.

3) Use Feynman parameters to express the product of propagators as a single propagator raised to some power of the form

$$\frac{1}{\left[L^2 - \Delta + i\epsilon\right]^q}. \quad \text{(A.12)}$$

Determine the shifted momentum L, the effective mass Δ and the power q. Add a small photon mass μ^2.

4) Verify the relations

$$(\gamma.p)\gamma^\mu = 2p^\mu - \gamma^\mu(\gamma.p)$$
$$\gamma^\mu(\gamma.p) = 2p^\mu - (\gamma.p)\gamma^\mu$$
$$(\gamma.p)\gamma^\mu(\gamma.p^{'}) = 2p^\mu(\gamma.p^{'}) - 2\gamma^\mu p.p^{'} + 2p^{'\mu}(\gamma.p) - (\gamma.p^{'})\gamma^\mu(\gamma.p). \quad \text{(A.13)}$$

5) We work in d dimensions. Use Lorentz invariance, the properties of the gamma matrices in d dimensions and the results of question 4) to show that we can replace [1]

$$\gamma^\lambda.i(\gamma.l^{'} + m_e).\gamma^\mu.i(\gamma.l + m_e)\gamma_\lambda \quad \longrightarrow \quad \gamma^\mu A + (p + p^{'})^\mu B + (p - p^{'})^\mu C. \quad \text{(A.14)}$$

[1] Very Difficult.

Determine the coefficients A, B and C.

6) Use Gordon's identity to show that the vertex function $\Gamma(p^{'},p)$ is of the form

$$\Gamma^{\mu}(p^{'},p)=\gamma^{\mu}F_1(q^2)+\frac{i\sigma^{\mu\nu}q_{\nu}}{2m_e}F_2(q^2). \tag{A.15}$$

Determine the form factors F_1 and F_2.

7) Compute the integrals

$$\int\frac{d^dL_E}{(2\pi)^d}\frac{L_E^2}{(L_E^2+\Delta)^3}\ ,\ \int\frac{d^dL_E}{(2\pi)^d}\frac{1}{(L_E^2+\Delta)^3}. \tag{A.16}$$

8) Calculate the form factor $F_1(q^2)$ explicitly in dimensional regularization. Determine the UV behavior.

9) Compute the renormalization constant Z_1 or equivalently the counter term $\delta_1=Z_1-1$ at one-loop.

10) Prove the Ward identity $\delta_1=\delta_2$ [2].

Problem 3

1) Write down using Feynman rules the photon self-energy $i\Pi_2^{\mu\nu}(q)$ at one-loop.

2) Use dimensional regularization to show that

$$\Pi_2^{\mu\nu}(q)=\Pi_2(q^2)(q^2\eta^{\mu\nu}-q^{\mu}q^{\nu}). \tag{A.17}$$

Determine $\Pi_2(q^2)$. What is the UV behavior.

3) Compute at one-loop the counter term $\delta_3=Z_3-1$.

4) Compute at one-loop the effective charge $e^2_{\rm eff}$. How does the effective charge behave at high energies.

Problem 4 Compute the unpolarized differential cross section of the process $e^-+e^+\longrightarrow\mu^-+\mu^+$ in the center of mass system.

[2] Difficult.

Final Examination QFT
Master 2
2011-2012
2.5 h

Problem 1**:**

1) Compute the electron 2−point function in configuration space up to one-loop using the Gell-Mann Low formula and Wick's theorem. Write down the corresponding Feynman diagrams.

2) Compute the electron 2−point function in momentum space up to one-loop using Feynman rules.

3) Use dimensional regularization to evaluate the electron self-energy. Add a small photon mass to regularize the IR behvavior. What is the UV behavior of the electron self-energy.

4) Determine the physical mass of the electron at one-loop.

5) Determine the wave-function renormalization Z_2 and the counter term $\delta_2 = 1 - Z_2$ up to one-loop.

Problem 2

1) Write down using Feynman rules the photon self-energy $i\Pi_2^{\mu\nu}(q)$ at one-loop.

2) Use dimensional regularization to show that

$$\Pi_2^{\mu\nu}(q) = \Pi_2(q^2)(q^2\eta^{\mu\nu} - q^\mu q^\nu). \tag{A.18}$$

   Determine $\Pi_2(q^2)$. What is the UV behavior.

3) Compute at one-loop the counter term $\delta_3 = Z_3 - 1$.

4) Compute at one-loop the effective charge e^2_{eff}. How does the effective charge behave at high energies.

Problem 3 Compute the unpolarized differential cross section of the process $e^- + e^+ \longrightarrow \mu^- + \mu^+$ in the center of mass system.

Recess Examination QFT
Master 2
2011-2012
2 h

Problem 1:

1) Compute the electron $2-$point function in momentum space up to one-loop using Feynman rules.

2) Use dimensional regularization to evaluate the electron self-energy. Add a small photon mass to regularize the IR behvavior. What is the UV behavior of the electron self-energy.

3) Determine the physical mass of the electron at one-loop.

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Problem 2:

1) Write down using Feynman rules the photon self-energy $i\Pi_2^{\mu\nu}(q)$ at one-loop.

2) Use dimensional regularization to show that

$$\Pi_2^{\mu\nu}(q) = \Pi_2(q^2)(q^2\eta^{\mu\nu} - q^\mu q^\nu).$$

   Determine $\Pi_2(q^2)$. What is the UV behavior.

3) Compute at one-loop the counter term $\delta_3 = Z_3 - 1$.

B
Problem Solutions

Chapter 1

Scalar Product Straightforward.

Relativistic Mechanics

- The trajectory of a particle in spacetime is called a world line. We take two infinitesimally close points on the world line given by (x^0, x^1, x^2, x^3) and $(x^0 + dx^0, x^1 + dx^1, x^2 + dx^2, x^3 + dx^3)$. Clearly $dx^1 = u^1 dt$, $dx^2 = u^2 dt$ and $dx^3 = u^3 dt$ where $\vec{u}$ is the velocity of the particle measured with respect to the observer O, viz

$$\vec{u} = \frac{d\vec{x}}{dt}.$$

The interval with respect to O is given by

$$ds^2 = -c^2 dt^2 + d\vec{x}^2 = (-c^2 + u^2)dt^2.$$

Let $O^{'}$ be the observer or inertial reference frame moving with respect to O with the velocity $\vec{u}$. We stress here that $\vec{u}$ is thought of as a constant velocity only during the infinitesimal time interval dt. The interval with respect to $O^{'}$ is given by

$$ds^2 = -c^2 d\tau^2. \tag{B.1}$$

Hence

$$d\tau = \sqrt{1 - \frac{u^2}{c^2}} dt.$$

The time interval $d\tau$ measured with respect to $O^{'}$ which is the observer moving with the particle is the proper time of the particle.

- The 4$-$vector velocity η is naturally defined by the components

$$\eta^\mu = \frac{dx^\mu}{d\tau}.$$

The spatial part of η is precisely the proper velocity $\vec{\eta}$ defined by

$$\vec{\eta} = \frac{d\vec{x}}{d\tau} = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}\vec{u}.$$

The temporal part is

$$\eta^0 = \frac{dx^0}{d\tau} = \frac{c}{\sqrt{1-\frac{u^2}{c^2}}}.$$

- The law of conservation of momentum and the principle of relativity put together forces us to define the momentum in relativity as mass times the proper velocity and not mass time the ordinary velocity, viz

$$\vec{p} = m\vec{\eta} = m\frac{d\vec{x}}{d\tau} = \frac{m}{\sqrt{1-\frac{u^2}{c^2}}}\vec{u}.$$

This is the spatial part of the 4$-$vector momentum

$$p^\mu = m\eta^\mu = m\frac{dx^\mu}{d\tau}.$$

The temporal part is

$$p^0 = m\eta^0 = m\frac{dx^0}{d\tau} = \frac{mc}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{E}{c}.$$

The relativistic energy is defined by

$$E = \frac{mc^2}{\sqrt{1-\frac{u^2}{c^2}}}.$$

The 4$-$vector p^μ is called the energy-momentum 4$-$vector.

- We note the identity

$$p_\mu p^\mu = -\frac{E^2}{c^2} + \vec{p}^2 = -m^2c^2.$$

Thus

$$E = \sqrt{\vec{p}^2c^2 + m^2c^4}.$$

The rest mass is m and the rest energy is clearly defined by

$$E_0 = mc^2.$$

- The first law of Newton is automatically satisfied because of the principle of relativity. The second law takes in the theory of special relativity the usual form provided we use the relativistic momentum, viz

$$\vec{F} = \frac{d\vec{p}}{dt}.$$

The third law of Newton does not in general hold in the theory of special relativity.

We can define a 4−vector proper force which is called the Minkowski force by the following equation

$$K^{\mu} = \frac{dp^{\mu}}{d\tau}.$$

The spatial part is

$$\vec{K} = \frac{d\vec{p}}{d\tau} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}\vec{F}.$$

Einstein's Velocity Addition Rule We consider a particle in the reference frame O moving a distance dx in the x direction during a time interval dt. The velocity with respect to O is

$$u = \frac{dx}{dt}.$$

In the reference frame $O^{'}$ the particle moves a distance $dx^{'}$ in a time interval $dt^{'}$ given by

$$dx^{'} = \gamma(dx - vdt).$$

$$dt^{'} = \gamma(dt - \frac{v}{c^2}dx).$$

The velocity with respect to $O^{'}$ is therefore

$$u^{'} = \frac{dx^{'}}{dt'} = \frac{u - v}{1 - \frac{vu}{c^2}}.$$

In general if $\vec{V}$ and $\vec{V}'$ are the velocities of the particle with respect to O and $O^{'}$ respectively and $\vec{v}$ is the velocity of $O^{'}$ with respect to O. Then

$$\vec{V}' = \frac{\vec{V} - \vec{v}}{1 - \frac{\vec{V}\vec{v}}{c^2}}.$$

Weyl Representation

- Straightforward.
- Straightforward.
- The Dirac equation can trivially be put in the form

$$i\hbar\frac{\partial\psi}{\partial t} = (\frac{\hbar c}{i}\gamma^0\gamma^i\partial_i + mc^2\gamma^0)\psi. \tag{B.2}$$

The Dirac Hamiltonian is

$$H = \frac{\hbar c}{i}\vec{\alpha}\vec{\nabla} + mc^2\beta \ , \ \alpha^i = \gamma^0\gamma^i \ , \ \beta = \gamma^0. \tag{B.3}$$

This is a Hermitian operator as it should be.

Lorentz Invariance of the D'Alembertian The invariance of the interval under Lorentz transformations reads

$$\eta_{\mu\nu}x^{\mu}x^{\nu}=\eta_{\mu\nu}x^{'\mu}x^{'\nu}=\eta_{\mu\nu}\Lambda^{\mu}\ _{\rho}x^{\rho}\Lambda^{\nu}\ _{\lambda}x^{\lambda}.$$

This leads immediately to

$$\eta=\Lambda^{T}\eta\Lambda.$$

Explicitly we write this as

$$\begin{aligned}\eta^{\mu}_{\nu} &= \Lambda_{\rho}\ ^{\mu}\eta^{\rho}_{\beta}\Lambda^{\beta}\ _{\nu}\\ &= \Lambda_{\rho}\ ^{\mu}\Lambda^{\rho}\ _{\nu}.\end{aligned}$$

But we also have

$$\delta^{\mu}_{\nu}=(\Lambda^{-1})^{\mu}\ _{\rho}\Lambda^{\rho}\ _{\nu}.$$

In other words

$$\Lambda_{\rho}\ ^{\mu}=(\Lambda^{-1})^{\mu}\ _{\rho}.$$

Since $x^{\mu}=(\Lambda^{-1})^{\mu}\ _{\nu}x^{'\nu}$ we have

$$\frac{\partial x^{\mu}}{\partial x^{'\nu}}=(\Lambda^{-1})^{\mu}\ _{\nu}.$$

Hence

$$\partial^{'}_{\nu}=(\Lambda^{-1})^{\mu}\ _{\nu}\partial_{\mu}.$$

Thus

$$\begin{aligned}\partial^{'}_{\mu}\partial^{'\mu} &= \eta^{\mu\nu}\partial^{'}_{\mu}\partial^{'}_{\nu}\\ &= \eta^{\mu\nu}(\Lambda^{-1})^{\rho}\ _{\mu}(\Lambda^{-1})^{\lambda}\ _{\nu}\partial_{\rho}\partial_{\lambda}\\ &= \eta^{\mu\nu}\Lambda_{\mu}\ ^{\rho}\Lambda_{\nu}\ ^{\lambda}\partial_{\rho}\partial_{\lambda}\\ &= (\Lambda^{T}\eta\Lambda)^{\rho\lambda}\partial_{\rho}\partial_{\lambda}\\ &= \partial_{\mu}\partial^{\mu}.\end{aligned}$$

Covariance of the Klein-Gordon equation Straightforward.

Vector Representations

- We have

$$V^{'i}(x^{'})=R^{ij}V^{j}(x).$$

The generators are given by the angular momentum operators J^{i} which satisfy the commutation relations

$$[J^{i},J^{j}]=i\hbar\epsilon^{ijk}J^{k}.$$

Thus a rotation with an angle $|\theta|$ about the axis $\hat{\theta}$ is obtained by exponentiation, viz

$$R = e^{-i\theta^i J^i}.$$

The matrices R form an $n-$dimensional representation with $n = 2j+1$ where j is the spin quantum number. The quantum numbers are therefore given by j and m.

- The angular momentum operators J^i are given by

$$J^i = -i\hbar\epsilon^{ijk}x^j\partial^k.$$

Thus

$$\begin{aligned} J^{ij} &= \epsilon^{ijk}J^k \\ &= -i\hbar(x^i\partial^j - x^j\partial^i). \end{aligned}$$

We compute

$$[J^{ij}, J^{kl}] = i\hbar\bigg(\eta^{jk}J^{il} - \eta^{ik}J^{jl} - \eta^{jl}J^{ik} + \eta^{il}J^{jk}\bigg).$$

- Generalization to 4−dimensional Minkowski space yields

$$J^{\mu\nu} = -i\hbar(x^\mu\partial^\nu - x^\nu\partial^\mu).$$

Now we compute the commutation relations

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\hbar\bigg(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\sigma}J^{\nu\rho}\bigg).$$

- A solution of is given by the 4×4 matrices

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\hbar(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha).$$

Equivalently

$$(\mathcal{J}^{\mu\nu})^\alpha{}_\beta = i\hbar(\eta^{\mu\alpha}\delta^\nu_\beta - \delta^\mu_\beta\eta^{\nu\alpha}).$$

We compute

$$(\mathcal{J}^{\mu\nu})^\alpha{}_\beta(\mathcal{J}^{\rho\sigma})^\beta{}_\lambda = (i\hbar)^2\bigg(\eta^{\mu\alpha}\eta^{\rho\nu}\delta^\sigma_\lambda - \eta^{\mu\alpha}\eta^{\sigma\nu}\delta^\rho_\lambda - \eta^{\nu\alpha}\eta^{\rho\mu}\delta^\sigma_\lambda + \eta^{\nu\alpha}\eta^{\sigma\mu}\delta^\rho_\lambda\bigg).$$

$$(\mathcal{J}^{\rho\sigma})^\alpha{}_\beta(\mathcal{J}^{\mu\nu})^\beta{}_\lambda = (i\hbar)^2\bigg(\eta^{\rho\alpha}\eta^{\mu\sigma}\delta^\nu_\lambda - \eta^{\rho\alpha}\eta^{\sigma\nu}\delta^\mu_\lambda - \eta^{\sigma\alpha}\eta^{\rho\mu}\delta^\nu_\lambda + \eta^{\sigma\alpha}\eta^{\nu\rho}\delta^\mu_\lambda\bigg).$$

Hence

$$\begin{aligned} [\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}]^\alpha{}_\lambda &= (i\hbar)^2\bigg(\eta^{\mu\sigma}\big[\eta^{\nu\alpha}\delta^\rho_\lambda - \eta^{\rho\alpha}\delta^\nu_\lambda\big] - \eta^{\nu\sigma}\big[\eta^{\mu\alpha}\delta^\rho_\lambda - \eta^{\rho\alpha}\delta^\mu_\lambda\big] - \eta^{\mu\rho}\big[\eta^{\nu\alpha}\delta^\sigma_\lambda - \eta^{\sigma\alpha}\delta^\nu_\lambda\big] \\ &+ \eta^{\nu\rho}\big[\eta^{\mu\alpha}\delta^\sigma_\lambda - \eta^{\sigma\alpha}\delta^\mu_\lambda\big]\bigg) \\ &= i\hbar\bigg[\eta^{\mu\sigma}(\mathcal{J}^{\nu\rho})^\alpha{}_\lambda - \eta^{\nu\sigma}(\mathcal{J}^{\mu\rho})^\alpha{}_\lambda - \eta^{\mu\rho}(\mathcal{J}^{\nu\sigma})^\alpha{}_\lambda + \eta^{\nu\rho}(\mathcal{J}^{\mu\sigma})^\alpha{}_\lambda\bigg]. \end{aligned}$$

