Remarques: Dans cette les numeros des references sont les memes que dans la version corrigee. Je n'ai pas change l'ordre.

Remarques: Tu lis tres tres attentivement cette section. J'ai trouve beaucoup d'eureurs dans l'ancienne version et des mauvaises tournures de phrases

1 System and Hamiltonian

1.1 The model

The optomechanical system depicted in Fig.1 comprises two Fabry-Perot cavities where each cavity is composed by two mirrors. The first mirror is fixed and partially transmitting, while the second is movable and perfectly reflecting. The j^{th} cavity is pumped by coherent laser field with an input power $\wp_j$, a phase φ_j and a frequency ω_{L_j}. In addition, the two cavities are also pumped by two-mode squeezed light which can be for instance produced by spontaneous parametric down-conversion source (SPDC) [?]. The first (respectively, the second) squeezed mode is sent towards the first (respectively, second) cavity. The mirrors are represented by harmonic oscillators [?] with an effective mass m_{μ_j}, a mechanical damping rate γ_j and a frequency ω_{μ_j}. The starting point of all subsequent discussions will be the Hamiltonian governing the dynamics of optical and mechanical modes of the system. This Hamiltonian reads, in the rotating frame at the lasers frequencies, as [?]

$$H = \sum_{j=1}^{2} \Big[\left(\omega_{c_j} - \omega_{L_j}\right) a_j^\dagger a_j + \omega_{\mu_j} b_j^\dagger b_j + g_j a_j^\dagger a_j (b_j^\dagger + b_j) + \varepsilon_j (e^{i\varphi_j} a_j^\dagger + e^{-i\varphi_j} a_j) \Big] . \tag{1}$$

where $b_j, b_j^\dagger$ are the annihilation and creation operators associated with the mechanical mode describing the mirror j (for $j = 1, 2$). They satisfy the usual commutation relations $[b_j, b_k^\dagger] = \delta_{jk}$. As we shall mainly concerned in Sec. 3 with the quantum correlations between the mechanical modes, we will refer to the mode 1 as Alice and to the mode 2 as Bob. In equation (1), the objects a_j and $a_j^\dagger$ (for $j = 1, 2$) denote the annihilation and creation operators of the optical modes. They satisfy also the commutation rules $[a_j, a_k^\dagger] = \delta_{jk}$. The quantity g_j in the equation (1) is the optomechanical single-photon coupling rate between the j^{th} mechanical mode and the j^{th} optical mode. It is given by $g_j = \left(\omega_{c_j}/l_j\right) \sqrt{\hbar/m_{\mu_j}\omega_{\mu_j}}$ where l_j is the j^{th} cavity length. The coupling strength between the j^{th} external laser and its corresponding cavity field is defined by $\varepsilon_j = \sqrt{2\kappa_j \wp_j/\hbar\omega_{L_j}}$; κ_j being the energy decay rate of the j^{th} cavity.

1.2 Quantum Langevin equation

In the Heisenberg picture, the nonlinear quantum Langevin equations for optical and mechanical modes are given by

$$\begin{aligned}
\partial_t b_j &= -\left(\gamma_j/2 + i\omega_{\mu_j}\right) b_j - i g_j a_j^{\dagger} a_j + \sqrt{\gamma_j} b_j^{in}, && (2)\\
\partial_t a_j &= -\left(\kappa_j/2 - i\Delta_j\right) a_j - i g_j a_j (b_j^{\dagger} + b_j) - i\varepsilon_j e^{i\varphi_j} + \sqrt{\kappa_j} a_j^{in}, && (3)
\end{aligned}$$

where $\Delta_j = \omega_{L_j} - \omega_{c_j}$ for) is the j^{th} laser detuning [?] with $j = 1, 2$. In equation (3) b_j^{in} is the j^{th} random Brownian operator with zero mean value ($\langle b_j^{in} \rangle = 0$) which describes the noise induced by the vacuum fluctuations of the continuum of modes outside the cavity. We assume that the mechanical baths are Markovian so that the noise operators b_j^{in} have the following nonzero time-domain correlation functions [?, ?]