- A finite Lorentz transformation in the vector representation is

$$\Lambda = e^{-\frac{i}{2\hbar}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}}.$$

$\omega_{\mu\nu}$ is an antisymmetric tensor. An infinitesimal transformation is given by

$$\Lambda = 1 - \frac{i}{2\hbar}\omega_{\mu\nu}\mathcal{J}^{\mu\nu}.$$

A rotation in the $xy-$plane corresponds to $\omega_{12} = -\omega_{21} = -\theta$ while the rest of the components are zero, viz

$$\Lambda^{\alpha}{}_{\beta} = (1 + \frac{i}{\hbar}\theta\mathcal{J}^{12})^{\alpha}{}_{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

A boost in the $x-$direction corresponds to $\omega_{01} = -\omega_{10} = -\beta$ while the rest of the components are zero, viz

$$\Lambda^{\alpha}{}_{\beta} = (1 + \frac{i}{\hbar}\beta\mathcal{J}^{01})^{\alpha}{}_{\beta} = \begin{pmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Dirac Spinors

- We compute

$$\sigma_{\mu}p^{\mu} = \frac{E}{c} - \vec{\sigma}\vec{p} = \begin{pmatrix} \frac{E}{c} - p^3 & -(p^1 - ip^2) \\ -(p^1 + ip^2) & \frac{E}{c} + p^3 \end{pmatrix}.$$

$$\bar{\sigma}_{\mu}p^{\mu} = \frac{E}{c} + \vec{\sigma}\vec{p} = \begin{pmatrix} \frac{E}{c} + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & \frac{E}{c} - p^3 \end{pmatrix}.$$

Thus

$$(\sigma_{\mu}p^{\mu})(\bar{\sigma}_{\mu}p^{\mu}) = m^2c^2.$$

- Recall the four possible solutions:

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow u^{(1)} = N^{(1)} \begin{pmatrix} 1 \\ 0 \\ \frac{\frac{E}{c} + p^3}{mc} \\ \frac{p^1 + ip^2}{mc} \end{pmatrix}.$$

$$u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow u^{(4)} = N^{(4)} \begin{pmatrix} 0 \\ 1 \\ \frac{p^1 - ip^2}{mc} \\ \frac{\frac{E}{c} - p^3}{mc} \end{pmatrix}.$$

$$u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow u^{(3)} = N^{(3)} \begin{pmatrix} \frac{\frac{E}{c} - p^3}{mc} \\ -\frac{p^1 + ip^2}{mc} \\ 1 \\ 0 \end{pmatrix}.$$

$$u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Leftrightarrow u^{(2)} = N^{(2)} \begin{pmatrix} -\frac{p^1 - ip^2}{mc} \\ \frac{\frac{E}{c} + p^3}{mc} \\ 0 \\ 1 \end{pmatrix}.$$

The normalization condition is

$$\bar{u}u = u^+\gamma^0 u = u_A^+ u_B + u_B^+ u_A = 2mc.$$

We obtain immediately

$$N^{(1)} = N^{(2)} = \sqrt{\frac{m^2c^2}{\frac{E}{c} + p^3}}.$$

- Recall that

$$v^{(1)}(E, \vec{p}) = u^{(3)}(-E, -\vec{p}) = N^{(3)} \begin{pmatrix} -\frac{\frac{E}{c} - p^3}{mc} \\ \frac{p^1 + ip^2}{mc} \\ 1 \\ 0 \end{pmatrix},$$

$$v^{(2)}(E, \vec{p}) = u^{(4)}(-E, -\vec{p}) = N^{(4)} \begin{pmatrix} 0 \\ 1 \\ -\frac{p^1 - ip^2}{mc} \\ -\frac{\frac{E}{c} - p^3}{mc} \end{pmatrix}.$$

The normalization condition in this case is

$$\bar{v}v = v^+\gamma^0 v = v_A^+ v_B + v_B^+ v_A = -2mc.$$

We obtain now

$$N^{(3)} = N^{(4)} = \sqrt{\frac{m^2c^2}{\frac{E}{c} - p^3}}.$$

- Let us define

$$\xi_0^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \xi_0^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We have

$$u^{(1)} = N^{(1)} \begin{pmatrix} \xi_0^1 \\ \frac{\frac{E}{c} + \vec{\sigma}\vec{p}}{mc} \xi_0^1 \end{pmatrix} = N^{(1)} \frac{1}{\sqrt{\sigma_\mu p^\mu}} \begin{pmatrix} \sqrt{\sigma_\mu p^\mu} \xi_0^1 \\ \sqrt{\bar{\sigma}_\mu p^\mu} \xi_0^1 \end{pmatrix} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu} \xi^1 \\ \sqrt{\bar{\sigma}_\mu p^\mu} \xi^1 \end{pmatrix}.$$

$$u^{(2)} = N^{(2)} \begin{pmatrix} \frac{\frac{E}{c} - \vec{\sigma}\vec{p}}{mc} \xi_0^2 \\ \xi_0^2 \end{pmatrix} = N^{(2)} \frac{1}{\sqrt{\bar{\sigma}_\mu p^\mu}} \begin{pmatrix} \sqrt{\sigma_\mu p^\mu} \xi_0^2 \\ \sqrt{\bar{\sigma}_\mu p^\mu} \xi_0^2 \end{pmatrix} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu} \xi^2 \\ \sqrt{\bar{\sigma}_\mu p^\mu} \xi^2 \end{pmatrix}.$$

The spinors ξ^1 and ξ^2 are defined by

$$\xi^1 = N^{(1)} \frac{1}{\sqrt{\sigma_\mu p^\mu}} \xi_0^1 = \sqrt{\frac{\bar{\sigma}_\mu p^\mu}{\frac{E}{c} + p^3}} \xi_0^1.$$

$$\xi^2 = N^{(2)} \frac{1}{\sqrt{\bar{\sigma}_\mu p^\mu}} \xi_0^2 = \sqrt{\frac{\sigma_\mu p^\mu}{\frac{E}{c} + p^3}} \xi_0^2.$$

They satisfy

$$(\xi^r)^+ \xi^s = \delta^{rs}.$$

Similarly let us define

$$\eta_0^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \ \eta_0^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then we have

$$v^{(1)} = N^{(3)} \begin{pmatrix} -\frac{\frac{E}{c} - \vec{\sigma}\vec{p}}{mc} \eta_0^1 \\ \eta_0^1 \end{pmatrix} = -N^{(3)} \frac{1}{\sqrt{\bar{\sigma}_\mu p^\mu}} \begin{pmatrix} \sqrt{\sigma_\mu p^\mu} \eta_0^1 \\ -\sqrt{\bar{\sigma}_\mu p^\mu} \eta_0^1 \end{pmatrix} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu} \eta^1 \\ -\sqrt{\bar{\sigma}_\mu p^\mu} \eta^1 \end{pmatrix}.$$

$$v^{(2)} = N^{(4)} \begin{pmatrix} \eta_0^2 \\ -\frac{\frac{E}{c} + \vec{\sigma}\vec{p}}{mc} \eta_0^2 \end{pmatrix} = N^{(4)} \frac{1}{\sqrt{\sigma_\mu p^\mu}} \begin{pmatrix} \sqrt{\sigma_\mu p^\mu} \eta_0^2 \\ -\sqrt{\bar{\sigma}_\mu p^\mu} \eta_0^2 \end{pmatrix} = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu} \eta^2 \\ -\sqrt{\bar{\sigma}_\mu p^\mu} \eta^2 \end{pmatrix}.$$

$$\eta^1 = -N^{(3)} \frac{1}{\sqrt{\bar{\sigma}_\mu p^\mu}} \eta_0^1 = -\sqrt{\frac{\sigma_\mu p^\mu}{\frac{E}{c} - p^3}} \eta_0^1.$$

$$\eta^2 = N^{(4)} \frac{1}{\sqrt{\sigma_\mu p^\mu}} \eta_0^2 = \sqrt{\frac{\bar{\sigma}_\mu p^\mu}{\frac{E}{c} - p^3}} \eta_0^2.$$

Again they satisfy

$$(\eta^r)^+ \eta^s = \delta^{rs}.$$

Spin Sums

- We have

$$u^{(r)}(E, \vec{p}) = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu} \xi^r \\ \sqrt{\bar{\sigma}_\mu p^\mu} \xi^r \end{pmatrix} \ , \ v^{(r)}(E, \vec{p}) = \begin{pmatrix} \sqrt{\sigma_\mu p^\mu} \eta^r \\ -\sqrt{\bar{\sigma}_\mu p^\mu} \eta^r \end{pmatrix}.$$

We compute

$$\bar{u}^{(r)} u^{(s)} = u^{(r)+} \gamma^0 u^{(s)} = 2\xi^{r+} \sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)} \xi^s = 2mc \xi^{r+} \xi^s = 2mc \delta^{rs}.$$

$$\bar{v}^{(r)}v^{(s)} = v^{(r)+}\gamma^0 v^{(s)} = -2\eta^{r+}\sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)}\eta^s = -2mc\eta^{r+}\eta^s = -2mc\delta^{rs}.$$

We have used

$$(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu) = m^2c^2.$$

$$\xi^{r+}\xi^s = \delta^{rs} \ , \ \eta^{r+}\eta^s = \delta^{rs}.$$

We also compute

$$\bar{u}^{(r)}v^{(s)} = u^{(r)+}\gamma^0 v^{(s)} = -\xi^{r+}\sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)}\eta^s + \xi^{r+}\sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)}\eta^s = 0.$$

A similar calculation yields

$$\bar{v}^{(r)}u^{(s)} = u^{(r)+}\gamma^0 v^{(s)} = 0.$$

- Next we compute

$$u^{(r)+}u^{(s)} = \xi^{r+}(\sigma_\mu p^\mu + \bar{\sigma}_\mu p^\mu)\xi^s = \frac{2E}{c}\xi^{r+}\xi^s = \frac{2E}{c}\delta^{rs}.$$

$$v^{(r)+}v^{(s)} = \eta^{r+}(\sigma_\mu p^\mu + \bar{\sigma}_\mu p^\mu)\eta^s = \frac{2E}{c}\eta^{r+}\eta^s = \frac{2E}{c}\delta^{rs}.$$

We have used

$$\sigma^\mu = (1, \sigma^i) \ , \ \sigma^\mu = (1, -\sigma^i).$$

We also compute

$$u^{(r)+}(E,\vec{p})v^{(s)}(E,-\vec{p}) = \xi^{r+}(\sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)} - \sqrt{(\sigma_\mu p^\mu)(\bar{\sigma}_\nu p^\nu)})\xi^s = 0.$$

Similarly we compute that

$$v^{(r)+}(E,-\vec{p})u^{(s)}(E,\vec{p}) = 0.$$

In the above two equation we have used the fact that

$$v^{(r)}(E,-\vec{p}) = \begin{pmatrix} \sqrt{\bar{\sigma}_\mu p^\mu}\eta^r \\ -\sqrt{\sigma_\mu p^\mu}\eta^r \end{pmatrix}.$$

- Next we compute

$$\begin{aligned}\sum_s u^{(s)}(E,\vec{p})\bar{u}^{(s)}(E,\vec{p}) &= \sum_s u^{(s)}(E,\vec{p})u^{(s)+}(E,\vec{p})\gamma^0 \\ &= \sum_s \begin{pmatrix} \sqrt{\sigma_\mu p^\mu}\xi^s\xi^{s+}\sqrt{\sigma_\mu p^\mu} & \sqrt{\sigma_\mu p^\mu}\xi^s\xi^{s+}\sqrt{\bar{\sigma}_\mu p^\mu} \\ \sqrt{\bar{\sigma}_\mu p^\mu}\xi^s\xi^{s+}\sqrt{\sigma_\mu p^\mu} & \sqrt{\bar{\sigma}_\mu p^\mu}\xi^s\xi^{s+}\sqrt{\bar{\sigma}_\mu p^\mu} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.\end{aligned}$$

We use

$$\sum_s \xi^s\xi^{s+} = 1.$$

We obtain

$$\sum_s u^{(s)}(E,\vec{p})\bar{u}^{(s)}(E,\vec{p}) = \begin{pmatrix} mc & \sigma_\mu p^\mu \\ \bar{\sigma}_\mu p^\mu & mc \end{pmatrix} = \gamma^\mu p_\mu + mc.$$

Similarly we use

$$\sum_s \eta^s \eta^{s+} = 1,$$

to calculate

$$\sum_s v^{(s)}(E,\vec{p})\bar{v}^{(s)}(E,\vec{p}) = \begin{pmatrix} -mc & \sigma_\mu p^\mu \\ \bar{\sigma}_\mu p^\mu & -mc \end{pmatrix} = \gamma^\mu p_\mu - mc.$$

Covariance of the Dirac Equation Under Lorentz transformations we have the following transformation laws

$$\psi(x) \longrightarrow \psi^{'}(x^{'}) = S(\Lambda)\psi(x).$$

$$\gamma_\mu \longrightarrow \gamma_\mu^{'} = \gamma_\mu.$$

$$\partial_\mu \longrightarrow \partial_\nu^{'} = (\Lambda^{-1})^\mu{}_\nu \partial_\mu.$$

Thus the Dirac equation $(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0$ becomes

$$(i\hbar\gamma^{'\mu}\partial_\mu^{'} - mc)\psi^{'} = 0,$$

or equivalently

$$(i\hbar(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\gamma^{'\mu}S(\Lambda)\partial_\nu - mc)\psi = 0.$$

We must have therefore

$$(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \gamma^\nu,$$

or equivalently

$$(\Lambda^{-1})^\nu{}_\mu S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \gamma^\nu.$$

We consider an infinitesimal Lorentz transformation

$$\Lambda = 1 - \frac{i}{2\hbar}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta} \ , \ \Lambda^{-1} = 1 + \frac{i}{2\hbar}\omega_{\alpha\beta}\mathcal{J}^{\alpha\beta}.$$

The corresponding $S(\Lambda)$ must also be infinitesimal of the form

$$S(\Lambda) = 1 - \frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta} \ , \ S^{-1}(\Lambda) = 1 + \frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta}.$$

By substitution we get

$$-(\mathcal{J}^{\alpha\beta})^\mu{}_\nu \gamma_\mu = [\gamma_\nu, \Gamma^{\alpha\beta}].$$

Explicitly this reads

$$-i\hbar(\delta^{\beta}_{\nu}\gamma^{\alpha}-\delta^{\alpha}_{\nu}\gamma^{\beta})=[\gamma_{\nu},\Gamma^{\alpha\beta}],$$

or equivalently

$$[\gamma_0,\Gamma^{0i}]=i\hbar\gamma^i$$
$$[\gamma_j,\Gamma^{0i}]=-i\hbar\delta^i_j\gamma^0$$
$$[\gamma_0,\Gamma^{ij}]=0$$
$$[\gamma_k,\Gamma^{ij}]=-i\hbar(\delta^j_k\gamma^i-\delta^i_k\gamma^j).$$

A solution is given by

$$\Gamma^{\mu\nu}=\frac{i\hbar}{4}[\gamma^{\mu},\gamma^{\nu}].$$

Spinor Bilinears The Dirac spinor ψ changes under Lorentz transformations as

$$\psi(x)\longrightarrow\psi^{'}(x^{'})=S(\Lambda)\psi(x).$$

$$S(\Lambda)=e^{-\frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta}}.$$

Since $(\gamma^{\mu})^+=\gamma^0\gamma^{\mu}\gamma^0$ we get $(\Gamma^{\mu\nu})^+=\gamma^0\Gamma^{\mu\nu}\gamma^0$. Therefore

$$S(\Lambda)^+=\gamma^0S(\Lambda)^{-1}\gamma^0.$$

In other words

$$\bar{\psi}(x)\longrightarrow\bar{\psi}^{'}(x^{'})=\bar{\psi}(x)S(\Lambda)^{-1}.$$

As a consequence

$$\bar{\psi}\psi\longrightarrow\bar{\psi}^{'}\psi^{'}=\bar{\psi}\psi.$$

$$\bar{\psi}\gamma^5\psi\longrightarrow\bar{\psi}^{'}\gamma^5\psi^{'}=\bar{\psi}\psi.$$

$$\bar{\psi}\gamma^{\mu}\psi\longrightarrow\bar{\psi}^{'}\gamma^{\mu}\psi^{'}=\Lambda^{\mu}\ _{\nu}\bar{\psi}\gamma^{\nu}\psi.$$

$$\bar{\psi}\gamma^{\mu}\gamma^5\psi\longrightarrow\bar{\psi}^{'}\gamma^{\mu}\gamma^5\psi^{'}=\Lambda^{\mu}\ _{\nu}\bar{\psi}\gamma^{\nu}\gamma^5\psi.$$

We have used $[\gamma^5,\Gamma^{\mu\nu}]=0$ and $S^{-1}\gamma^{\mu}S=\Lambda^{\mu}\ _{\nu}\gamma^{\nu}$. Finally we compute

$$\begin{aligned}\bar{\psi}\Gamma^{\mu\nu}\psi\longrightarrow\bar{\psi}^{'}\Gamma^{\mu\nu}\psi^{'}&=\bar{\psi}S^{-1}\Gamma^{\mu\nu}S\psi\\&=\bar{\psi}\frac{i\hbar}{4}[S^{-1}\gamma^{\mu}S,S^{-1}\gamma^{\nu}S]\psi\\&=\Lambda^{\mu}\ _{\alpha}\Lambda^{\nu}\ _{\beta}\bar{\psi}\Gamma^{\alpha\beta}\psi.\end{aligned}$$

Clifford Algebra

- The Clifford algebra in three Euclidean dimensions is solved by Pauli matrices, viz

$$\{\gamma^i,\gamma^j\}=2\delta^{ij}\ ,\ \gamma^i\equiv\sigma^i.$$

Any 2×2 matrix can be expanded in terms of the Pauli matrices and the identity. In other words

$$M_{2\times 2}=M_0\mathbf{1}+M_i\sigma_i.$$

- Any 4×4 matrix can be expanded in terms of a 16 antisymmetric combinations of the Dirac gamma matrices.

The 4−dimensional identity and the Dirac matrices provide the first five independent 4×4 matrices. The product of two Dirac gamma matrices yield six different matrices which because of $\{\gamma^\mu,\gamma^\nu\}=2\eta^{\mu\nu}$ can be encoded in the six matrices $\Gamma^{\mu\nu}$ defined by

$$\Gamma^{\mu\nu}=\frac{i\hbar}{4}[\gamma^\mu,\gamma^\nu].$$

There are four independent 4×4 matrices formed by the product of three Dirac gamma matrices. They are

$$\gamma^0\gamma^1\gamma^2\ ,\ \gamma^0\gamma^1\gamma^3\ ,\ \gamma^0\gamma^2\gamma^3\ ,\ \gamma^1\gamma^2\gamma^3.$$

These can be rewritten as

$$i\epsilon^{\mu\nu\alpha\beta}\gamma_\beta\gamma^5.$$

The product of four Dirac gamma matrices leads to an extra independent 4×4 matrix which is precisely the gamma five matrix. In total there are $1+4+6+4+1=16$ antisymmetric combinations of Dirac gamma matrices. Hence any 4×4 matrix can be expanded as

$$M_{4\times 4}=M_0\mathbf{1}+M_\mu\gamma^\mu+M_{\mu\nu}\Gamma^{\mu\nu}+M_{\mu\nu\alpha}i\epsilon^{\mu\nu\alpha\beta}\gamma_\beta\gamma^5+M_5\gamma^5.$$

Chirality Operator and Weyl Fermions

- We have

$$\gamma^5=i\gamma^0\gamma^1\gamma^2\gamma^3.$$

Thus

$$\begin{aligned}
-\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma &= -\frac{i}{4!}(4)\epsilon_{0abc}\gamma^0\gamma^a\gamma^b\gamma^c\\
&= -\frac{i}{4!}(4.3)\epsilon_{0ij3}\gamma^0\gamma^i\gamma^j\gamma^3\\
&= -\frac{i}{4!}(4.3.2)\epsilon_{0123}\gamma^0\gamma^1\gamma^2\gamma^3\\
&= i\gamma^0\gamma^1\gamma^2\gamma^3\\
&= \gamma^5.
\end{aligned}$$

We have used

$$\epsilon_{0123} = -\epsilon^{0123} = -1.$$

We also verify

$$\begin{aligned}(\gamma^5)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3.\gamma^0\gamma^1\gamma^2\gamma^3\\ &= \gamma^1\gamma^2\gamma^3.\gamma^1\gamma^2\gamma^3\\ &= -\gamma^2\gamma^3.\gamma^2\gamma^3\\ &= 1.\end{aligned}$$

$$\begin{aligned}(\gamma^5)^+ &= -i(\gamma^3)^+(\gamma^2)^+(\gamma^1)^+(\gamma^0)^+\\ &= i\gamma^3\gamma^2\gamma^1\gamma^0\\ &= -i\gamma^0\gamma^3\gamma^2\gamma^1\\ &= -i\gamma^0\gamma^1\gamma^3\gamma^2\\ &= i\gamma^0\gamma^1\gamma^2\gamma^3\\ &= \gamma^5.\end{aligned}$$

$$\{\gamma^5,\gamma^0\} = \{\gamma^5,\gamma^1\} = \{\gamma^5,\gamma^2\} = \{\gamma^5,\gamma^3\} = 0.$$

From this last property we conclude directly that

$$[\gamma^5,\Gamma^{\mu\nu}] = 0.$$

- Hence the Dirac representation is reducible. To see this more clearly we work in the Weyl or chiral representation given by

$$\gamma^0 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}\ ,\ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

In this representation we compute

$$\gamma^5 = i\begin{pmatrix} \sigma^1\sigma^2\sigma^3 & 0 \\ 0 & \sigma^1\sigma^2\sigma^3 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence by writing the Dirac spinor as

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix},$$

we get

$$\Psi_R = \frac{1+\gamma^5}{2}\psi = \begin{pmatrix} 0 \\ \psi_R \end{pmatrix},$$

and

$$\Psi_L = \frac{1-\gamma^5}{2}\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}.$$

In other words

$$\gamma^5\Psi_L = -\Psi_L \ , \ \gamma^5\Psi_R = \Psi_R.$$

The spinors Ψ_L and Ψ_R do not mix under Lorentz transformations since they are eigen-spinors of γ^5 which commutes with Γ^{ab}. In other words

$$\Psi_L(x) \longrightarrow \Psi^{'}_L(x^{'}) = S(\Lambda)\Psi_L(x).$$

$$\Psi_R(x) \longrightarrow \Psi^{'}_R(x^{'}) = S(\Lambda)\Psi_R(x).$$

- The dirac equation is

$$(i\hbar\gamma^\mu\partial_\mu - mc)\psi = 0.$$

In terms of ψ_L and ψ_R this becomes

$$i\hbar(\partial_0 + \sigma^i\partial_i)\psi_R = mc\psi_L \ , \ i\hbar(\partial_0 - \sigma^i\partial_i)\psi_L = mc\psi_R.$$

For a massless theory we get two fully decoupled equations

$$i\hbar(\partial_0 + \sigma^i\partial_i)\psi_R = 0 \ , \ i\hbar(\partial_0 - \sigma^i\partial_i)\psi_L = 0.$$

These are known as Weyl equations. They are relevant in describing neutrinos. It is clear that ψ_L describes a left-moving particles and ψ_R describes a right-moving particles.

Chapter 2

Scalars Commutation Relations Straightforward.

The One-Particle States

- The Hamiltonian operator of a real scalar field is given by (ignoring an infinite constant due to vacuum energy)

$$\hat{H}_{\rm KG} = \int \frac{d^3p}{(2\pi\hbar)^3}\omega(\vec{p})\hat{a}(\vec{p})^+\hat{a}(\vec{p}).$$

It satisfies

$$\hat{H}_{\rm KG}|0> = 0.$$

$$[\hat{H}_{\rm KG}, \hat{a}(\vec{p})^+] = \hbar\omega(\vec{p})\hat{a}(\vec{p})^+ \ , \ [\hat{H}, \hat{a}(\vec{p})] = -\hbar\omega(\vec{p})\hat{a}(\vec{p}).$$

Thus we compute

$$\begin{aligned}
\hat{H}_{\rm KG}|\vec{p}> &= \frac{1}{c}\sqrt{2\omega(\vec{p})}\hat{H}_{\rm KG}\hat{a}(\vec{p})^+|0> \\
&= \frac{1}{c}\sqrt{2\omega(\vec{p})}[\hat{H}_{\rm KG}, \hat{a}(\vec{p})^+]|0> \\
&= \frac{1}{c}\sqrt{2\omega(\vec{p})}\hbar\omega(\vec{p})\hat{a}(\vec{p})^+|0> \\
&= \hbar\omega(\vec{p})|\vec{p}> .
\end{aligned}$$

- Next we compute

$$< \vec{p}|\vec{q} >= \frac{2}{c^2}(2\pi\hbar)^3 E(\vec{p})\delta^3(\vec{p}-\vec{q}).$$

We have assumed that $< 0|0 >= 1$. This is Lorentz invariant since $E(\vec{p})\delta^3(\vec{p}-\vec{q})$ is Lorentz invariant. Let us consider a Lorentz boost along the $x-$direction, viz

$$x^{0^{'}} = \gamma(x^0 - \beta x^1)\ ,\ x^{1^{'}} = \gamma(x^1 - \beta x^0)\ ,\ x^{2^{'}} = x^2\ ,\ x^{3^{'}} = x^3.$$

The energy-momentum $4-$vector $p^\mu = (p^0, p^i) = (E/c, p^i)$ will transform as

$$p^{0^{'}} = \gamma(p^0 - \beta p^1)\ ,\ p^{1^{'}} = \gamma(p^1 - \beta p^0)\ ,\ p^{2^{'}} = p^2\ ,\ p^{3^{'}} = p^3.$$

We compute

$$\begin{aligned}
\delta(p^1 - q^1) &= \delta(p^{1^{'}} - q^{1^{'}})\frac{dp^{1^{'}}}{dp^1} \\
&= \delta(p^{1^{'}} - q^{1^{'}})\gamma(1 - \beta\frac{dp^0}{dp^1}) \\
&= \delta(p^{1^{'}} - q^{1^{'}})\gamma(1 - \beta\frac{p^1}{p^0}) \\
&= \delta(p^{1^{'}} - q^{1^{'}})\frac{p^{0^{'}}}{p^0}.
\end{aligned}$$

Hence we have

$$p^0\delta(\vec{p}-\vec{q}) = p^{0^{'}}\delta(\vec{p}^{'} - \vec{q}^{'}).$$

- The completeness relation on the Hilbert subspace of one-particle states is

$$\mathbf{1}_{\rm one-particle} = c^2\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}|\vec{p}>< \vec{p}|. \tag{B.4}$$

It is straightforward to compute

$$\hat{\phi}(x^0, \vec{x})|0 >= c^2\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}|\vec{p}>\ \ e^{\frac{i}{\hbar}(E(\vec{p})t - \vec{p}\vec{x})}. \tag{B.5}$$

This is a linear combination of one-particle states. For small $\vec{p}$ we can make the approximation $E(\vec{p}) \simeq mc^2$ and as a consequence

$$\hat{\phi}(x^0, \vec{x})|0 >= \frac{e^{\frac{i}{\hbar}mc^2t}}{2m}\int\frac{d^3p}{(2\pi\hbar)^3}|\vec{p}>\ \ e^{-\frac{i}{\hbar}\vec{p}\vec{x}}. \tag{B.6}$$

In this case the Dirac orthonormalization and the completeness relations read

$$< \vec{p}|\vec{q} >= 2m(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}). \tag{B.7}$$

$$\mathbf{1}_{\rm one-particle} = \frac{1}{2m}\int\frac{d^3p}{(2\pi\hbar)^3}|\vec{p}>< \vec{p}|. \tag{B.8}$$

The eigenstates $|\vec{x}>$ of the position operator can be defined by

$$< \vec{p}|\vec{x} >= \sqrt{2m}e^{-\frac{i}{\hbar}\vec{p}\vec{x}}. \quad (B.9)$$

Hence

$$\hat{\phi}(x^0,\vec{x})|0 >= \frac{e^{\frac{i}{\hbar}mc^2t}}{\sqrt{2m}}|\vec{x} > . \quad (B.10)$$

In other words in the relativistic theory the operator $\hat{\phi}(x^0,\vec{x})|0>$ should be interpreted as the eigenstate $|\vec{x}>$ of the position operator. Indeed we can compute in the relativistic theory

$$< 0|\hat{\phi}(x^0,\vec{x})|\vec{p} >= e^{-\frac{i}{\hbar}px} \ , \ px = E(\vec{p})t - \vec{p}\vec{x}. \quad (B.11)$$

We say that the field operator $\hat{\phi}(x^0,\vec{x})$ creates a particle at the point $\vec{x}$ at time $t = x^0/c$.

Momentum Operator

- For a real scalar field

$$\begin{aligned}\hat{P}_i &= c\int d^3x\hat{\pi}\partial_i\hat{\phi}\\ &= \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\vec{p}\hat{a}(\vec{p})^+\hat{a}(\vec{p}).\end{aligned}$$

- For a Dirac field

$$\hat{P}_i = \frac{1}{\hbar}\int\frac{d^3p}{(2\pi\hbar)^3}\vec{p}\sum_i\bigg(\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},i) + \hat{d}(\vec{p},i)^+\hat{d}(\vec{p},i)\bigg).$$

Fermions Anticommutation Relations

- We have

$$\hat{\chi}(x^0,\vec{p}) = \sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i\bigg(e^{-i\omega(\vec{p})t}u^{(i)}(\vec{p})\hat{b}(\vec{p},i) + e^{i\omega(\vec{p})t}v^{(i)}(-\vec{p})\hat{d}(-\vec{p},i)^+\bigg).$$

We compute

$$\begin{aligned}[\hat{\chi}_\alpha(x^0,\vec{p}),\hat{\chi}^+_\beta(x^0,\vec{q})]_\pm &= \frac{c}{2\sqrt{\omega(\vec{p})\omega(\vec{q})}}\sum_{i,j}e^{i(\omega(\vec{p})-\omega(\vec{q}))t}u^{(i)}_\alpha(\vec{p})u^{(j)*}_\beta(\vec{q})[\hat{b}(\vec{p},i),\hat{b}(\vec{q},j)^+]_\pm\\ &+ \frac{c}{2\sqrt{\omega(\vec{p})\omega(\vec{q})}}\sum_{i,j}e^{-i(\omega(\vec{p})+\omega(\vec{q}))t}u^{(i)}_\alpha(\vec{p})v^{(j)*}_\beta(-\vec{q})[\hat{b}(\vec{p},i),\hat{d}(-\vec{q},j)]_\pm\\ &+ \frac{c}{2\sqrt{\omega(\vec{p})\omega(\vec{q})}}\sum_{i,j}e^{i(\omega(\vec{p})+\omega(\vec{q}))t}v^{(i)}_\alpha(-\vec{p})u^{(j)*}_\beta(\vec{q})[\hat{d}(-\vec{p},i)^+,\hat{b}(\vec{q},j)^+]_\pm\\ &+ \frac{c}{2\sqrt{\omega(\vec{p})\omega(\vec{q})}}\sum_{i,j}e^{i(\omega(\vec{p})-\omega(\vec{q}))t}v^{(i)}_\alpha(-\vec{p})v^{(j)*}_\beta(-\vec{q})[\hat{d}(-\vec{p},i)^+,\hat{d}(-\vec{q},j)]_\pm.\end{aligned}$$

We impose

$$[\hat{b}(\vec{p},i),\hat{b}(\vec{q},j)^{+}]_{\pm}=\hbar\delta_{ij}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}),$$

$$[\hat{d}(\vec{p},i)^{+},\hat{d}(\vec{q},j)]_{\pm}=\hbar\delta_{ij}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}),$$

and

$$[\hat{b}(\vec{p},i),\hat{d}(\vec{q},j)]_{\pm}=[\hat{d}(\vec{q},j)^{+},\hat{b}(\vec{p},i)]_{\pm}=0.$$

Thus we get

$$\begin{aligned}[\hat{\chi}_{\alpha}(x^0,\vec{p}),\hat{\chi}^{+}_{\beta}(x^0,\vec{q})]_{\pm} &= \frac{c\hbar}{2\omega(\vec{p})}\sum_i u^{(i)}_{\alpha}(\vec{p})u^{(i)*}_{\beta}(\vec{p})(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q})\\ &+ \frac{c\hbar}{2\omega(\vec{p})}\sum_i v^{(i)}_{\alpha}(-\vec{p})v^{(i)*}_{\beta}(-\vec{p})(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}).\end{aligned}$$

By using the completeness relations $\sum_s u^{(s)}(E,\vec{p})\bar{u}^{(s)}(E,\vec{p})=\gamma^{\mu}p_{\mu}+mc$ and $\sum_s v^{(s)}(E,\vec{p})\bar{v}^{(s)}(E,\vec{p})=\gamma^{\mu}p_{\mu}-mc$ we derive

$$\sum_i u^{(i)}_{\alpha}(E,\vec{p})u^{(i)*}_{\beta}(E,\vec{p})+\sum_i v^{(i)}_{\alpha}(E,-\vec{p})v^{(i)*}_{\beta}(E,-\vec{p})=\frac{2E(\vec{p})}{c}\delta_{\alpha\beta}.$$

We get then the desired result

$$[\hat{\chi}_{\alpha}(x^0,\vec{p}),\hat{\chi}^{+}_{\beta}(x^0,\vec{q})]_{\pm}=\hbar^2\delta_{\alpha\beta}(2\pi\hbar)^3\delta^3(\vec{p}-\vec{q}).$$

- Straightforward.