$$\begin{aligned}
\langle b_j^{in\dagger}(t) b_j^{in}(t') \rangle &= n_{\mathrm{th},j}\delta(t - t'), && (4)\\
\langle b_j^{in}(t) b_j^{in\dagger}(t') \rangle &= (n_{\mathrm{th},j} + 1)\delta(t - t'), && (5)
\end{aligned}$$

where $n_{\mathrm{th},j} = \left[\exp(\hbar\omega_{\mu_j}/k_B T_j) - 1\right]^{-1}$ is the mean thermal photons number, T_j is the temperature of the j^{th} mirror environment and k_B is the Boltzmann constant. Another kind of noise affecting the system is the j^{th} input squeezed vacuum noise operator a_j^{in} with zero mean value. They have the following non zero correlation properties [?]

$$\begin{aligned}
\langle a_j^{in^{\dagger}}(t) a_j^{in}(t') \rangle &= N\delta(t - t') \quad \text{for} \quad j = 1, 2, && (6)\\
\langle a_j^{in}(t) a_j^{in^{\dagger}}(t') \rangle &= (N + 1)\delta(t - t') \ \text{for} \quad j = 1, 2, && (7)\\
\langle a_j^{in}(t) a_k^{in}(t') \rangle &= M e^{-i\omega_{\mu}(t+t')}\delta(t - t') \quad \text{for} \quad j \neq k = 1, 2, && (8)\\
\langle a_j^{in^{\dagger}}(t) a_k^{in^{\dagger}}(t') \rangle &= M e^{i\omega_{\mu}(t+t')}\delta(t - t') \ \text{for} \quad j \neq k = 1, 2, && (9)
\end{aligned}$$

where $N = \sinh^2 r$, $M = \sinh r \cosh r$; r being the squeezing parameter (we have assumed that $\omega_{\mu_1} = \omega_{\mu_2} = \omega_{\mu}$).

1.3 Linearization of quantum Langevin equations

Due to the nonlinear nature of the radiation pressure, the exact solution coupled nonlinear quantum Langevin equations (2)-(3) is in general very challenging. To overcome this difficulty, we adopt the linearization scheme discussed in [?, ?]. In this scheme, the optical and mechanical operators a_j and b_j are decomposed as the sum of their mean value of the steady state plus fluctuation with zero mean value so that $\mathcal{O}_j = \langle \mathcal{O}_j \rangle + \delta\mathcal{O}_j = \mathcal{O}_{js} + \delta\mathcal{O}_j$ where $\mathcal{O}_j \equiv a_j, b_j$. The mean values b_{js} and a_{js} are obtained by solving the equations (2) and (3) in the steady state

$$\langle a_j \rangle = a_{js} = \frac{-i\varepsilon_j e^{i\varphi_j}}{\kappa_j/2 - i\Delta_j'} \quad \text{and} \quad \langle b_j \rangle = b_{js} = \frac{-i g_j \left|a_{js}\right|^2}{\gamma_j/2 + i\omega_{\mu_j}} \tag{10}$$

where $\Delta_j' = \Delta_j - g_j(b_{js}^* + b_{js})$ is the j^{th} effective cavity detuning including the radiation pressure effects [?, ?]. To simplify further our purpose, we assume that the double-cavity system is intensely driven ($|a_{js}| \gg 1$, for $j = 1, 2$). This assumption can be realized considering lasers with a large input power $\wp_j$ [?]. Therefore, the contributions arising from the nonlinear terms $\delta a_j^\dagger \delta a_j$, $\delta a_j \delta b_j$ and $\delta a_j \delta b_j^\dagger$ can be ignored. As result, one gets the following linearized Langevin equations