Retarded Propagator Straightforward.

Feynman Propagator Straightforward.

The Dirac Propagator

- We compute

$$\begin{aligned}S_{ab}(x-y) &= c\int\frac{d^3p}{(2\pi\hbar)^3}\int\frac{d^3q}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}\frac{1}{2E(\vec{q})}\sum_{i,j}e^{\frac{i}{\hbar}py}e^{-\frac{i}{\hbar}qx}u^{(i)}_a(\vec{q})\bar{u}^{(j)}_b(\vec{p}) < \vec{q},ib|\vec{p},jb >\\ &= c\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}p(x-y)}\sum_i u^{(i)}_a(\vec{p})\bar{u}^{(i)}_b(\vec{p})\\ &= c\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}p(x-y)}(\gamma^{\mu}p_{\mu}+mc)_{ab}\\ &= c(i\hbar\gamma^{\mu}\partial^x_{\mu}+mc)_{ab}\int\frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{-\frac{i}{\hbar}p(x-y)}\\ &= \frac{1}{c}(i\hbar\gamma^{\mu}\partial^x_{\mu}+mc)_{ab}D(x-y).\end{aligned}$$

Similarly

$$\begin{aligned}
\bar{S}_{ba}(y-x) &= c\int \frac{d^3p}{(2\pi\hbar)^3}\int \frac{d^3q}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}\frac{1}{2E(\vec{q})}\sum_{i,j} e^{-\frac{i}{\hbar}py}e^{\frac{i}{\hbar}qx}v_a^{(i)}(\vec{q})\bar{v}_b^{(j)}(\vec{p}) <\vec{p},jd|\vec{q},id> \\
&= c\int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{\frac{i}{\hbar}p(x-y)}\sum_i v_a^{(i)}(\vec{p})\bar{v}_b^{(i)}(\vec{p}) \\
&= c\int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{\frac{i}{\hbar}p(x-y)}(\gamma^\mu p_\mu - mc)_{ab} \\
&= -c(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}\int \frac{d^3p}{(2\pi\hbar)^3}\frac{1}{2E(\vec{p})}e^{\frac{i}{\hbar}p(x-y)} \\
&= -\frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D(y-x).
\end{aligned}$$

- The retarded Green's function of the Dirac equation can be defined by

$$(S_R)_{ab}(x-y) = \frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D_R(x-y).$$

We compute

$$\begin{aligned}
(S_R)_{ab}(x-y) &= \frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}\bigg(\theta(x^0-y^0) <0|[\hat{\phi}(x),\hat{\phi}(y)]|0>\bigg) \\
&= \frac{1}{c}\theta(x^0-y^0)(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab} <0|[\hat{\phi}(x),\hat{\phi}(y)]|0> \\
&+ \frac{i\hbar}{c}\gamma_{ab}^0\partial_0^x\theta(x^0-y^0). <0|[\hat{\phi}(x),\hat{\phi}(y)]|0> \\
&= \frac{1}{c}\theta(x^0-y^0)(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab} <0|[\hat{\phi}(x),\hat{\phi}(y)]|0> \\
&+ \frac{i\hbar}{c}\gamma_{ab}^0\delta(x^0-y^0). <0|[\hat{\phi}(x),\hat{\phi}(y)]|0> .
\end{aligned}$$

By inspection we will find that the second term will vanish. Thus we get

$$\begin{aligned}
(S_R)_{ab}(x-y) &= \frac{1}{c}\theta(x^0-y^0)(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab} <0|[\hat{\phi}(x),\hat{\phi}(y)]|0> \\
&= \frac{1}{c}\theta(x^0-y^0)(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D(x-y) \\
&- \frac{1}{c}\theta(x^0-y^0)(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D(y-x) \\
&= \theta(x^0-y^0) <0|\hat{\psi}_a(x)\bar{\hat{\psi}}_b(y)|0> +\theta(x^0-y^0) <0|\bar{\hat{\psi}}_b(y)\hat{\psi}_a(x)|0> \\
&= \theta(x^0-y^0) <0|\{\hat{\psi}_a(x),\bar{\hat{\psi}}_b(y)\}|0> .
\end{aligned}$$

- From the Fourier expansion of the retarded Green's function $D_R(x-y)$ we obtain

$$(S_R)_{ab}(x-y) = \hbar\int \frac{d^4p}{(2\pi\hbar)^4}\frac{i(\gamma^\mu p_\mu + mc)_{ab}}{p^2-m^2c^2}e^{-\frac{i}{\hbar}p(x-y)}.$$

We can immediately compute

$$(i\hbar\gamma^\mu\partial_\mu^x - mc)_{ca}(S_R)_{ab}(x-y) = \hbar\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i(\gamma^\mu p_\mu - mc)_{ca}(\gamma^\mu p_\mu + mc)_{ab}}{p^2-m^2c^2}e^{-\frac{i}{\hbar}p(x-y)}$$
$$= i\hbar\delta^4(x-y)\delta_{cb}.$$

- The Feynman propagator is defined by

$$(S_F)_{ab}(x-y) = \frac{1}{c}(i\hbar\gamma^\mu\partial_\mu^x + mc)_{ab}D_F(x-y).$$

We compute

$$(S_F)_{ab}(x-y) = \theta(x^0-y^0) < 0|\hat{\psi}_a(x)\bar{\hat{\psi}}_b(y)|0> -\theta(y^0-x^0) < 0|\bar{\hat{\psi}}_b(y)\hat{\psi}_a(x)|0>$$
$$+ \frac{i\hbar}{c}(\gamma^0)_{ab}\delta(x^0-y^0)(D(x-y)-D(y-x)).$$

Again the last term is zero and we end up with

$$(S_F)_{ab}(x-y) = < 0|T\hat{\psi}_a(x)\bar{\hat{\psi}}_b(y)|0> .$$

T is the time-ordering operator. The Fourier expansion of $S_F(x-y)$ is

$$(S_F)_{ab}(x-y) = \hbar\int\frac{d^4p}{(2\pi\hbar)^4}\frac{i(\gamma^\mu p_\mu + mc)_{ab}}{p^2-m^2c^2+i\epsilon}e^{-\frac{i}{\hbar}p(x-y)}.$$

Dirac Hamiltonian Straightforward.

Energy-Momentum Tensor We consider spacetime translations

$$x^\mu \longrightarrow x^{'\mu} = x^\mu + a^\mu.$$

The field ϕ transforms as

$$\phi \longrightarrow \phi^{'}(x^{'}) = \phi(x+a) = \phi(x) + a^\mu\partial_\mu\phi.$$

The Lagrangian density $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ is a scalar and therefore it will transform as $\phi(x)$, viz

$$\mathcal{L} \longrightarrow \mathcal{L}^{'} = \mathcal{L} + \delta\mathcal{L}\ ,\ \delta\mathcal{L} = \delta x^\mu\frac{\partial\mathcal{L}}{\partial x^\mu} = a^\mu\partial_\mu\mathcal{L}.$$

This equation means that the action changes by a surface term and hence it is invariant under spacetime translations and as a consequence Euler-Lagrange equations of motion are not affected.

From the other hand the Lagrangian density $\mathcal{L} = \mathcal{L}(\phi, \partial_\mu\phi)$ transforms as

$$\delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\partial_\mu\phi$$
$$= \left(\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\delta\phi + \partial_\mu(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi).$$

By using Euler-Lagrange equations of motion we get

$$\delta\mathcal{L} = \partial_\mu(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\delta\phi).$$

Hence by comparing we get

$$a^\nu \partial^\mu \bigg(-\eta_{\mu\nu}\mathcal{L} + \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)} \delta\phi \bigg) = 0.$$

Equivalently

$$\partial^\mu T_{\mu\nu} = 0.$$

The four conserved currents $j_\mu^{(0)} = T_{\mu 0}$ (which is associated with time translations) and $j_\mu^{(i)} = T_{\mu i}$ (which are associated with space translations) are given by

$$T_{\mu\nu} = -\eta_{\mu\nu}\mathcal{L} + \frac{\delta \mathcal{L}}{\delta(\partial^\mu \phi)} \partial_\nu \phi.$$

The conserved charges are (with $\pi = \delta\mathcal{L}/\delta(\partial_t \phi)$)

$$Q^{(0)} = \int d^3x j_0^{(0)} = \int d^3x T_{00} = \int d^3x (\pi \partial_t \phi - \mathcal{L}).$$

$$Q^{(i)} = \int d^3x j_0^{(i)} = \int d^3x T_{0i} = c \int d^3x \pi \partial_i \phi.$$

Clearly T_{00} is a Hamiltonian density and hence $Q^{(0)}$ is the Hamiltonian of the scalar field. By analogy T_{0i} is the momentum density and hence $Q^{(i)}$ is the momentum of the scalar field. We have then

$$Q^{(0)} = H \ , \ Q^{(i)} = P_i.$$

We compute

$$\frac{dH}{dt} = \int d^3x \frac{\partial T_{00}}{\partial t} = -c \int d^3x \partial^i T_{i0} = 0.$$

Similarly

$$\frac{dP_i}{dt} = 0.$$

In other words H and P_i are constants of the motion.

Electric Charge

- The Dirac Lagrangian density and as a consequence the action are invariant under the global gauge transformations

  $$\psi \longrightarrow e^{i\alpha}\psi.$$

  Under a local gauge transformation the Dirac Lagrangian density changes by

  $$\delta\mathcal{L}_{\rm Dirac} = -\hbar c \partial_\mu (\bar{\psi}\gamma^\mu \psi \alpha) + \hbar c \partial_\mu (\bar{\psi}\gamma^\mu \psi)\alpha.$$

  The total derivative leads to a surface term in the action and thus it is irrelevant. We get then

  $$\delta\mathcal{L}_{\rm Dirac} = \hbar c \partial_\mu (\bar{\psi}\gamma^\mu \psi)\alpha.$$

  Imposing $\delta\mathcal{L}_{\rm Dirac} = 0$ leads immediately to $\partial_\mu J^\mu = 0$.

- We compute

$$\hat{Q} = \frac{1}{\hbar}\int \frac{d^3p}{(2\pi\hbar)^3}\sum_i \bigg(\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},i) - \hat{d}(\vec{p},i)^+\hat{d}(\vec{p},i)\bigg).$$

$\hat{Q}$ is the electric charge.

Chiral Invariance

- The Dirac Lagrangian in terms of ψ_L and ψ_R reads

$$\begin{aligned}\mathcal{L}_{\rm Dirac} &= \bar{\psi}(i\hbar c\gamma^\mu\partial_\mu - mc^2)\psi \\ &= i\hbar c\bigg(\psi_R^+(\partial_0+\sigma^i\partial_i)\psi_R + \psi_L^+(\partial_0-\sigma^i\partial_i)\psi_L\bigg) - mc^2\bigg(\psi_R^+\psi_L + \psi_L^+\psi_R\bigg).\end{aligned}$$

- This Lagrangian is invariant under the vector transformations

$$\psi \longrightarrow e^{i\alpha}\psi \Leftrightarrow \psi_L \longrightarrow e^{i\alpha}\psi_L \text{ and } \psi_R \longrightarrow e^{i\alpha}\psi_R.$$

The variation of the Dirac Lagrangian under these transformations is

$$\delta\mathcal{L}_{\rm Dirac} = \hbar c(\partial_\mu j^\mu)\alpha + \text{surface term}\ ,\ j^\mu = \bar{\psi}\gamma^\mu\psi.$$

According to Noether's theorem each invariance of the action under a symmetry transformation corresponds to a conserved current. In this case the conserved current is the electric current density

$$j^\mu = \bar{\psi}\gamma^\mu\psi.$$

- The Dirac Lagrangian is also almost invariant under the axial vector (or chiral) transformations

$$\psi \longrightarrow e^{i\gamma^5\alpha}\psi \Leftrightarrow \psi_L \longrightarrow e^{i\gamma^5\alpha}\psi_L \text{ and } \psi_R \longrightarrow e^{i\gamma^5\alpha}\psi_R.$$

The variation of the Dirac Lagrangian under these transformations is

$$\delta\mathcal{L}_{\rm Dirac} = \bigg(\hbar c(\partial_\mu j^{\mu 5}) - 2imc^2\bar{\psi}\gamma^5\psi\bigg)\alpha + \text{surface term}\ ,\ j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi.$$

Imposing $\delta\mathcal{L}_{\rm Dirac} = 0$ yields

$$\partial_\mu j^{\mu 5} = 2i\frac{mc}{\hbar}\bar{\psi}\gamma^5\psi.$$

Hence the current $j^{\mu 5}$ is conserved only in the massless limit.

- In the massless limit we have two conserved currents j^μ and $j^{\mu 5}$. They can be rewritten as

$$j^\mu = j^\mu_L + j^\mu_R\ ,\ j^{\mu 5} = -j^\mu_L + j^\mu_R.$$

$$j^\mu_L = \bar{\Psi}_L\gamma^\mu\Psi_L\ ,\ j^\mu_R = \bar{\Psi}_R\gamma^\mu\Psi_R.$$

These are electric current densities associated with left-handed and right-handed particles.

Parity and Time Reversal Under parity we have

$$U(P)^+\hat{\psi}(x)U(P) = \eta_b\gamma^0\hat{\psi}(\tilde{x}).$$

Immediately we get

$$U(P)^+\bar{\hat{\psi}}(x)U(P) = \eta_b^*\bar{\hat{\psi}}(\tilde{x})\gamma^0.$$

Hence

$$U(P)^+\bar{\hat{\psi}}\hat{\psi}(x)U(P) = |\eta_b|^2\bar{\hat{\psi}}\hat{\psi}(\tilde{x}) = \bar{\hat{\psi}}\hat{\psi}(\tilde{x}).$$

$$U(P)^+i\bar{\hat{\psi}}\gamma^5\hat{\psi}(x)U(P) = -|\eta_b|^2 i\bar{\hat{\psi}}\gamma^5\hat{\psi}(\tilde{x}) = -i\bar{\hat{\psi}}\gamma^5\hat{\psi}(\tilde{x}).$$

$$\begin{aligned} U(P)^+\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(x)U(P) &= +|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(\tilde{x}) = +\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(\tilde{x}) \ , \ \mu = 0 \\ &= -|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(\tilde{x}) = -\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(\tilde{x}) \ , \ \mu \neq 0. \end{aligned}$$

$$\begin{aligned} U(P)^+\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(x)U(P) &= -|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(\tilde{x}) = -\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(\tilde{x}) \ , \ \mu = 0 \\ &= +|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(\tilde{x}) = +\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(\tilde{x}) \ , \ \mu \neq 0. \end{aligned}$$

Under time reversal we have

$$U(T)^+\hat{\psi}(x)U(T) = \eta_b\gamma^1\gamma^3\hat{\psi}(-x^0,\vec{x}).$$

We get

$$U(T)^+\bar{\hat{\psi}}(x)U(T) = \eta_b^*\bar{\hat{\psi}}(-x^0,\vec{x})\gamma^3\gamma^1.$$

We compute

$$U(T)^+\bar{\hat{\psi}}\hat{\psi}(x)U(T) = |\eta_b|^2\bar{\hat{\psi}}\hat{\psi}(-x^0,\vec{x}) = \bar{\hat{\psi}}\hat{\psi}(-x^0,\vec{x}).$$

$$\begin{aligned} U(T)^+i\bar{\hat{\psi}}\gamma^5\hat{\psi}(x)U(T) &= -iU(T)^+\bar{\hat{\psi}}\gamma^5\hat{\psi}(x)U(T) \\ &= -|\eta_b|^2 i\bar{\hat{\psi}}\gamma^5\hat{\psi}(-x^0,\vec{x}) = -i\bar{\hat{\psi}}\gamma^5\hat{\psi}(-x^0,\vec{x}). \end{aligned}$$

$$\begin{aligned} U(T)^+\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(x)U(T) &= U(T)^+\bar{\hat{\psi}}(x)U(T).(\gamma^\mu)^*.U(T)^+\hat{\psi}(x)U(T) \\ &= +|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(-x^0,\vec{x}) = +\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(-x^0,\vec{x}) \ , \ \mu = 0 \\ &= -|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(-x^0,\vec{x}) = -\bar{\hat{\psi}}\gamma^\mu\hat{\psi}(-x^0,\vec{x}) \ , \ \mu \neq 0. \end{aligned}$$

$$\begin{aligned} U(T)^+\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(x)U(T) &= U(T)^+\bar{\hat{\psi}}(x)U(T).(\gamma^\mu)^*\gamma^5.U(T)^+\hat{\psi}(x)U(T) \\ &= +|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(-x^0,\vec{x}) = +\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(-x^0,\vec{x}) \ , \ \mu = 0 \\ &= -|\eta_b|^2\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(-x^0,\vec{x}) = -\bar{\hat{\psi}}\gamma^\mu\gamma^5\hat{\psi}(-x^0,\vec{x}) \ , \ \mu \neq 0. \end{aligned}$$

Angular Momentum of Dirac Field

- An infintesimal rotation aroun the z axis with an angle θ is given by the Lorentz transformation

$$\Lambda = 1 + \frac{i}{\hbar}\theta\mathcal{J}^{12} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta & 0 \\ 0 & -\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Clearly

$$t^{'} = t \ , \ x^{'} = x + \theta y \ , \ y^{'} = -\theta x + y \ , \ z^{'} = z.$$

- Under this rotation the spinor transforms as

$$\psi^{'}(x^{'}) = S(\Lambda)\psi(x).$$

From one hand

$$\begin{aligned} \psi^{'}(x^{'}) &= \psi^{'}(t, x+\theta y, y-\theta x, z) \\ &= \psi^{'}(x) - \theta(x\partial_y - y\partial_x)\psi^{'}(x) \\ &= \psi^{'}(x) - \frac{i\theta}{\hbar}(\vec{x}\times\vec{p})^3\psi^{'}(x). \end{aligned}$$

From the other hand

$$\begin{aligned} \psi^{'}(x^{'}) &= S(\Lambda)\psi^{'}(x) \\ &= \psi(x) - \frac{i}{2\hbar}\omega_{\alpha\beta}\Gamma^{\alpha\beta}\psi(x) \\ &= \psi(x) - \frac{i}{\hbar}\omega_{12}\Gamma^{12}\psi(x) \\ &= \psi(x) + \frac{i}{\hbar}\theta\Gamma^{12}\psi(x) \\ &= \psi(x) + i\theta\frac{\Sigma^3}{2}\psi(x), \end{aligned}$$

where

$$\Sigma^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}.$$

Hence

$$\delta\psi(x) = \psi^{'}(x) - \psi(x) = \frac{i\theta}{\hbar}[\vec{x}\times\vec{p} + \frac{\hbar}{2}\vec{\Sigma}]^3\psi.$$

The quantity $\vec{x}\times\vec{p} + \frac{\hbar}{2}\vec{\Sigma}$ is the total angular momentum.

- Under the change $\psi(x) \longrightarrow \psi^{'}(x) = \psi(x) + \delta\psi(x)$ the Dirac Lagrangian $\mathcal{L}_{\rm Dirac} = \bar{\psi}(i\hbar c\gamma^\mu\partial_\mu - mc^2)\psi$ changes by

$$\begin{aligned} \delta\mathcal{L}_{\rm Dirac} &= \partial_\mu\left(\frac{\delta\mathcal{L}_{\rm Dirac}}{\delta(\partial_\mu\psi)}\delta\psi\right) + {\rm h.c} \\ &= -c\theta\partial_\mu j^\mu + {\rm h.c.} \end{aligned}$$

The current j^μ is given by

$$j^\mu = \bar{\psi}\gamma^\mu[\vec{x}\times\vec{p}+\frac{\hbar}{2}\vec{\Sigma}]^3\psi.$$

Assuming that the Lagrangian is invariant under the above rotation we have $\delta\mathcal{L}_{\rm Dirac}=0$ and as a consequence the current j^μ is conserved. This is an instance of Noether's theorem. The integral over space of the zero-component of the current j^0 is the conserved charge which is identified with the angular momentum along the z axis since we are considering the invariance under rotations about the z axis. Hence the angular momentum of the Dirac field along the z direction is defined by

$$\begin{aligned} J^3 &= \int d^3x j^0 \\ &= \int d^3x \psi^+(x)[\vec{x}\times\vec{p}+\frac{\hbar}{2}\vec{\Sigma}]^3\psi. \end{aligned}$$

This is conserved since

$$\begin{aligned} \frac{dJ^3}{dt} &= \int d^3x\partial_t j^0 \\ &= -\int d^3x\partial_i j^i \\ &= -\oint_S \vec{j}d\vec{S}. \end{aligned}$$

The surface S is at infinity where the Dirac field vanishes and hence the surface integral vanishes. For a general rotation the conserved charge will be the angular momentum of the Dirac field given by

$$\vec{J} = \int d^3x\psi^+(x)[\vec{x}\times\vec{p}+\frac{\hbar}{2}\vec{\Sigma}]\psi.$$

- In the quantum theory the angular momentum operator of the Dirac field along the z direction is

$$\hat{J}^3 = \int d^3x\hat{\psi}^+(x)[\vec{\hat{x}}\times\vec{\hat{p}}+\frac{\hbar}{2}\Sigma^3]\hat{\psi}(x).$$

It is clear that the angular momentum of the vacuum is zero, viz

$$\hat{J}^3|0>=0. \tag{B.12}$$

- Next we consider a one-particle zero-momentum state. This is given by

$$|\vec{0},sb>=\sqrt{\frac{2mc^2}{\hbar}}\hat{b}(\vec{0},s)^+|0>.$$

Hence

$$\begin{aligned} \hat{J}^3|\vec{0},sb> &= \sqrt{\frac{2mc^2}{\hbar}}\hat{J}^3\hat{b}(\vec{0},s)^+|0> \\ &= \sqrt{\frac{2mc^2}{\hbar}}[\hat{J}^3,\hat{b}(\vec{0},s)^+]|0>. \end{aligned}$$

Clearly for a Dirac particle at rest the orbital piece of the angular momentum operator vanishes and thus

$$\hat{J}^3 = \int d^3x \hat{\psi}^+(x)[\frac{\hbar}{2}\Sigma^3]\hat{\psi}(x).$$

We have

$$\hat{\psi}(x^0,\vec{x}) = \frac{1}{\hbar}\int \frac{d^3p}{(2\pi\hbar)^3}\hat{\chi}(x^0,\vec{p})e^{\frac{i}{\hbar}\vec{p}\vec{x}}.$$

We compute

$$\hat{J}^3 = \frac{1}{\hbar^2}\int \frac{d^3p}{(2\pi\hbar)^3}\hat{\chi}^+(x^0,\vec{p})[\frac{\hbar}{2}\Sigma^3]\hat{\chi}(x^0,\vec{p}).$$

Next we have

$$\hat{\chi}(x^0,\vec{p}) = \sqrt{\frac{c}{2\omega(\vec{p})}}\sum_i \bigg(e^{-i\omega(\vec{p})t}u^{(i)}(\vec{p})\hat{b}(\vec{p},i) + e^{i\omega(\vec{p})t}v^{(i)}(-\vec{p})\hat{d}(-\vec{p},i)^+\bigg).$$

We get

$$\hat{J}^3 = \int \frac{d^3p}{(2\pi\hbar)^3}\frac{c}{4E(\vec{p})}\sum_i\sum_j \bigg[u^{(i)+}(\vec{p})\Sigma^3u^{(j)}(\vec{p})\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},j) + v^{(i)+}(\vec{p})\Sigma^3v^{(j)}(\vec{p})\hat{d}(\vec{p},i)\hat{d}(\vec{p},j)^+$$
$$+ e^{2i\omega(\vec{p})t}u^{(i)+}(\vec{p})\Sigma^3v^{(j)}(-\vec{p})\hat{b}(\vec{p},i)^+\hat{d}(-\vec{p},j)^+ + e^{-2i\omega(\vec{p})t}v^{(i)+}(-\vec{p})\Sigma^3u^{(j)}(\vec{p})\hat{d}(-\vec{p},i)\hat{b}(\vec{p},j)\bigg].$$

We can immediately compute

$$[\hat{b}(\vec{p},i)^+\hat{b}(\vec{p},j),\hat{b}(\vec{0},s)^+] = \hbar\delta_{sj}(2\pi\hbar)^3\delta^3(\vec{p})\hat{b}(\vec{p},i)^+$$
$$[\hat{d}(\vec{p},i)\hat{d}(\vec{p},j)^+,\hat{b}(\vec{0},s)^+] = 0$$
$$[\hat{b}(\vec{p},i)^+\hat{d}(-\vec{p},j)^+,\hat{b}(\vec{0},s)^+] = 0$$
$$[\hat{d}(-\vec{p},i)\hat{b}(\vec{p},j),\hat{b}(\vec{0},s)^+] = \hbar\delta_{sj}(2\pi\hbar)^3\delta^3(\vec{p})\hat{d}(-\vec{p},i).$$

Thus (by using $u^{(i)+}(\vec{0})\Sigma^3u^{(s)}(\vec{0}) = (2E(\vec{0})\xi^{i+}\sigma^3\xi^s)/c$)

$$[\hat{J}^3,\hat{b}(\vec{0},s)^+]|0> = \sum_i \xi^{i+}\frac{\hbar\sigma^3}{2}\xi^s\hat{b}(\vec{0},i)^+|0>.$$

Hence

$$\hat{J}^3|\vec{0},sb> = \sum_i \xi^{i+}\frac{\hbar\sigma^3}{2}\xi^s|\vec{0},ib>.$$

Let us choose the basis

$$\xi_0^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\ ,\ \xi_0^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus one-particle zero-momentum states have spins given by

$$\hat{J}^3|\vec{0},1b> = \frac{\hbar}{2}|\vec{0},1b>\ ,\ \hat{J}^3|\vec{0},2b> = -\frac{\hbar}{2}|\vec{0},2b>.$$

- A similar calculation will lead to the result that one-antiparticle zero-momentum states have spins given by

$$\hat{J}^3|\vec{0},1d>=-\frac{\hbar}{2}|\vec{0},1d> \ , \ \hat{J}^3|\vec{0},2d>=\frac{\hbar}{2}|\vec{0},2d> .$$

Chapter 3

Asymptotic Solutions

- Straightforward.
- Straightforward. This is a different solution in which we do not have the constraint $t-t^{'}>0$ in the Feynman Green's function $G_{\vec{p}}(t-t^{'})$.
-

$$\begin{aligned}\int \frac{d^3p}{(2\pi)^3}G_{\vec{p}}(t-t^{'})e^{i\vec{p}(\vec{x}-\vec{x}^{'})} &= \int \frac{d^3p}{(2\pi)^3}\int\frac{dp^0}{(2\pi)^3}\frac{i}{(p^0)^2-E_{\vec{p}}^2+i\epsilon}e^{-ip^0(t-t^{'})+i\vec{p}(\vec{x}-\vec{x}^{'})}\\ &= \int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}e^{-ip(x-x^{'})}\\ &= D_F(x-x^{'}).\end{aligned}$$

- Thus the second solution corresponds to the causal Feynman propagator. Indeed by integrating both sides of the equation over $\vec{p}$ we obtain

$$\begin{aligned}\hat{\phi}(x) &= \hat{\phi}^+_{\rm in}(x)+\hat{\phi}^-_{\rm out}(x)+i\int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\vec{x}}\int_{-\infty}^{+\infty}dt^{'}G_{\vec{p}}(t-t^{'})j(t^{'},\vec{p})\\ &= \hat{\phi}^+_{\rm in}(x)+\hat{\phi}^-_{\rm out}(x)+i\int\frac{d^3p}{(2\pi)^3}\int d^4x^{'}G_{\vec{p}}(t-t^{'})J(x^{'})e^{i\vec{p}(\vec{x}-\vec{x}^{'})}.\end{aligned}$$

In other words

$$\hat{\phi}(x) = \hat{\phi}^+_{\rm in}(x)+\hat{\phi}^-_{\rm out}(x)+i\int d^4x^{'}D_F(x-x^{'})J(x^{'}).$$

Feynman Scalar Propagator Perform the integral using the residue theorem.

Fourier Transform Straightforward.

Forced Harmonic Oscillator

- Verify that

$$\sum_l S^*_{lm}S_{ln}=\delta_{mn}.$$

- We get

$$S = \exp(\alpha\hat{a}_{\rm in}^{+} - \alpha^*\hat{a}_{\rm in} + i\beta) = e^{\alpha\hat{a}_{\rm in}^{+}}e^{-\alpha^*\hat{a}_{\rm in}}e^{+i\beta-\frac{1}{2}|\alpha|^2}.$$

$$\alpha = \frac{i}{\sqrt{2E}}j(E).$$

In this result β is still arbitrary. We use $[\hat{a}_{\rm in},\hat{a}_{\rm in}^{+}]=1$ and the BHC formula

$$e^Ae^B = e^{A+B}e^{\frac{1}{2}[A,B]}.$$

In particular

$$\hat{a}_{\rm in}e^{\alpha\hat{a}_{\rm in}^{+}} = e^{\alpha\hat{a}_{\rm in}^{+}}(\hat{a}_{\rm in}+\alpha).$$

- We find

$$|<n\ out|0\ in>|^2 = \frac{x^n}{n!}e^{-x}\ ,\ x=|\alpha|^2.$$

We use $|n$ in $>=((\hat{a}_{\rm in}^{+})^n/\sqrt{n!})|0$ in $>$ and $<n$ in$|m$ in $>=\delta_{nm}$.