$$\delta \dot{b}_j = -\left(\gamma_j/2 + i\omega_{\mu_j}\right)\delta b_j + G_j\left(\delta a_j - \delta a_j^\dagger\right) + \sqrt{\gamma_j} b_j^{in}, \tag{11}$$

$$\delta \dot{a}_j = -\left(\kappa_j/2 - i\Delta_j'\right)\delta a_j - G_j\left(\delta b_j^\dagger + \delta b_j\right) + \sqrt{\kappa_j} a_i^{in}, \tag{12}$$

where the parameter G_j, defined by $G_j = g_j |a_{js}| = g_j \sqrt{\bar{n}_{\rm cav}^j}$, is the j^{th} light-enhanced optomechanical coupling for the linearized regime [?]. The quantity $\bar{n}_{\rm cav}^j$ is the number of photons circulating inside the j^{th} cavity [?]. We notice that the Eqs. (11) and (12) have been obtained by setting $a_{js} = -i |a_{js}|$ or equivalently by taking the phase φ_j of the j^{th} input laser field equal to $\varphi_j = -\arctan(2\Delta_j'/\kappa_j)$. Introducing the operators $\delta\tilde{b}_j$ and $\delta\tilde{a}_j$ defined respectively by $\delta b_j = \delta\tilde{b}_j e^{-i\omega_\mu t}$ and $\delta a_j = \delta\tilde{a}_j e^{i\Delta_j' t}$, the equations (11) and (12) rewrite

$$\delta\dot{\tilde{b}}_j = -\frac{\gamma_j}{2}\delta\tilde{b}_j + G_j\left(\delta\tilde{a}_j e^{i(\Delta_j'+\omega_\mu)t} - \delta\tilde{a}_j^\dagger e^{-i(\Delta_j'-\omega_\mu)t}\right) + \sqrt{\gamma_j}\tilde{b}_j^{in}, \tag{13}$$

$$\delta\dot{\tilde{a}}_j = -\frac{\kappa_j}{2}\delta\tilde{a}_j - G_j\left(\delta\tilde{b}_j e^{-i(\Delta_j'+\omega_\mu)t} + \delta\tilde{b}_j^\dagger e^{-i(\Delta_j'-\omega_\mu)t}\right) + \sqrt{\kappa_j}\tilde{a}_i^{in}. \tag{14}$$

Next, we assume that the two cavities are driven at *the red sideband* ($\Delta_j' = -\omega_\mu$ for $j = 1, 2$) which corresponds to quantum states transfer regime [?, ?]. We note also that, in the resolved-sideband regime where the mechanical frequency ω_μ of the movable mirror is larger than the j^{th} cavity decay rate κ_j ($\omega_\mu \gg \kappa_1, \kappa_2$), one can use the rotating wave approximation (RWA) [?, ?]. Therefore in a frame rotating with frequency $\pm 2\omega_\mu$, the equations (13) and (14) give

$$\delta\dot{\tilde{b}}_j = -\frac{\gamma_j}{2}\delta\tilde{b}_j + G_j\delta\tilde{a}_j + \sqrt{\gamma_j}\tilde{b}_j^{in}, \tag{15}$$

$$\delta\dot{\tilde{a}}_j = -\frac{\kappa_j}{2}\delta\tilde{a}_j - G_j\delta\tilde{b}_j + \sqrt{\kappa_j}\tilde{a}_j^{in}, \tag{16}$$

when the the fast oscillating terms are neglected.