- We use

$$\hat{Q}_I(t) = \hat{Q}_{\rm in}(t) = \frac{1}{\sqrt{2E}}(\hat{a}_{\rm in}e^{-iEt}+\hat{a}_{\rm in}^{+}e^{iEt}).$$

We find

$$\Omega(t) = \exp(\alpha(t)\hat{a}_{\rm in}^{+} - \alpha^*(t)\hat{a}_{\rm in} + i\beta(t)) = e^{\alpha(t)\hat{a}_{\rm in}^{+}}e^{-\alpha^*(t)\hat{a}_{\rm in}}e^{+i\beta(t)-\frac{1}{2}|\alpha(t)|^2}.$$

$$\alpha(t) = \frac{i}{\sqrt{2E}}\int_{-\infty}^{t}dsJ(s)e^{iEs}.$$

The Schrodinger equation $i\partial_t\Omega(t)=\hat{V}_I(t)\Omega(t)$ becomes

$$i\partial_t\Omega = i\bigg(\partial_t\alpha\hat{a}_{\rm in}^{+} - \partial_t\alpha^*\hat{a}_{\rm in} + i\partial_t\beta - \frac{1}{2}\partial_t\alpha.\alpha^* + \frac{1}{2}\partial_t\alpha^*.\alpha\bigg)\Omega.$$

This reduces to

$$\partial_t\beta(t) = \frac{i}{2}(\alpha\partial_t\alpha^* - \alpha^*\partial_t\alpha).$$

Thus

$$\beta(t) = \frac{i}{2}\int_{-\infty}^{t}ds(\alpha\partial_s\alpha^* - \alpha^*\partial_s\alpha).$$

- In the limit $t\longrightarrow\infty$ we obtain

$$\alpha(+\infty) = \frac{i}{\sqrt{2E}}\int_{-\infty}^{+\infty}dsJ(s)e^{iEs} = \frac{i}{\sqrt{2E}}j(E) = \alpha.$$

$$-\frac{1}{2}|\alpha(+\infty)|^2 = -\frac{1}{4E}\int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} ds^{'}\, J(s)J(s^{'})e^{iE(s-s^{'})}.$$

Also

$$\begin{aligned} i\beta(+\infty) &= -\frac{1}{4E}\int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} ds^{'}\, J(s)J(s^{'})e^{-iE(s-s^{'})}\theta(s-s^{'}) \\ &+ \frac{1}{4E}\int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} ds^{'}\, J(s)J(s^{'})e^{iE(s-s^{'})}\theta(s-s^{'}). \end{aligned}$$

Hence (by using $1-\theta(s-s^{'}) = \theta(s^{'}-s)$)

$$\begin{aligned} i\beta(+\infty) - \frac{1}{2}|\alpha(+\infty)|^2 &= -\frac{1}{4E}\int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} ds^{'}\, J(s)J(s^{'})e^{-iE(s-s^{'})}\theta(s-s^{'}) \\ &- \frac{1}{4E}\int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} ds^{'}\, J(s)J(s^{'})e^{iE(s-s^{'})}\theta(s^{'}-s) \\ &= -\frac{1}{2}\int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} ds^{'}\, J(s)J(s^{'})G(s-s^{'}). \end{aligned}$$

The Feynman propagator in one-dimension is

$$G(s-s^{'}) = \frac{1}{2E}\bigg(e^{-iE(s-s^{'})}\theta(s-s^{'}) + e^{iE(s-s^{'})}\theta(s^{'}-s)\bigg).$$

The $S-$matrix is

$$S = e^{\alpha\hat{a}_{\rm in}^{+}}e^{-\alpha^{*}\hat{a}_{\rm in}}e^{-\frac{1}{2}\int_{-\infty}^{+\infty} ds \int_{-\infty}^{+\infty} ds^{'}\, J(s)J(s^{'})G(s-s^{'})}.$$

This is the same formula obtained in the second question except that β is completely fixed in this case.

Interaction Picture From one hand we compute that

$$i\partial_t\hat{Q}_I(t,\vec{p}) = -[\hat{Q}_I(t,\vec{p}),\hat{V}_I(t,\vec{p})] + \Omega(t)i\partial_t\hat{Q}_I(t,\vec{p})\Omega^{-1}(t).$$

From the other hand we compute

$$\begin{aligned} i\partial_t\hat{Q}(t,\vec{p}) &= U^{-1}(t)[\hat{Q}(\vec{p}),\hat{\mathcal{H}}_{\vec{p}}]U(t) + U^{-1}(t)[\hat{Q}(\vec{p}),\hat{V}(t,\vec{p})]U(t) \\ &= \Omega^{-1}(t)[\hat{Q}_I(t,\vec{p}),\hat{\mathcal{H}}_{\vec{p}}]\Omega(t) + \Omega^{-1}(t)[\hat{Q}_I(t,\vec{p}),\hat{V}_I(t,\vec{p})]\Omega(t). \end{aligned}$$

We can then compute immediately that

$$i\partial_t\hat{Q}_I(t,\vec{p}) = [\hat{Q}_I(t,\vec{p}),\hat{\mathcal{H}}_{\vec{p}}].$$

Next we compute

$$\begin{aligned} i\partial_t\hat{Q}_I(t,\vec{p}) = [\hat{Q}_I(t,\vec{p}),\hat{\mathcal{H}}_{\vec{p}}] &= e^{it\hat{\mathcal{H}}_{\vec{p}}}[\hat{Q}(\vec{p}),\hat{\mathcal{H}}_{\vec{p}}]e^{-it\hat{\mathcal{H}}_{\vec{p}}} \\ &= ie^{it\hat{\mathcal{H}}_{\vec{p}}}\hat{P}(\vec{p})e^{-it\hat{\mathcal{H}}_{\vec{p}}} \\ &= i\hat{P}_I(t,\vec{p}). \end{aligned}$$

Similarly we compute

$$
\begin{aligned}
i\partial_t\hat{P}_I(t,\vec{p})=[\hat{P}_I(t,\vec{p}),\hat{\mathcal{H}}_{\vec{p}}] &= e^{it\hat{\mathcal{H}}_{\vec{p}}}[\hat{P}(\vec{p}),\hat{\mathcal{H}}_{\vec{p}}]e^{-it\hat{\mathcal{H}}_{\vec{p}}}\\
&= -iE_{\vec{p}}^2 e^{it\hat{\mathcal{H}}_{\vec{p}}}\hat{Q}(\vec{p})e^{-it\hat{\mathcal{H}}_{\vec{p}}}\\
&= -iE_{\vec{p}}^2\hat{Q}_I(t,\vec{p}).
\end{aligned}
$$

Thus the operators $\hat{Q}_I(t,\vec{p})$ and $\hat{P}_I(t,\vec{p})$ describe free oscillators.

Time Ordering Operator We have

$$
\begin{aligned}
&T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3))=\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3)\ ,\ \text{if } t_1>t_2>t_3\\
&T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3))=\hat{V}_I(t_2)\hat{V}_I(t_1)\hat{V}_I(t_3)\ ,\ \text{if } t_2>t_1>t_3\\
&T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3))=\hat{V}_I(t_1)\hat{V}_I(t_3)\hat{V}_I(t_2)\ ,\ \text{if } t_1>t_3>t_2\\
&T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3))=\hat{V}_I(t_3)\hat{V}_I(t_1)\hat{V}_I(t_2)\ ,\ \text{if } t_3>t_1>t_2\\
&T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3))=\hat{V}_I(t_2)\hat{V}_I(t_3)\hat{V}_I(t_1)\ ,\ \text{if } t_2>t_3>t_1\\
&T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3))=\hat{V}_I(t_3)\hat{V}_I(t_2)\hat{V}_I(t_1)\ ,\ \text{if } t_3>t_2>t_1.
\end{aligned}
$$

Thus $T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3))$ is a function of t_1, t_2 and t_3 which is symmetric about the axis $t_1=t_2=t_3$. Therefore the integral of $T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3))$ in the different six regions $t_1>t_2>t_3$, $t_2>t_1>t_3$, etc gives the same result. Hence

$$
\frac{1}{6}\int_{-\infty}^t dt_1\int_{-\infty}^t dt_2\int_{-\infty}^t dt_3 T(\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3)) = \int_{-\infty}^t dt_1\int_{-\infty}^{t_1} dt_2\int_{-\infty}^{t_2} dt_3\hat{V}_I(t_1)\hat{V}_I(t_2)\hat{V}_I(t_3).
$$

Wick's Theorem For Forced Scalar Field In order to compute $i\beta_{\vec{p}}(t)$ when $t\longrightarrow+\infty$ we start from

$$
\partial_t\Omega_{\vec{p}}(t)\Omega_{\vec{p}}(t)^{-1}=\dot{\alpha}_{\vec{p}}\ \hat{a}_{\rm in}(\vec{p})^+-\dot{\alpha}_{\vec{p}}^*\ \hat{a}_{\rm in}(\vec{p})+\frac{V}{2}\dot{\alpha}_{\vec{p}}^*\ \alpha_{\vec{p}}-\frac{V}{2}\dot{\alpha}_{\vec{p}}\ \alpha_{\vec{p}}^*+i\dot{\beta}_{\vec{p}}.
$$

In deriving this last result we used

$$
e^{\alpha_{\vec{p}}(t)\hat{a}_{\rm in}(\vec{p})^+}\hat{a}_{\rm in}(\vec{p})=(\hat{a}_{\rm in}(\vec{p})-V\alpha_{\vec{p}}(t))e^{\alpha_{\vec{p}}(t)\hat{a}_{\rm in}(\vec{p})^+}.
$$

Clearly we must have

$$
\partial_t\Omega_{\vec{p}}(t)\Omega_{\vec{p}}(t)^{-1}=-iV_I(t,\vec{p}).
$$

From the second line of (4.58) we have

$$
\Omega(t) = T\bigg(e^{\frac{i}{V}\int_{-\infty}^t ds\sum_{\vec{p}}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(j(s,\vec{p})^*\hat{a}_{\rm in}(\vec{p})e^{-iE_{\vec{p}}s}+j(s,\vec{p})\hat{a}_{\rm in}(\vec{p})^+e^{iE_{\vec{p}}s}\right)}\bigg).
$$

The potential $\hat{V}_I(t,\vec{p})$ can then be defined by

$$
\hat{V}_I(t,\vec{p}) = -\frac{1}{V}\frac{1}{\sqrt{2E_{\vec{p}}}}\bigg(j(t,\vec{p})^*\hat{a}_{\rm in}(\vec{p})e^{-iE_{\vec{p}}t}+j(t,\vec{p})\hat{a}_{\rm in}(\vec{p})^+e^{iE_{\vec{p}}t}\bigg).
$$

The differential equation $\partial_t \Omega_{\vec{p}}(t)\Omega_{\vec{p}}(t)^{-1} = -iV_I(t,\vec{p})$ yields then the results

$$\dot{\alpha}_{\vec{p}} = \frac{i}{V}\frac{j(t,\vec{p})}{\sqrt{2E_{\vec{p}}}}e^{iE_{\vec{p}}t}.$$

$$\dot{\beta}_{\vec{p}} = \frac{iV}{2}\big(\dot{\alpha}^*_{\vec{p}}\,\alpha_{\vec{p}} - \dot{\alpha}_{\vec{p}}\,\alpha^*_{\vec{p}}\big).$$

The first equation yields precisely the formula (4.64). The second equation indicates that the phase $\beta(t)$ is actually not zero. The integration of the second equation gives

$$\begin{aligned}\beta_{\vec{p}} &= \frac{1}{4iVE_{\vec{p}}}\int_{-\infty}^{t}ds\int_{-\infty}^{s}ds^{'}j(s,\vec{p})j(s^{'},\vec{p})^*e^{iE_{\vec{p}}(s-s^{'})}\\ &- \frac{1}{4iVE_{\vec{p}}}\int_{-\infty}^{t}ds\int_{-\infty}^{s}ds^{'}j(s,\vec{p})^*j(s^{'},\vec{p})e^{-iE_{\vec{p}}(s-s^{'})}.\end{aligned}$$

By summing over $\vec{p}$ and taking the limit $t\longrightarrow\infty$ we obtain

$$i\sum_{\vec{p}}\beta_{\vec{p}}(+\infty) = \frac{1}{2}\int d^4x\int d^4x^{'}J(x)J(x^{'})\bigg(\frac{\theta(t-t^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{ip(x-x^{'})} - \frac{\theta(t-t^{'})}{V}\sum_{\vec{p}}\frac{1}{2E_{\vec{p}}}e^{-ip(x-x^{'})}\bigg).$$

Unitarity of The $S-$Matrix

- The solution $\Omega(t)$ can be written explicitly as

$$\Omega(t) = \sum_{n=0}^{\infty}(-i)^n\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2...\int_{-\infty}^{t_{n-1}}dt_n\hat{V}_I(t_1)\hat{V}_I(t_2)...\hat{V}_I(t_n).$$

The first few terms of this expansion are

$$\Omega(t) = 1 - i\int_{-\infty}^{t}dt_1\hat{V}_I(t_1) + (-i)^2\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\hat{V}_I(t_1)\hat{V}_I(t_2) + ...$$

Let us rewrite the different terms as follows

$$\int_{-\infty}^{t}dt_1\hat{V}_I(t_1) = \int_{-\infty}^{+\infty}dt_1\hat{V}_I(t_1) - \int_{t}^{+\infty}dt_1\hat{V}_I(t_1).$$

$$\begin{aligned}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\hat{V}_I(t_1)\hat{V}_I(t_2) &= \int_{-\infty}^{+\infty}dt_1\int_{-\infty}^{t_1}dt_2\hat{V}_I(t_1)\hat{V}_I(t_2) + \int_{t}^{+\infty}dt_1\int_{t_1}^{+\infty}dt_2\hat{V}_I(t_1)\hat{V}_I(t_2)\\ &- \int_{t}^{+\infty}dt_1\int_{-\infty}^{+\infty}dt_2\hat{V}_I(t_1)\hat{V}_I(t_2).\end{aligned}$$

Hence to this order we have

$$\begin{aligned}\Omega(t) &= \bigg(1 + i\int_{t}^{+\infty}dt_1\hat{V}_I(t_1) + i^2\int_{t}^{+\infty}dt_1\int_{t_1}^{+\infty}dt_2\hat{V}_I(t_1)\hat{V}_I(t_2) + ...\bigg)\\ &\times \bigg(1 - i\int_{-\infty}^{+\infty}dt_1\hat{V}_I(t_1) + (-i)^2\int_{-\infty}^{+\infty}dt_1\int_{-\infty}^{t_1}dt_2\hat{V}_I(t_1)\hat{V}_I(t_2) + ...\bigg)\\ &= \bar{T}\bigg(e^{i\int_t^{+\infty}ds\hat{V}_I(s)}\bigg)S.\end{aligned}$$

The operator $\bar{T}$ is the anti time-ordering operator, i.e. it orders earlier times to the left and later times to the right. This result is actually valid to all orders in perturbation theory. Taking the limit $t \longrightarrow -\infty$ in this equation we obtain

$$S^{-1} = \bar{T}\bigg(e^{i\int_{-\infty}^{+\infty} ds \hat{V}_I(s)}\bigg).$$

- Recall that

$$\Omega(t) = T\bigg(e^{-i\int_{-\infty}^{t} ds \hat{V}_I(s)}\bigg).$$

By taking the Hermitian conjugate we obtain

$$S^{+} = \bar{T}\bigg(e^{i\int_{-\infty}^{+\infty} ds \hat{V}_I(s)}\bigg).$$

In other words S is unitary as it should be. This is expected since by construction the operators $U(t)$ and $\Omega(t)$ are unitary.

Evolution Operator $\Omega(t)$ and Gell-Mann Low Formula Straightforward.

Interaction Fields are Free Fields We compute

$$\begin{aligned}
i\partial_t \hat{\phi}_I(t,\vec{x}) &= [\hat{\phi}_I(t,\vec{x}), \hat{H}_0] \\
&= e^{it\hat{H}_0}[\hat{\phi}(\vec{x}), \hat{H}_0]e^{-it\hat{H}_0} \\
&= e^{it\hat{H}_0}\int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\vec{x}} \int_{+} \frac{d^3\vec{q}}{(2\pi)^3}[\hat{Q}(\vec{p}), \hat{P}^{+}(\vec{q})]\hat{P}(\vec{q})e^{-it\hat{H}_0} \\
&= ie^{it\hat{H}_0}\int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\vec{x}}\hat{P}(\vec{p})e^{-it\hat{H}_0} \\
&= ie^{it\hat{H}_0}\hat{\pi}(\vec{x})e^{-it\hat{H}_0} \\
&= i\hat{\pi}_I(t,\vec{x}).
\end{aligned}$$

Similarly

$$\begin{aligned}
i\partial_t \hat{\pi}_I(t,\vec{x}) &= [\hat{\pi}_I(t,\vec{x}), \hat{H}_0] \\
&= e^{it\hat{H}_0}[\hat{\pi}(\vec{x}), \hat{H}_0]e^{-it\hat{H}_0} \\
&= e^{it\hat{H}_0}\int \frac{d^3\vec{p}}{(2\pi)^3} e^{i\vec{p}\vec{x}} \int_{+} \frac{d^3\vec{q}}{(2\pi)^3} E_{\vec{q}}^2[\hat{P}(\vec{p}), \hat{Q}^{+}(\vec{q})]\hat{Q}(\vec{q})e^{-it\hat{H}_0} \\
&= -ie^{it\hat{H}_0}\int \frac{d^3\vec{p}}{(2\pi)^3} E_{\vec{p}}^2 e^{i\vec{p}\vec{x}}\hat{Q}(\vec{p})e^{-it\hat{H}_0} \\
&= i(\vec{\nabla}^2 - m^2)e^{it\hat{H}_0}\hat{\phi}(\vec{x})e^{-it\hat{H}_0} \\
&= i(\vec{\nabla}^2 - m^2)\hat{\phi}_I(t,\vec{x}).
\end{aligned}$$

These last two results indicates that the interaction field $\hat{\phi}_I$ is a free field since it obeys the equation of motion

$$(\partial_t^2 - \vec{\nabla}^2 + m^2)\hat{\phi}_I(t,\vec{x}) = 0.$$

LSZ Reduction Formulae

- Let us consider the integral

$$\int_{-\infty}^{+\infty} dt\partial_t\bigg(e^{iE_{\vec{p}}t}(i\partial_t + E_{\vec{p}})T(\hat{Q}(t,\vec{p})\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)\bigg).$$