1.4 The adiabatic elimination of the optical modes

Being interested only in the quantum correlations between mechanical modes, the ideal configuration is the adiabatic regime which corresponds to the situation where the mirrors have a large mechanical quality factor and weak effective optomechanical coupling ($\kappa_j \gg G_j, \gamma_j$) [?]. In this limiting configuration, by inserting the steady state solution of (16) into (15), one shows that the j^{th} mirror dynamics reduces to

$$\delta\dot{\tilde{b}}_j = -\frac{\Gamma_j}{2}\delta\tilde{b}_j + \sqrt{\gamma_j}\tilde{b}_j^{in} + \sqrt{\Gamma_{{\rm a}_j}}\tilde{a}_j^{in} = -\frac{\Gamma_j}{2}\delta\tilde{b}_j + \tilde{F}_j^{in}, \tag{17}$$

where $\Gamma_{{\rm a}_j} = 4G_j^2/\kappa_j$ is the effective relaxation rate induced by radiation pressure [?], $\Gamma_j = \Gamma_{{\rm a}_j} + \gamma_j$ and $\tilde{F}_j^{in} = \sqrt{\gamma_j}\tilde{b}_j^{in} + \sqrt{\Gamma_{{\rm a}_j}}\tilde{a}_j^{in}$. In terms of the quadratures

$$\delta\tilde{q}_j = (\delta\tilde{b}_j^\dagger + \delta\tilde{b}_j)/\sqrt{2}, \quad \delta\tilde{p}_j = i(\delta\tilde{b}_j^\dagger - \delta\tilde{b}_j)/\sqrt{2}, \tag{18}$$

$$\tilde{F}_{q_j}^{in} = (\tilde{F}_j^{in,\dagger} + \tilde{F}_j^{in})/\sqrt{2}, \qquad \tilde{F}_{p_j}^{in} = i(\tilde{F}_j^{in,\dagger} - \tilde{F}_j^{in})/\sqrt{2}, \tag{19}$$

the linear quantum Langevin equations (17) can be cast in matricial form [?]

$$\dot{u}(t) = Su(t) + n(t), \tag{20}$$

where $S = \mathrm{diag}(-\frac{\Gamma_1}{2}, -\frac{\Gamma_1}{2}, -\frac{\Gamma_2}{2}, -\frac{\Gamma_2}{2})$, $u(t)^{\mathrm{T}} = (\delta\tilde{q}_1, \delta\tilde{p}_1, \delta\tilde{q}_2, \delta\tilde{p}_2)$ and $n(t)^{\mathrm{T}} = (\tilde{F}^{in}_{q_1}, \tilde{F}^{in}_{p_1}, \tilde{F}^{in}_{q_2}, \tilde{F}^{in}_{p_2})$. Needless to say, the form of the matrix S guarantees the full stability of the system and in this case the use of the Routh-Hurwitz criterion [?] is not necessary. Thus, we end up with linear evolution equations for the mechanical modes with zero-mean Gaussian noises. We notice that the mechanical fluctuations in the stable regime will also evolve to an asymptotic zero-mean Gaussian state. It follows that the state of the system is completely described by the correlation matrix $V(t)$ whose elements are given by

$$V_{ii'}(t) = \frac{1}{2}(\langle u_i(t)u_{i'}(t) + u_{i'}(t)u_i(t)\rangle). \tag{21}$$

Using Eqs. (20) and (21), it is simple to check that the matrix $V(t)$ satisfies the following evolution equation [?]

$$\frac{d}{dt}V(t) = SV(t) + V(t)S^{\mathrm{T}} + D, \tag{22}$$

where D is the noise correlation matrix defined by $D_{kk'}\delta(t-t') = (\langle n_k(t)n_{k'}(t') + n_{k'}(t')n_k(t)\rangle)/2$. Using the correlation properties of the noise operators given by the set of equations (4)-(9), one shows that the matrix D takes the form

$$D = \begin{pmatrix} D_{11} & 0 & D_{13} & 0 \\ 0 & D_{22} & 0 & D_{24} \\ D_{13} & 0 & D_{33} & 0 \\ 0 & D_{24} & 0 & D_{44} \end{pmatrix}, \tag{23}$$