We compute

$$\int_{-\infty}^{+\infty} dt\partial_t\bigg(e^{iE_{\vec{p}}t}(i\partial_t + E_{\vec{p}})T(\hat{Q}(t,\vec{p})\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)\bigg) = \sqrt{2E_{\vec{p}}}\,\bigg(\hat{a}_{\rm out}(\vec{p})T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...) - T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)\hat{a}_{\rm in}(\vec{p})\bigg).$$

On the other hand we compute

$$\int_{-\infty}^{+\infty} dt\partial_t\bigg(e^{iE_{\vec{p}}t}(i\partial_t + E_{\vec{p}})T(\hat{Q}(t,\vec{p})\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)\bigg) = i\int_{-\infty}^{+\infty} dte^{iE_{\vec{p}}t}(\partial_t^2 + E_{\vec{p}}^2)T(\hat{Q}(t,\vec{p})\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...).$$

Hence we obtain the LSZ reduction formulae

$$i\int_{-\infty}^{+\infty} dte^{iE_{\vec{p}}t}(\partial_t^2 + E_{\vec{p}}^2)T(\hat{Q}(t,\vec{p})\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...) = \sqrt{2E_{\vec{p}}}\,\bigg(\hat{a}_{\rm out}(\vec{p})T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...) - T(\hat{Q}(t_1,\vec{p}_1)\hat{Q}(t_2,\vec{p}_2)...)\hat{a}_{\rm in}(\vec{p})\bigg).$$

- We use the identity (with the notation $\partial^2 = \partial_\mu\partial^\mu$)

$$\int d^3xe^{-i\vec{p}\vec{x}}(\partial^2 + m^2)\hat{\phi}(x) = (\partial_t^2 + E_{\vec{p}}^2)\hat{Q}(t,\vec{p}).$$

The above LSZ reduction formulae can then be put in the form

$$i\int d^4xe^{ipx}(\partial_\mu\partial^\mu + m^2)T(\hat{\phi}(x)\hat{\phi}(x_1)\hat{\phi}(x_2)...) = \sqrt{2E_{\vec{p}}}\,\bigg(\hat{a}_{\rm out}(\vec{p})T(\hat{\phi}(x_1)\hat{\phi}(x_2)...) - T(\hat{\phi}(x_1)\hat{\phi}(x_2)...)\hat{a}_{\rm in}(\vec{p})\bigg).$$

- Straightforward.

Wick's Theorem Straightforward.

The $4-$**Point Function in** $\Phi-$**Four Theory** The first order in perturbation theory is given by

$$i\int d^4y_1 < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_4)\mathcal{L}_{\rm int}(y_1)\bigg)|0> = i(-\frac{\lambda}{4!})\int d^4y_1 < 0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_4)\hat{\phi}_{\rm in}(y_1)^4\bigg)|0> .$$

In total we $7.5.3 = 105$ contractions which we can divide into three classes

- We contract only two external points together and the other two external points are contracted with the internal points. Here we have six diagrams corresponding to contracting (x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4) and (x_3, x_4). Each diagram corresponds to 12 contractions coming from the 4 possibilities opened to the first external point to be contracted with the internal points times the 3 possibilities opened to the second external point when contracted with the remaining internal points. See figure 9a). The value of these diagrams is

$$\begin{aligned}12i(-\frac{\lambda}{4!})\int d^4y_1 D_F(0) \quad \times \quad & \bigg[D_F(x_1-x_2)D_F(x_3-y_1)D_F(x_4-y_1)\\ &+D_F(x_1-x_3)D_F(x_2-y_1)D_F(x_4-y_1)\\ &+D_F(x_1-x_4)D_F(x_3-y_1)D_F(x_2-y_1)\\ &+D_F(x_2-x_3)D_F(x_1-y_1)D_F(x_4-y_1)\\ &+D_F(x_2-x_4)D_F(x_3-y_1)D_F(x_1-y_1)\\ &+D_F(x_3-x_4)D_F(x_1-y_1)D_F(x_2-y_1)\bigg].\end{aligned}$$

  The corresponding Feynman diagram is shown on figure 10a).

- We can contract all the internal points among each other. In this case we have three distinct diagrams corresponding to contracting x_1 with x_2 and x_3 with x_4 or x_1 with x_3 and x_2 with x_4 or x_1 with x_4 and x_2 with x_3. Each diagram corresponds to 3 contractions coming from the three possibilities of contracting the internal points among each other. See figure 9b). The value of these diagrams is

$$\begin{aligned}3i(-\frac{\lambda}{4!})\int d^4y_1 D_F(0)^2\bigg[D_F(x_1-x_2)D_F(x_3-x_4) \quad &+\\ D_F(x_1-x_3)D_F(x_2-x_4)+D_F(x_1-x_4)D_F(x_2-x_3)\bigg]&.\end{aligned}$$

  The corresponding Feynman diagram is shown on figure 10b).

- The last possibility is to contract all the internal points with the external points. The first internal point can be contracted in 4 different ways with the external points, the second internal point will have 3 possibilities, the third internal point will have two possibilities and the fourth internal point will have one possibility. Thus there are $4.3.2 = 24$ contractions corresponding to a single diagram. See figure 9c). The value of this diagram is

$$24i(-\frac{\lambda}{4!})\int d^4y_1\bigg[D_F(x_1-y_1)D_F(x_2-y_1)D_F(x_3-y_1)D_F(x_4-y_1)\bigg].$$

  The corresponding Feynman diagram is shown on figure 10c).

The second order in perturbation theory is given by

$$\begin{aligned}\frac{i^2}{2!}\int d^4y_1\int d^4y_2 <0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_4)\mathcal{L}_{\rm int}(y_1)\mathcal{L}_{\rm int}(y_2)\bigg)|0> \quad &=\\ -\frac{1}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 <0|T\bigg(\hat{\phi}_{\rm in}(x_1)...\hat{\phi}_{\rm in}(x_4)\hat{\phi}_{\rm in}(y_1)^4\hat{\phi}_{\rm in}(y_2)^4\bigg)|0>&.\end{aligned}$$

There are in total 11.9.7.5.3 contractions.

- We contract two of the internal points together whereas we contract the other two with the external points. We have 6 possibilities corresponding to the 6 contractions (x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_3), (x_2, x_4) and (x_3, x_4). Thus we have (6).8.7.5.3 contractions in all involved. We focus on the contraction (x_3, x_4) since the other ones are similar. In this case we obtain 4 contractions which are precisely $a)_1$, $b)_1$,$a)_2$ and $b)_2$ shown on figure 3). The value of these diagrams is

$$-\frac{1}{2}(\frac{\lambda}{4!})^2 \int d^4y_1 \int d^4y_2 D_F(x_3 - x_4) \quad \times \quad \bigg[8.3.3 D_F(x_1 - y_1) D_F(x_2 - y_1) D_F(0)^3$$
$$+8.3.4.3 D_F(x_1 - y_1) D_F(x_2 - y_1) D_F(y_1 - y_2)^2 D_F(0)$$
$$+8.4.3.3 D_F(x_1 - y_1) D_F(x_2 - y_2) D_F(y_1 - y_2) D_F(0)^2$$
$$+8.4.3.2 D_F(x_1 - y_1) D_F(x_2 - y_2) D_F(y_1 - y_2)^3\bigg].$$

Clearly these diagrams are given by

$$D_F(x_3 - x_4) \times \bigg(a)_1 + b)_1 + a)_2 + b)_2 \text{ of figure 4}\bigg).$$

To get the other 5 possibilities we should permute the points x_1,x_2,x_3 and x_4 appropriately.

- Next we can contract the 4 internal points together giving

$$D_F(x_1 - x_2) D_F(x_3 - x_4) + D_F(x_1 - x_3) D_F(x_2 - x_4) + D_F(x_1 - x_4) D_F(x_2 - x_3).$$

This should be multiplied by the sum of 7.5.3 contractions of the external points given on figure 11. Compare with the contractions on figure $3a)_3$, $3b)_3$ and $3c)_3$. The value of these diagrams is

$$-\frac{1}{2}(\frac{\lambda}{4!})^2 \bigg(D_F(x_1 - x_2) D_F(x_3 - x_4) + D_F(x_1 - x_3) D_F(x_2 - x_4) \quad +$$
$$D_F(x_1 - x_4) D_F(x_2 - x_3)\bigg) \int d^4y_1 \int d^4y_2 \bigg(3.3 D_F(0)^4 \quad +$$
$$6.4.3 D_F(0)^2 D_F(y_1 - y_2)^2 + 4.3.2 D_F(y_1 - y_2)^4\bigg).$$

The corresponding Feynman diagrams are shown on figure 12.

- There remains 48.7.5.3 contractions which must be accounted for. These correspond to the contraction of all of the internal points with the external points. The set of all these contractions is shown on figure 13. The corresponding Feynman diagrams are shown on figure 14. The value of these diagrams is

$$\begin{aligned}
-\frac{1}{2}(\frac{\lambda}{4!})^2\int d^4y_1\int d^4y_2 D_F(x_1-y_1) &\times \Big[\\
8.3.2.3.4D_F(x_2-y_1)D_F(x_3-y_1)D_F(x_4-y_2)D_F(y_1-y_2)D_F(0) &+ \\
8.3.2.3D_F(x_2-y_1)D_F(x_3-y_1)D_F(x_4-y_1)D_F(0)^2 &+ \\
8.3.4.2.3D_F(x_2-y_1)D_F(x_3-y_2)D_F(x_4-y_1)D_F(y_1-y_2)D_F(0) &+ \\
8.3.4.3D_F(x_2-y_1)D_F(x_3-y_2)D_F(x_4-y_2)D_F(0)^2 &+ \\
8.3.4.3.2D_F(x_2-y_1)D_F(x_3-y_2)D_F(x_4-y_2)D_F(y_1-y_2)^2 &+ \\
8.4.3.3D_F(x_2-y_2)D_F(x_3-y_1)D_F(x_4-y_2)D_F(0)^2 &+ \\
8.4.3.3.2D_F(x_2-y_2)D_F(x_3-y_1)D_F(x_4-y_2)D_F(y_1-y_2)^2 &+ \\
8.4.3.2.3D_F(x_2-y_2)D_F(x_3-y_1)D_F(x_4-y_1)D_F(y_1-y_2)D_F(0) &+ \\
8.4.3.3D_F(x_2-y_2)D_F(x_3-y_2)D_F(x_4-y_1)D_F(0)^2 &+ \\
8.4.3.2.3D_F(x_2-y_2)D_F(x_3-y_2)D_F(x_4-y_2)D_F(y_1-y_2)D_F(0) &+ \\
8.4.3.3.2D_F(x_2-y_2)D_F(x_3-y_2)D_F(x_4-y_1)D_F(y_1-y_2)^2\Big]&.
\end{aligned}$$

Evolution Operator $\Omega(t,t')$ Straightforward.

$\Phi-$**Cube Theory** Straightforward.

Examination QFT
Master 2
2011-2012
Take Home

Exercise 1**:**

1)

$$\begin{aligned}
<\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega> &= S_F(x-y)+(-ie)^2\int dz_1\int dz_2 D_F^{\mu\nu}(z_1-z_2).S_F(x-z_1)\gamma_\mu S_F(z_1-z_2) \\
&\times \gamma_\nu S_F(z_2-y). \qquad \text{(B.13)}
\end{aligned}$$

2)

$$\int d^4x\, e^{ip(x-y)}<\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega> = \frac{i}{\gamma.p-m_e+i\epsilon}+\frac{i}{\gamma.p-m_e+i\epsilon}(-i\Sigma_2(p))\frac{i}{\gamma.p-m_e+i\epsilon}. \qquad \text{(B.14)}$$

$$i\Sigma_2(p)=e^2\int\frac{d^4k}{(2\pi)^4}\gamma^\mu\frac{\gamma.k+m_e}{k^2-m_e^2+i\epsilon}\gamma_\mu\frac{1}{(p-k)^2+i\epsilon}. \qquad \text{(B.15)}$$

3) By employing Feynman parameters, Wick rotation and gamma matrices in d dimensions and then going to spherical coordinates we find

$$\begin{aligned}\Sigma_2(p) &= e^2\int\frac{d^dk}{(2\pi)^d}\gamma^\mu\frac{\gamma.k+m_e}{k^2-m_e^2+i\epsilon}\gamma_\mu\frac{1}{(p-k)^2+i\epsilon}\\ &= \frac{e^2}{(4\pi)^{\frac{d}{2}}}\Gamma(2-\frac{d}{2})\int_0^1dx\;\frac{m_e(4-\epsilon)-(1-x)(2-\epsilon)\gamma.p}{(xm_e^2+(1-x)\mu^2-x(1-x)p^2)^{2-\frac{d}{2}}}.\end{aligned}\quad\text{(B.16)}$$

Since for $d\longrightarrow 4$ or equivalently $\epsilon=4-d\longrightarrow 0$ we have

$$\Gamma(2-\frac{d}{2})=\frac{2}{\epsilon}-\gamma+O(\epsilon).\quad\text{(B.17)}$$

The UV divergence is logarithmic.

4) The physical (renormalized) mass m_r is the pole of the dressed propagator which near $p^2=m_r^2$ is known to behave as

$$\int d^4x\;e^{ip(x-y)}<\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega> \;=\; \frac{iZ_2}{\gamma.p-m_r+i\epsilon}.\quad\text{(B.18)}$$

By considering the one-particle irreducible (1PI) diagrams with two electron lines the exact electron propagator becomes (by dropping the Feynman prescription)

$$\begin{aligned}\int d^4x\;e^{ip(x-y)}<\Omega|T(\hat{\psi}(x)\bar{\hat{\psi}}(y))|\Omega> &= \frac{i}{\gamma.p-m_e}+\frac{i}{\gamma.p-m_e}(-i\Sigma(p))\frac{i}{\gamma.p-m_e}\\ &+ \frac{i}{\gamma.p-m_e}(-i\Sigma(p))\frac{i}{\gamma.p-m_e}(-i\Sigma(p))\frac{i}{\gamma.p-m_e}+...\\ &= \frac{i}{\gamma.p-m_e-\Sigma(p)}.\end{aligned}\quad\text{(B.19)}$$

The pole is given by the equation

$$(\gamma.p-m_e-\Sigma(p))|_{\gamma.p=m_r}=0.\quad\text{(B.20)}$$

The renormalized mass at one-loop is thus

$$m_r=m_e+\Sigma_2(m_e).\quad\text{(B.21)}$$

5) It is not difficult to show that

$$\frac{i}{\gamma.p-m_e-\Sigma(p)}=\frac{iZ_2}{\gamma.p-m_r}.\quad\text{(B.22)}$$

The wave-function renormalization Z_2 is given in terms of the electron self-energy by

$$Z_2^{-1}=1-\frac{d\Sigma(p)}{d\gamma.p}|_{\gamma.p=m_r}.\quad\text{(B.23)}$$

At one-loop we get

$$\begin{aligned}Z_2^{-1} &= 1-\frac{d\Sigma_2(p)}{d\gamma.p}|_{\gamma.p=m_r}\\ &= 1+\frac{e^2}{(4\pi)^{\frac{d}{2}}}\Gamma(2-\frac{d}{2})\int_0^1\frac{dx}{(x^2m_e^2+(1-x)\mu^2)^{2-\frac{d}{2}}}\Bigg[(1-x)(2-\epsilon)\\ &- \frac{\epsilon x(1-x)m_e^2}{x^2m_e^2+(1-x)\mu^2}(2+2x-\epsilon x)\Bigg].\end{aligned}\quad\text{(B.24)}$$

$$\begin{aligned}
\delta_2 &= -\frac{e^2}{(4\pi)^{\frac{d}{2}}}\Gamma(2-\frac{d}{2})\int_0^1 \frac{dx}{(x^2m_e^2+(1-x)\mu^2)^{2-\frac{d}{2}}}\bigg[(1-x)(2-\epsilon)\\
&- \frac{\epsilon x(1-x)m_e^2}{x^2m_e^2+(1-x)\mu^2}(2+2x-\epsilon x)\bigg]\\
&= -\frac{e^2}{(4\pi)^{\frac{d}{2}}}\Gamma(2-\frac{d}{2})\int_0^1 \frac{dx}{((1-x)^2m_e^2+x\mu^2)^{2-\frac{d}{2}}}\bigg[x(2-\epsilon)\\
&+ \frac{\epsilon x(1-x)m_e^2}{(1-x)^2m_e^2+x\mu^2}(2x-4+\epsilon(1-x))\bigg].
\end{aligned} \tag{B.25}$$

Exercise 2

1) There are 11 diagrams in total. Eight of them are given by VERTEX, WAVEFUNCTION and PHOTONVACUUM. There is a digaram in which the electron internal loop in PHOTONVACUUM is replaced by a muon internal loop. The remaining two diagrams are of the same type as RAD5 and RAD6.