where $D_{11} \;= D_{22} = \Gamma_{\mathrm{a}_1}(N+1/2) + \gamma_1(n_{\mathrm{th},1}+1/2)$, $D_{33} = D_{44} = \Gamma_{\mathrm{a}_2}(N+1/2) + \gamma_2(n_{\mathrm{th},2}+1/2)$ and $D_{13} \;= -D_{24} = M\sqrt{\Gamma_{\mathrm{a}_1}\Gamma_{\mathrm{a}_2}}$. The equation (22) is easily solvable and the solution writes as

$$V(t) = \begin{pmatrix} v_{11}(t) & 0 & v_{13}(t) & 0 \\ 0 & v_{22}(t) & 0 & v_{24}(t) \\ v_{13}(t) & 0 & v_{33}(t) & 0 \\ 0 & v_{24}(t) & 0 & v_{44}(t) \end{pmatrix} \equiv \begin{pmatrix} V_1(t) & V_3(t) \\ V_3^{\mathrm{T}}(t) & V_2(t) \end{pmatrix}, \tag{24}$$

with $V_1(t) = \mathrm{diag}(v_{11}(t), v_{22}(t))$, $V_2(t) = \mathrm{diag}(v_{33}(t), v_{44}(t))$ and $V_3(t) = \mathrm{diag}(v_{13}(t), v_{24}(t))$. Notice that $V(t)$ is a real, symmetric and positive definite matrix. The 2×2 matrices $V_1(t)$ and $V_2(t)$ represent the first and second mechanical modes respectively, while the information about the correlations between them is encoded in the sub-matrix $V_3(t)$. Considering identical damping rates ($\gamma_1 = \gamma_2 = \gamma$), the explicit expressions of the covariance matrix elements are given by

$$v_{11}(t) = v_{22}(t) = \frac{(2N+1)\mathcal{C}_1 + 2n_{\mathrm{th},1}+1}{2(\mathcal{C}_1+1)} + \frac{(-2N+1)\mathcal{C}_1 - 2n_{\mathrm{th},1}+1}{2(\mathcal{C}_1+1)}e^{-\gamma(\mathcal{C}_1+1)t}, \tag{25}$$

$$v_{33}(t) = v_{44}(t) = \frac{(2N+1)\mathcal{C}_2 + 2n_{\mathrm{th},2}+1}{2(\mathcal{C}_2+1)} + \frac{(-2N+1)\mathcal{C}_2 - 2n_{\mathrm{th},2}+1}{2(\mathcal{C}_2+1)}e^{-\gamma(\mathcal{C}_2+1)t}, \tag{26}$$

$$v_{13}(t) = -v_{24}(t) = \frac{2M\sqrt{\mathcal{C}_1\mathcal{C}_2}}{\mathcal{C}_1+\mathcal{C}_2+2}\left(1 - e^{-\frac{\gamma}{2}(\mathcal{C}_1+\mathcal{C}_2+2)t}\right), \tag{27}$$

in terms of the j^{th} optomechanical cooperativity $\mathcal{C}_j$ defined by [?]

$$\mathcal{C}_j = \Gamma_{\mathrm{a}_j}/\gamma = 4G_j^2/\gamma\kappa_j = \frac{8\omega_{c_j}^2}{\gamma m_{\mu_j}\omega_\mu\omega_{L_j}l_j^2}\frac{\wp_j}{\left[\left(\frac{\kappa_j}{2}\right)^2+\omega_\mu^2\right]}. \tag{28}$$

Remak that for $r=0$, the equation (27) gives $v_{13}(t)=v_{24}(t)=0$. Accordingly, without squeezed light, the two mechanical modes are separable and they are not steerable in any direction [?]. Hence, to detect the steerability in the adiabatic regime, the squeezing parameter must take non vanishing values. In fact, if $r\neq 0$, we have $\det V_3(t)<0$ which is a necessary condition for two-mode Gaussian state to be entangled [?]. This reflects the crucial role of the squeezed light in the transfer quantum correlations from light to mechanical modes.