2) The probability amplitude is

$$(2\pi)^4\delta^4(k+p-k^{'}-p^{'})\frac{ie^2}{q^2}(\bar{u}^{s^{'}}(p^{'})\Gamma^\mu(p^{'},p)u^s(p))(\bar{u}^{r^{'}}(k^{'})\gamma_\mu u^r(k)). \tag{B.26}$$

$$\Gamma^\mu(p^{'},p)=\gamma^\mu+ie^2\int\frac{d^4l}{(2\pi)^4}\frac{1}{(l-p)^2+i\epsilon}\bigg(\gamma^\lambda\frac{i(\gamma.l^{'}+m_e)}{l'^2-m_e^2+i\epsilon}\gamma^\mu\frac{i(\gamma.l+m_e)}{l^2-m_e^2+i\epsilon}\gamma_\lambda\bigg). \tag{B.27}$$

$$l^{'}=l-q\ ,\ q=p-p^{'}. \tag{B.28}$$

3) We find

$$\frac{1}{((l-p)^2-\mu^2+i\epsilon)(l'^2-m_e^2+i\epsilon)(l^2-m_e^2+i\epsilon)}=2\int_0^1 dxdydz\ \delta(x+y+z-1)\frac{1}{\big[L^2-\Delta+i\epsilon\big]^3}. \tag{B.29}$$

$$L=l-xp-yq\ ,\ \Delta=(1-x)^2m_e^2+x\mu^2-yzq^2. \tag{B.30}$$

4) Straightforward.

5) By using Lorentz invariance we can make the replacement

$$\begin{aligned}
\gamma^\lambda.i(\gamma.l^{'}+m_e).\gamma^\mu.i(\gamma.l+m_e)\gamma_\lambda \quad\longrightarrow\quad & -\bigg[\frac{1}{d}\gamma^\lambda\gamma^\rho\gamma^\mu\gamma_\rho\gamma_\lambda.L^2+(xp+yq)_\rho(xp+yq)_\sigma\gamma^\lambda\gamma^\rho\gamma^\mu\gamma^\sigma\gamma_\lambda\\
+\quad & m_e(xp+yq)_\rho\gamma^\lambda\gamma^\rho\gamma^\mu\gamma_\lambda-q_\rho(xp+yq)_\sigma\gamma^\lambda\gamma^\rho\gamma^\mu\gamma^\sigma\gamma_\lambda\\
-\quad & m_eq_\rho\gamma^\lambda\gamma^\rho\gamma^\mu\gamma_\lambda+m_e(xp+yq)_\rho\gamma^\lambda\gamma^\mu\gamma^\rho\gamma_\lambda+m_e^2\gamma^\lambda\gamma^\mu\gamma_\lambda\bigg].
\end{aligned} \tag{B.31}$$

By using the properties of the gamma matrices in d dimensions (equation (7.89) of Peskin and Schroeder) we get

$$\begin{aligned}\gamma^\lambda.i(\gamma.l^{'}+m_e).\gamma^\mu.i(\gamma.l+m_e)\gamma_\lambda &\longrightarrow -\bigg[\frac{(2-\epsilon)^2}{d}\gamma^\mu.L^2+(\epsilon-2)(xp+yq)_\rho(xp+yq)_\sigma\gamma^\rho\gamma^\mu\gamma^\sigma\\ &+ 2m_e(4-\epsilon)(xp+yq)^\mu-4m_eq^\mu-\epsilon q_\rho(xp+yq)_\sigma\gamma^\rho\gamma^\mu\gamma^\sigma\\ &+ 2q_\rho(xp+yq)_\sigma\gamma^\sigma\gamma^\mu\gamma^\rho+\epsilon m_eq_\rho\gamma^\rho\gamma^\mu-(2-\epsilon)m_e^2\gamma^\mu\bigg].\end{aligned} \tag{B.32}$$

This expression is sandwiched between $\bar{u}^{s^{'}}(p^{'})$ and $u^s(p)$ and thus we can use the on-shell conditions $\gamma.pu^s(p)=m_eu^s(p)$ and $\bar{u}^{s^{'}}(p^{'})\gamma.p^{'}=m_e\bar{u}^{s^{'}}(p^{'})$. We recall also that $x+y+z=1$. We then get

$$\begin{aligned}\gamma^\lambda.i(\gamma.l^{'}+m_e).\gamma^\mu.i(\gamma.l+m_e)\gamma_\lambda &\longrightarrow -\bigg[\frac{(2-\epsilon)^2}{d}\gamma^\mu.L^2+2m_e(4-\epsilon)(xp+yq)^\mu-4m_eq^\mu\\ &+ m_e\Big(2z(x+y)-\epsilon z(x+y)+\epsilon\Big)(\gamma.p)\gamma^\mu\\ &+ m_e(2-\epsilon)y(1-y)\gamma^\mu(\gamma.p^{'})+m_e^2\gamma^\mu\Big(-2y-2+\epsilon(x+y)\\ &- (\epsilon-2)y(x+y)\Big)+\Big(-2(1-y)(1-z)+\epsilon yz\Big)(\gamma.p)\gamma^\mu(\gamma.p^{'})\\ &\bigg].\end{aligned} \tag{B.33}$$

By using the results of question 4) we obtain

$$\begin{aligned}\gamma^\lambda.i(\gamma.l^{'}+m_e).\gamma^\mu.i(\gamma.l+m_e)\gamma_\lambda &\longrightarrow \gamma^\mu\bigg[-\frac{(2-\epsilon)^2}{d}L^2+m_e^2\Big(2(1-x^2-2x)+\epsilon(1-x)^2\Big)\\ &+ q^2\Big(2(1-y)(1-z)-\epsilon yz\Big)\bigg]+m_e\Big[2x(x-1)-\epsilon(x-1)^2\Big]\\ &\times (p+p^{'})^\mu.\end{aligned} \tag{B.34}$$

The term proportional to $q^\mu=(p-p^{'})^\mu$ vanishes by the symmetry $y\leftrightarrow z$. This is Ward identity in this case.

6) Next we use the Gordon's identity

$$\bar{u}^{s^{'}}(p^{'})(p+p^{'})^\mu u^s(p)\longrightarrow \bar{u}^{s^{'}}(p^{'})\Big[2m_e\gamma^\mu+i\sigma^{\mu\nu}q_\nu\Big]u^s(p). \tag{B.35}$$

We get then

$$\begin{aligned}\gamma^\lambda.i(\gamma.l^{'}+m_e).\gamma^\mu.i(\gamma.l+m_e)\gamma_\lambda &\longrightarrow \gamma^\mu\bigg[-\frac{(2-\epsilon)^2}{d}L^2+m_e^2\Big(2(1+x^2-4x)-\epsilon(1-x)^2\Big)\\ &+ q^2\Big(2(1-y)(1-z)-\epsilon yz\Big)\bigg]+m_e\Big[2x(x-1)-\epsilon(x-1)^2\Big]\\ &\times i\sigma^{\mu\nu}q_\nu.\end{aligned} \tag{B.36}$$

From here it is easy to see that

$$\Gamma^{\mu}(p^{'},p)=\gamma^{\mu}F_1(q^2)+\frac{i\sigma^{\mu\nu}q_{\nu}}{2m_e}F_2(q^2). \tag{B.37}$$

$$\begin{aligned}F_1(q^2) &= 1+2ie^2\int dxdydz\;\delta(x+y+z-1)\int\frac{d^dL}{(2\pi)^d}\frac{1}{\big[L^2-\Delta+i\epsilon\big]^3}\bigg[-\frac{(2-\epsilon)^2}{d}L^2\\ &+ m_e^2\bigg(2(1+x^2-4x)-\epsilon(1-x)^2\bigg)+q^2\bigg(2(1-y)(1-z)-\epsilon yz\bigg)\bigg].\end{aligned} \tag{B.38}$$

$$F_2(q^2)=2ie^2\int dxdydz\;\delta(x+y+z-1)\int\frac{d^dL}{(2\pi)^d}\frac{1}{\big[L^2-\Delta+i\epsilon\big]^3}2m_e^2\bigg[2x(x-1)-\epsilon(x-1)^2\bigg]. \tag{B.39}$$

7)

$$\int\frac{d^dL_E}{(2\pi)^d}\frac{L_E^2}{(L_E^2+\Delta)^3} = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Delta^{2-\frac{d}{2}}}\Gamma(2-\frac{d}{2}).\frac{d}{4}. \tag{B.40}$$

$$\int\frac{d^dL_E}{(2\pi)^d}\frac{1}{(L_E^2+\Delta)^3} = \frac{1}{(4\pi)^{\frac{d}{2}}}\frac{1}{\Delta^{3-\frac{d}{2}}}\Gamma(3-\frac{d}{2}).\frac{1}{2}. \tag{B.41}$$

8) The form factor $F_1(q^2)$ is given explicitly by

$$\begin{aligned}F_1(q^2) &= 1+2e^2\int dxdydz\;\delta(x+y+z-1)\int\frac{d^dL_E}{(2\pi)^d}\frac{1}{\big[L_E^2+\Delta\big]^3}\bigg[\frac{(2-\epsilon)^2}{d}L_E^2\\ &+ m_e^2\bigg(2(1+x^2-4x)-\epsilon(1-x)^2\bigg)+q^2\bigg(2(1-y)(1-z)-\epsilon yz\bigg)\bigg]\\ &= 1+\frac{e^2}{(4\pi)^{\frac{d}{2}}}\int dxdydz\;\delta(x+y+z-1)\bigg[\frac{(2-\epsilon)^2}{2}\frac{\Gamma(2-\frac{d}{2})}{\Delta^{2-\frac{d}{2}}}\\ &+ \frac{\Gamma(3-\frac{d}{2})}{\Delta^{3-\frac{d}{2}}}\bigg(m_e^2\bigg(2(1+x^2-4x)-\epsilon(1-x)^2\bigg)+q^2\bigg(2(1-y)(1-z)-\epsilon yz\bigg)\bigg)\bigg].\end{aligned} \tag{B.42}$$

The gamma function $\Gamma(2-\frac{d}{2})$ has a pole at $d=4$ which goes as $1/\epsilon$. Thus $F_1(q^2)$ is logarithmically divergent.

9) The renormalization constant Z_1 is defined by

$$\Gamma^{\mu}(p^{'},p)|_{q=0}=\gamma^{\mu}Z_1^{-1}. \tag{B.43}$$

We conclude that

$$Z_1^{-1}=F_1(0). \tag{B.44}$$

The counter term $\delta_1 = Z_1 - 1$ is therefore given at one-loop by (with $\delta F_1(q^2) = 1 - F_1(q^2)$ and $\Delta_0 = (1-x)^2 m_e^2 + x\mu^2$)

$$\begin{aligned}\delta_1 = -\delta F_1(0) &= -\frac{e^2}{(4\pi)^{\frac{d}{2}}}\int dxdydz\ \delta(x+y+z-1)\bigg[\frac{(2-\epsilon)^2}{2}\frac{\Gamma(2-\frac{d}{2})}{\Delta_0^{2-\frac{d}{2}}}\\ &+ \frac{\Gamma(3-\frac{d}{2})}{\Delta_0^{3-\frac{d}{2}}}.m_e^2\Big(2(1+x^2-4x)-\epsilon(1-x)^2\Big)\bigg].\end{aligned} \tag{B.45}$$

10) We use the identity

$$\int_0^1 dy \int_0^1 dz\ \delta(x+y+z-1) = 1-x. \tag{B.46}$$

Then

$$\begin{aligned}\delta_1 &= -\frac{e^2}{(4\pi)^{\frac{d}{2}}}\int dx(1-x)\bigg[\frac{(2-\epsilon)^2}{2}\frac{\Gamma(2-\frac{d}{2})}{\Delta_0^{2-\frac{d}{2}}} + \frac{\Gamma(3-\frac{d}{2})}{\Delta_0^{3-\frac{d}{2}}}.m_e^2\Big(2(1+x^2-4x)-\epsilon(1-x)^2\Big)\bigg]\\ &= -\frac{e^2}{(4\pi)^{\frac{d}{2}}}\int dx(1-x)\bigg[(2-\epsilon)\frac{\Gamma(2-\frac{d}{2})}{\Delta_0^{2-\frac{d}{2}}} + \frac{\Gamma(3-\frac{d}{2})}{\Delta_0^{3-\frac{d}{2}}}.m_e^2\Big(2(1+x^2-4x)-2(1-x)^2\Big)\bigg]\\ &= -\frac{e^2}{(4\pi)^{\frac{d}{2}}}\int dx\bigg[x(2-\epsilon)\frac{\Gamma(2-\frac{d}{2})}{\Delta_0^{2-\frac{d}{2}}} + (1-2x)(2-\epsilon)\frac{\Gamma(2-\frac{d}{2})}{\Delta_0^{2-\frac{d}{2}}} + (1-x)\frac{\Gamma(3-\frac{d}{2})}{\Delta_0^{3-\frac{d}{2}}}m_e^2 \times\\ &\Big(2(1+x^2-4x)-2(1-x)^2\Big)\bigg]\\ &= -\frac{e^2}{(4\pi)^{\frac{d}{2}}}\int dx\bigg[x(2-\epsilon)\frac{\Gamma(2-\frac{d}{2})}{\Delta_0^{2-\frac{d}{2}}} - 2m_e^2(2-\epsilon)x(1-x)^2\frac{\Gamma(3-\frac{d}{2})}{\Delta_0^{3-\frac{d}{2}}} + (1-x)\frac{\Gamma(3-\frac{d}{2})}{\Delta_0^{3-\frac{d}{2}}}m_e^2 \times\\ &\times \Big(2(1+x^2-4x)-2(1-x)^2\Big)\bigg]\\ &= -\frac{e^2}{(4\pi)^{\frac{d}{2}}}\int dx\bigg[x(2-\epsilon)\frac{\Gamma(2-\frac{d}{2})}{\Delta_0^{2-\frac{d}{2}}} + 2m_e^2 x(1-x)\Big(2x-4+\epsilon(1-x)\Big)\frac{\Gamma(3-\frac{d}{2})}{\Delta_0^{3-\frac{d}{2}}}\bigg]\\ &= \delta_2.\end{aligned} \tag{B.47}$$

Exercise 3

1) The self-energy of the photon at one-loop is

$$i\Pi_2^{\mu\nu}(q) = (-1)\int \frac{d^4k}{(2\pi)^4}\mathrm{tr}(-ie\gamma^\mu)\frac{i(\gamma.k+m_e)}{k^2-m_e^2+i\epsilon}(-ie\gamma^\nu)\frac{i(\gamma.(k+q)+m_e)}{(k+q)^2-m_e^2+i\epsilon}. \tag{B.48}$$

2) We find

$$\Pi_2(q^2) = -4e^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dx\frac{2x(1-x)}{\Delta^{2-\frac{d}{2}}}\ ,\ \Delta = m_e^2 - x(1-x)q^2. \tag{B.49}$$

This is logarithmically divergent.

3) We know that at one-loop $Z_3 = 1/(1-\Pi_2(0)) = 1+\Pi_2(0)$. Hence $\delta_3 = \Pi_2(0)$. In other words

$$\delta_3 = -4e^2 \frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}} \int_0^1 dx \frac{2x(1-x)}{m_e^{4-d}}. \tag{B.50}$$

4) We know that

$$e_{\rm eff}^2 = \frac{e_R^2}{1-\Pi_2(q^2)+\Pi_2(0)}. \tag{B.51}$$

$$e_R = e\sqrt{Z_3}. \tag{B.52}$$

Thus

$$e_{\rm eff}^2 = \frac{e_R^2}{1-\frac{\alpha_R}{3\pi}\big[\ln\frac{-q^2}{m_e^2}-\frac{5}{3}+O(\frac{m_e^2}{-q^2})\big]}. \tag{B.53}$$

The effective charge becomes large at high energies $-q^2 >> m_e^2$.

Exercise 4 See lecture.